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DARBOUX-INTEGRABILITY AND UNIFORM CONVERGENCE

Abstract

In 1992, Šikić gives a characterization of Riemann-integrable functions as uniform limits of simple functions. The aim of this article is to prove an extension to the case of functions defined on a *basic space* $(X, \mathcal{D}, \mu)$ and with values in *any* Banach space F.

0 Introduction

In the article [6], the author gives a characterization of Riemann-integrable functions as uniform limits of simple functions; more exactly, he proves the following assertion:

Theorem (Šikić). *The function* $f : [a, b] \mapsto \mathbb{R}$ *is Riemann-integrable if and only if* f *is the uniform limit of a sequence of functions*

$$f_n = \sum_{i=1}^{l_n} a_{i,n} \cdot 1_{A_{i,n}}$$

where $A_{i,n} \in \mathcal{A}$, *the algebra of subsets of* $[a, b]$ *formed by the Lebesgue-measurable subsets* A *of* $[a, b]$ *with* $\Lambda(\mathrm{Fr}(A)) = 0$, *where* Fr *denotes the boundary and* Λ *is the Lebesgue measure.*

Note that exercise 116 of § 7 from [2] presents a generalization of this result to the case of functions with values in a Banach space of *finite* dimension. The aim of this article is to prove Theorem 2.5, which gives an extension to the case of functions defined on a *basic space* $(X, \mathcal{D}, \mu)$ and with values in *any* Banach space F. We precise that the proofs - most of them are simple - of the results quoted in this paper are in the thesis [1] .

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1 Preliminaries

1.1 Conventions and Notation

If a_i denotes an element of a vector space and A_i a subset of a set, we use the following conventions: $\sum_{i\in\emptyset} a_i = 0$, and $\bigcup_{i\in\emptyset} A_i = \emptyset$. Moreover, the notation $\coprod$ denotes *disjoint* union.

The Banach spaces we consider are over the field $\mathbb{R}$ of real numbers. Let F be a Banach space with norm $||\cdot||$, and P a non-empty subset of F; we call *diameter* of P the quantity $\mathrm{diam}(P) = \sup_{y,z\in P} ||y-z||$.

1.2 Semi-Ring

Given a set X, a *semi-ring* $\mathcal{D}$ of subsets of X is a family of subsets of X such that

- $\emptyset \in \mathcal{D}$;
- if $A, B \in \mathcal{D}$, then $A \bigcap B \in \mathcal{D}$;
- if $A, B \in \mathcal{D}$, then $A \setminus B = A \bigcap B^c = \coprod_{j=1}^{n} C_j$, where $C_j \in \mathcal{D}, 1 \leq j \leq n$.

Note that, generally, $A \setminus B \notin \mathcal{D}$.

1.3 Finite $\mathcal{D}$-Partition

Given a non-empty set X and $\mathcal{D}$ a semi-ring of subsets of X, every finite family $\pi = \{D_1, \ldots, D_n\}$ of non-empty disjoint elements of $\mathcal{D}$ and such that $X = \coprod_{j=1}^{n} D_j$, is called a *finite $\mathcal{D}$-partition* of X. We write Π_X for the set of all the finite $\mathcal{D}$-partitions of X.

1.4 Fineness on Π_X

Suppose that π_1 is a finite $\mathcal{D}$-partition of X; a finite $\mathcal{D}$-partition π_2 of X is said to be *finer* than π_1, which we note by $\pi_2 \gg \pi_1$, if every element of π_1 is the union of elements of π_2.

1.5 Remark

Given π_1 and π_2 any two finite $\mathcal{D}$-partitions of X, there exists a finite $\mathcal{D}$-partition π of X finer than π_1 and π_2. Indeed, if $\pi_1 = \{D_1, \ldots, D_m\}$ and $\pi_2 = \{E_1, \ldots, E_n\}$, it suffices to consider the set of the $D_i \bigcap E_j$ which are non-empty, $1 \leq i \leq m, 1 \leq j \leq n$.

1.6 Lemma

Let X be a non-empty set, $\mathcal{D}$ a semi-ring of subsets of X such that there exists a finite $\mathcal{D}$-partition of X. Then,

$$\mathcal{A}(\mathcal{D}) = \left\{ \coprod_{i=1}^{n} D_i \,:\, D_i \in \mathcal{D}\,,\, 1 \leq i \leq n\,,\, n \in \mathbb{N}^* \right\}$$

is the algebra (of subsets of X) generated by $\mathcal{D}$.

1.7 Remark

(to be used in the proof of Theorem 2.5)

In the hypothesis of Lemma 1.6, if $m \in \mathbb{N}^*$ and $D_1, \ldots, D_m \in \mathcal{D} \setminus \{\emptyset\}$ with $D_i \bigcap D_j = \emptyset$ if $i \neq j$, then there exists $\pi \in \Pi_X$ such that every $D_i \in \pi$, $1 \leq i \leq m$. Indeed, if $A = \coprod_{i=1}^{m} D_i = X$, then $\pi = \{D_1, \ldots, D_m\}$. And if $A = \coprod_{i=1}^{m} D_i \neq X$, then $A^c \in \mathcal{A}(\mathcal{D})$ and $A^c \neq \emptyset$, thus there exists $D_{m+1}, \ldots, D_n \in \mathcal{D} \setminus \{\emptyset\}$ such that $A^c = \coprod_{i=m+1}^{n} D_i$; so that $\pi = \{D_1, \ldots, D_n\} \in \Pi_X$.

1.8 Functions $\mathcal{D}$-Simple

Let $\mathcal{D}$ be a semi-ring of subsets of a set X (such that $\Pi_X \neq \emptyset$), and F a Banach space. Consider $V = \mathbb{R}_+$ or $V = F$, and let

$$\mathcal{S}_V(\mathcal{D}) = \left\{ \sum_{i=1}^{m} v_i \cdot 1_{D_i} : v_i \in V,\ \{D_1, \ldots, D_m\} \in \Pi_X \right\},$$

where 1_D denotes the indicator function of D. The elements of $S_V(\mathcal{D})$ are called $\mathcal{D}$-*simple functions* with values in V.

1.9 (Jordan) Content

Given X a non-empty set, and $\mathcal{D}$ a semi-ring of subsets of X such that there exists a finite $\mathcal{D}$-partition of X, we call (Jordan) *content*, any monotone function of sets μ defined on $\mathcal{A}(\mathcal{D})$ which is finite, positive and additive, that is $\mu : \mathcal{A}(\mathcal{D}) \mapsto \mathbb{R}_+$, $\mu(\emptyset) = 0$, $\mu(A) \leq \mu(B)$ if $A \subset B$, and $\mu\Big(\coprod_{i=1}^{n} A_i \Big) = \sum_{i=1}^{n} \mu(A_i), n \in \mathbb{N}^*$.

1.10 Basic Space

We call *basic space* any triple $(X, \mathcal{D}, \mu)$, where X is a non-empty set provided with a semi-ring $\mathcal{D}$ of subsets of X such that there exists a finite $\mathcal{D}$-partition of X, and μ is a (Jordan) content defined on $\mathcal{A}(\mathcal{D})$.

1.11 Lemma

Let $(X, \mathcal{D}, \mu)$ be a basic space. Then, $\mu\Big(\bigcup\limits_{i=1}^{n} A_i\Big) \leq \sum\limits_{i=1}^{n} \mu(A_i)$ *, for every* $A_i \in \mathcal{A}(\mathcal{D})$, $1 \leq i \leq n$, $n \in \mathbb{N}^*$.

1.12 Example

Consider $a < b \in \mathbb{R}$, and $X = [a,b]$; let

$$\begin{aligned} \mathcal{D} &= \Big\{ [a,b] \bigcap]\alpha,\beta] : \alpha < \beta \in \mathbb{R} \Big\} \\ &= \{ [a,\beta] : a \leq \beta \leq b \} \bigcup \{]\alpha,\beta] : a \leq \alpha \leq \beta \leq b \} , \text{ with} \end{aligned}$$

$\mu([a,\beta]) = (\beta - a)$ and $\mu(]\alpha,\beta]) = (\beta - \alpha)$. Then, $(X, \mathcal{D}, \mu)$ is a basic space.

1.13 Darboux-Integrability

Consider $(X, \mathcal{D}, \mu)$ a basic space and F a Banach space. A function $f : X \mapsto F$ is said to be *Darboux-integrable*, what we will note by $\mathfrak{D}$-*integrable* or $\mathfrak{D}$-F-*integrable*, or $\mathfrak{D}$-$(X, \mathcal{D}, \mu; F)$-*integrable* if there is a risk of confusion, if

(a) $\operatorname{diam}(f(X)) < \infty$ (what is equivalent to f bounded);

(b) for every $\varepsilon > 0$, there exists $\pi_\varepsilon = \{D_1, \ldots, D_n\}$ a finite $\mathcal{D}$-partition of X such that $\sum\limits_{i=1}^{n} \operatorname{diam}(f(D_i))\mu(D_i) < \varepsilon$.

1.14 Lemma

The set $\mathcal{I}_{\mathfrak{D}}(X, \mathcal{D}, \mu; F)$ of the $\mathfrak{D}$-$(X, \mathcal{D}, \mu; F)$-integrable functions is a vector subspace of $B(X; F)$, the set of the bounded functions from X to F, and moreover $\mathcal{S}_F(\mathcal{D})$ is a subset of $\mathcal{I}_{\mathfrak{D}}(X, \mathcal{D}, \mu; F)$.

The proposition 1.16 below will allow us to give the definition of the Darboux-integral of a Darboux-integrable function.

1.15 Notations and Remarks

Let $(X, \mathcal{D}, \mu)$ be a basic space, F a Banach space, and $f : X \mapsto F$ a bounded function. For each $D \in \mathcal{D}$ with $D \neq \emptyset$, let

$$E_f(D) = \{y \in F \, : \, \exists a \ \in \text{Conv}(f(D)) \text{ such that } ||y - a|| \leq \text{diam}(f(D))\} \ ,$$

where $\text{Conv}(f(D))$ is the convex hull of $f(D)$, that is,

$$\text{Conv}(f(D)) = \left\{ \sum_{i=1}^{m} \lambda_i f(x_i) : 0 \leq \lambda_i \leq 1, \sum_{i=1}^{m} \lambda_i = 1, x_i \in D, 1 \leq i \leq m, \, m \in \mathbb{N}^* \right\} .$$

Note that $f(D) \subset \text{Conv}(f(D)) \subset E_f(D)$; moreover, for every $x \in D$, we have $\{y \in F \, : \, ||y - f(x)|| \leq \text{diam}(f(D))\} \subset E_f(D)$. In addition, we observe that if $f(D) = \{a\}$, then $E_f(D) = \{a\}$.

1.16 Proposition

Given $(X, \mathcal{D}, \mu)$ a basic space and F a Banach space, a bounded function $f : X \mapsto F$ is $\mathfrak{D}$-F-integrable if and only if there exists $I \in F$ such that for every $\varepsilon > 0$, there exists $\pi_\varepsilon = \{D_1, \ldots, D_l\}$ a finite $\mathcal{D}$-partition of X such that $\left|\left| \sum_{i=1}^{l} y_i \cdot \mu(D_i) - I \right|\right| < \varepsilon$ for every $y_i \in E_f(D_i), 1 \leq i \leq l$. Moreover, in that case, I is unique.

1.17 Darboux-Integral

Let $f : X \mapsto F$ be a $\mathfrak{D}$-F-integrable function. Then, the unique element $I = I(f)$ established in Proposition 1.16 is called the *Darboux-integral* of f and is noted $\mathfrak{D}$-$\int_X f(x) d\mu(x)$ or $I_{\mathfrak{D}}(f)$ (to simplify the writing).

1.18 Remark

Given f a $\mathfrak{D}$-F-integrable function and $\varepsilon_n \searrow 0$, if $\pi_n = \{D_{1,n}, \ldots, D_{l_n,n}\} \in \Pi_X$ with $\sum_{i=1}^{l_n} \text{diam}(f(D_{i,n}))\mu(D_{i,n}) < \varepsilon_n$, then if $x_{i,n} \in D_{i,n}$, $1 \leq i \leq l_n$, $n \geq 1$, we obtain

$$I_{\mathfrak{D}}(f) = \lim_{n \to \infty} \left(\sum_{i=1}^{l_n} f(x_{i,n}) \mu(D_{i,n}) \right) .$$

1.19 Proposition

Let $(X, \mathcal{D}, \mu)$ *be a basic space and* Θ *a topology on* X. *Suppose that for every* $D \in \mathcal{D}$ *with* $\overline{D} \neq D$ *and for every* $\delta > 0$, *there exists* $E_1, \ldots, E_{K(D,\delta)} \in \mathcal{D} \setminus \{\emptyset\}$ *(which depend on* D *and* δ*) pairwise disjoint,* $E_k \subset D$, $1 \leq k \leq K(D,\delta)$, *with*

$$\sum_{k=1}^{K(D,\delta)} \mu(E_k) < \delta, \text{ and } \overline{\left(D \setminus \coprod_{k=1}^{K(D,\delta)} E_k\right)} \subset D.$$

Consider a Banach space F *and* $f : X \mapsto F$ *a bounded function. Then,* f *is* $\mathfrak{D}$-F*-integrable if and only if for every* $\varepsilon > 0$, *there exists* $\pi_\varepsilon = \{D_1, \ldots, D_m\} \in \Pi_X$ *such that* $\sum\limits_{i=1}^{m} \operatorname{diam}(f(\overline{D_i}))\mu(D_i) < \varepsilon$.

Moreover, in that case, $I_{\mathfrak{D}}(f) = \lim\limits_{n\to\infty} \left(\sum\limits_{i=1}^{l_n} f(x_{i,n})\mu(D_{i,n}) \right)$, *where* $x_{i,n} \in \overline{D_{i,n}}$, $1 \leq i \leq l_n$, *and* $\sum\limits_{i=1}^{l_n} \operatorname{diam}(f(\overline{D_{i,n}}))\mu(D_{i,n}) < \varepsilon_n \searrow 0$.

1.20 An Application

Consider the basic space $(X = [a,b], \mathcal{D}, \mu)$ of Example 1.12, and X provided with the usual topology.

If $D = [\alpha, \beta]$, then $\overline{D} = D$. If $D =]\alpha, \beta]$ with $\alpha < \beta$, and if $\delta > 0$, let $E_\delta =]\alpha, \gamma_\delta]$, where $\gamma_\delta = \min\{\alpha + \frac{\delta}{2}; \frac{\alpha+\beta}{2}\}$; then $\overline{D \setminus E_\delta} = [\gamma_\delta, \beta] \subset]\alpha, \beta] = D$, and moreover $\mu(E_\delta) \leq \frac{\delta}{2} < \delta$.

Consequently, by Proposition 1.19, if $f : X \mapsto F$ is a bounded function with values in a Banach space F, then f is $\mathfrak{D}$-F-integrable if and only if for every $\varepsilon > 0$, there exists $\pi_\varepsilon = \{D_1, \ldots, D_m\} \in \Pi_X$ with $\sum\limits_{i=1}^{m} \operatorname{diam}(f(\overline{D_i}))\mu(D_i) < \varepsilon$. In particular, if $F = \mathbb{R}$, then f is $\mathfrak{D}$-F-integrable if and only if for each $\varepsilon > 0$, there exists $\pi_\varepsilon = \{D_1, \ldots, D_m\} \in \Pi_X$ such that

$$\begin{aligned} \varepsilon &> \sum_{i=1}^{m} \operatorname{diam}(f(\overline{D_i} = [\alpha_i, \beta_i]))\mu(D_i) = \sum_{i=1}^{m} \sup_{x,y \in [\alpha_i,\beta_i]} |f(x) - f(y)|(\beta_i - \alpha_i) \\ &= \sum_{i=1}^{m} \Big(\sup_{x \in [\alpha_i,\beta_i]} f(x) - \inf_{x \in [\alpha_i,\beta_i]} f(x) \Big)(\beta_i - \alpha_i), \end{aligned}$$

in other words, f is $\mathfrak{D}$-$\mathbb{R}$-integrable if and only if f is Riemann-integrable. Moreover, we have $I_{\mathfrak{D}}(f) = \int\limits_a^b f(x)dx$.

1.21 Proposition

Consider a basic space $(X, \mathcal{D}, \mu)$ and F a Banach space. Then, a bounded function $f : X \mapsto F$ is $\mathfrak{D}$-F-integrable if and only if there exists a sequence $\Big(\pi_n = \{D_{1,n}, \dots, D_{k_n,n}\}\Big)_{n\geq 1}$ of finite $\mathcal{D}$-partitions of X such that $\pi_{n+1} \gg \pi_n$, $n \in \mathbb{N}^*$, and such that for every $\varepsilon > 0$, $\lim\limits_{n\to\infty} \mu(A_n(f;\varepsilon)) = 0$, where $A_n(f;\varepsilon) = \coprod\limits_{j\in J_n(\varepsilon)} D_{j,n}$, with $J_n(\varepsilon) = \{1 \leq j \leq k_n : \operatorname{diam}(f(D_{j,n})) > \varepsilon\}$, $n \in \mathbb{N}^*$.

2 Darboux-Integrability and Uniform Convergence

The aim of this paragraph is Theorem 2.5. However we first give some preliminary results. Add that Lemma 2.1 can be proved in a classical way, but Corollary 3.4 gives another proof.

2.1 Lemma

Given a basic space $(X, \mathcal{D}, \mu)$ and a Banach space F, let $(f_n)_{n\geq 1}$ be a sequence of $\mathfrak{D}$-F-integrable functions and f be a function such that f is the uniform limit of the f_n. Then, f is $\mathfrak{D}$-F-integrable.

2.2 Remark

If f is the uniform limit of $\mathcal{C}$-simple functions, where $\mathcal{C}$ is a semi-ring of subsets of X, then $f(X)$ is totally bounded and then, as F is a Banach space, we deduce that $\overline{f(X)}$ is compact. Indeed, let $\varepsilon > 0$; we have $||f - f_n||_\infty < \varepsilon$, $n \geq n_0 = n_0(\varepsilon) \in \mathbb{N}^*$, where $f_n = \sum\limits_{i=1}^{l_n} c_{i,n} \cdot 1_{C_{i,n}} \in \mathcal{S}_F(\mathcal{C})$. Then, as $\{C_{1,n_0}, \dots, C_{l_{n_0},n_0}\} \in \Pi_X$, we have $\overline{f(X)} \subset \bigcup\limits_{i=1}^{l_{n_0}} B(c_{i,n_0}, \varepsilon)$.

2.3 Definition of the Algebra $\mathcal{B}$ (of Subsets of X)

Given $B \subset X$ and $\pi = \{D_1, \dots, D_n\}$ a finite $\mathcal{D}$-partition of X, let

$$\Delta_{\pi,B} = \Big\{1 \leq i \leq n : \ D_i \bigcap B \neq \emptyset \ \text{ and } \ D_i \bigcap B^c \neq \emptyset\Big\}.$$

Let $\mathcal{B} = \Big\{B \subset X$ such that for every $\varepsilon > 0$, there exists $\pi_\varepsilon = \{D_1, \dots, D_n\}$ a finite $\mathcal{D}$-partition of X such that $\sum\limits_{i\in\Delta_{\pi_\varepsilon,B}} \mu(D_i) < \varepsilon\Big\}$.

2.4 Lemma

(a) *The family $\mathcal{B}$ is an algebra (of subsets of X) containing $\mathcal{D}$.*

(b) *If F is a Banach space, then $\overline{\mathcal{S}_F(\mathcal{B})}^{||\cdot||_\infty} \subset \mathcal{I}_{\mathfrak{D}}(X, \mathcal{D}, \mu; F)$.*

2.5 Theorem

Given a basic space $(X, \mathcal{D}, \mu)$ and a Banach space F, let $f : X \mapsto F$ be a function. Then, $f \in \overline{\mathcal{S}_F(\mathcal{B})}^{||\cdot||_\infty}$ **if and only if** *$\overline{f(X)}$ is compact and f is $\mathfrak{D}$-F-integrable.*

PROOF.

Necessity. From (b) of Lemma 2.4, f is $\mathfrak{D}$-F-integrable; moreover, from Remark 2.2, we deduce that $\overline{f(X)}$ is compact.

Sufficiency. Suppose that f is $\mathfrak{D}$-F-integrable; from Proposition 1.21, there exists a sequence $\Big(\pi_n = \{D_{1,n}, \ldots, D_{l_n,n}\}\Big)_{n\geq 1}$ of finite $\mathcal{D}$-partitions of X such that $\pi_{n+1} \gg \pi_n$, $n \in \mathbb{N}^*$, and such that for every $\varepsilon > 0$, $\lim\limits_{n\to\infty} \mu(A_n(\varepsilon)) = 0$, where $A_n(\varepsilon) = \coprod\limits_{j\in J_n(\varepsilon)} D_{j,n}$, where $J_n(\varepsilon) = \{1 \leq j \leq l_n : \operatorname{diam}(f(D_{j,n})) > \varepsilon\}$.

Consider $\varepsilon > 0$ and let $B_\varepsilon = \bigcap\limits_{n=1}^{\infty} A_n(\varepsilon)$. Prove that $B \in \mathcal{B}$ for every $B \subset B_\varepsilon$. Now, for each $\eta > 0$, there exists $n_0 = n_0(\eta) \in \mathbb{N}^*$ such that $\mu(A_n(\varepsilon)) < \eta$ for every $n \geq n_0$; consider $A_{n_0}(\varepsilon)$.

As $B \subset A_{n_0}(\varepsilon)$, we deduce $\Delta_{\pi_{n_0},B} \subset J_{n_0}(\varepsilon)$ (because if $D_{j,n_0} \bigcap B \neq \emptyset$, then $D_{j,n_0} \bigcap A_{n_0}(\varepsilon) \neq \emptyset$, and therefore $D_{j,n_0} \subset A_{n_0}(\varepsilon)$). It follows that $\sum\limits_{j\in\Delta_{\pi_{n_0},B}} \mu(D_{j,n_0}) \leq \mu(A_{n_0}(\varepsilon)) < \eta$. Thus, as $\eta > 0$ is arbitrary, we obtain $B \in \mathcal{B}$ for every $B \subset B_\varepsilon$.

Considering first (if necessary) $g = f - f(x_0)$, where $x_0 \in X$, we can suppose, without loss of generality, that there exists $x \in X$ with $f(x) = 0$. As $\overline{f(X)}$ is compact, there exists $a_1, \ldots, a_p \in F$ such that $f(X) \subset \bigcup\limits_{i=1}^{p} B(a_i; \varepsilon) = \coprod\limits_{j=1}^{q} V_j$, where $q \leq p, V_j \neq \emptyset$, and $||y - z|| < 2\varepsilon$ if $y, z \in V_j, 1 \leq j \leq q$, (where $B(a_i; \varepsilon)$ denotes the open ball of center a_i and radius ε). Indeed, let $U_m = \bigcup\limits_{i=1}^{m} B(a_i; \varepsilon)$, $1 \leq m \leq p$. Then,

$$\bigcup_{i=1}^{p} B(a_i; \varepsilon) = U_1 \coprod \coprod_{m=2}^{p} (U_m \setminus U_{m-1}),$$

and we have the existence of the V_j.

Let $f = f_1 + f_2$, where $f_1 = f \cdot 1_{B_\varepsilon}$ and $f_2 = f \cdot 1_{{B_\varepsilon}^c}$. For every $1 \le j \le q$, let $B_j = f_1^{-1}(V_j)$. There exists (one and only one) $j_0 \in \{1, \ldots, q\}$ with $0 \in V_{j_0}$. So, for every $1 \le j \le q$ with $j \ne j_0$, we have $B_j \subset B_\varepsilon$; therefore, $B_j \in \mathcal{B}$, $j \ne j_0$. As $\mathcal{B}$ is an algebra and from the fact that $X = \coprod_{j=1}^{q} B_j$, it follows that $B_{j_0} \in \mathcal{B}$.

For each $1 \le j \le q$, consider $b_j \in V_j$ and let $\varphi_\varepsilon = \sum_{j=1}^{q} b_j \cdot 1_{B_j}$. We obtain $\varphi_\varepsilon \in \mathcal{S}_F(\mathcal{B})$ and $||f_1 - \varphi_\varepsilon||_\infty \le 2\varepsilon$. Consider the case of $f_2 = f \cdot 1_{{B_\varepsilon}^c}$. Observe that $B_\varepsilon^c = \bigcup_{n=1}^{\infty} (A_n(\varepsilon))^c = (A_1(\varepsilon))^c \coprod \coprod_{n=1}^{\infty} \Big(A_n(\varepsilon) \backslash A_{n+1}(\varepsilon)\Big)$. Now, we have $(A_1(\varepsilon))^c = D_{i_1(1),1} \coprod \ldots \coprod D_{i_{k_1}(1),1} = E_1 \coprod \ldots \coprod E_{k_1}$ with $E_k = D_{i_k(1),1} \in \mathcal{D}$ (maybe $\emptyset$, but only if $(A_1(\varepsilon))^c = \emptyset$), and $\operatorname{diam}(f(D_{i_k(1),1})) \le \varepsilon$ $(1 \le k \le k_1)$; with the convention $\operatorname{diam}(\emptyset) = 0$; and for every $n \in \mathbb{N}^*$, we have

$$\begin{aligned} A_n(\varepsilon) \backslash A_{n+1}(\varepsilon) &= D_{i_1(n+1),n+1} \coprod \cdots \coprod D_{i_{k_{n+1}}(n+1),n+1} \\ &= E_{\left(\sum_{r=1}^{n} k_r\right)+1} \coprod \cdots \coprod E_{\left(\sum_{r=1}^{n} k_r\right)+k_{n+1}}, \end{aligned}$$

with $\operatorname{diam}(f(D_{i_k(n+1),n+1})) \le \varepsilon$ $(1 \le k \le k_{n+1})$; in other words $B_\varepsilon^c = \coprod_{l=1}^{\infty} E_l$ with $E_l \in \mathcal{D}$ and $\operatorname{diam}(f(E_l)) \le \varepsilon$, $l \in \mathbb{N}^*$. Let $l \in \mathbb{N}^*$; E_l corresponds to a $D_{i_{j(l)}(n_l),n_l}$ (which can be $\emptyset$), for a $n_l \in \mathbb{N}^*$; if $E_l = D_{i_{j(l)}(n_l),n_l} \ne \emptyset$, let $\alpha_l = f(\tilde{x}_l)$ for $\tilde{x}_l \in E_l$; if $E_l = \emptyset$, let $\alpha_l = 0$.

Note that $||f \cdot 1_{E_l} - \alpha_l \cdot 1_{E_l}||_\infty \le \operatorname{diam}(f(E_l)) \le \varepsilon$. Let $f_3 = \sum_{l=1}^{\infty} \alpha_l \cdot 1_{E_l}$; so, we have $||f_2 - f_3||_\infty = \Big|\Big| \sum_{l=1}^{\infty} f \cdot 1_{E_l} - \sum_{l=1}^{\infty} \alpha_l \cdot 1_{E_l} \Big|\Big|_\infty \le \varepsilon$. Given $S \subset \mathbb{N}^*$, $S \ne \emptyset$, let $B_S = \coprod_{s \in S} E_s$. Prove that $B_S \in \mathcal{B}$. If S is finite, then $B_S \in \mathcal{B}$ (because $\mathcal{B}$ is an algebra containing $\mathcal{D}$ and $E_s \in \mathcal{D}, s \in S$). If S is infinite, write $S = \{s_1, s_2, \ldots\}$ with $s_i < s_j$ if $i < j$.

Let $\eta > 0$; there exists $n_0 = n_0(\eta) \in \mathbb{N}^*$ such that for every $n \ge n_0$, we have $\mu(A_n(\varepsilon)) < \eta$. Now, for each $p \in \mathbb{N}^*$, $E_{s_p} = D_{i_{j(s_p)}(n_{s_p}),n_{s_p}}$ for a $n_{s_p} \in \mathbb{N}^*$. Observe that from the "construction" of the E_l, if $p_1 < p_2$, then $n_{s_{p_1}} \le n_{s_{p_2}}$. Consider $n_1 \ge \max\{n_0(\eta), n_{s_1}\}$ and let $p_0 = \min\{p \in \mathbb{N}^* : n_{s_p} > n_1\}$. So, we have $n_{s_{p_0}} > n_1$, $p_0 \ge 2$ (because $n_{s_1} \le n_1$), and $n_{s_{p_0-1}} \le n_1$. Moreover, $B_S = \coprod_{p=1}^{p_0-1} E_{s_p} \coprod \coprod_{p=p_0}^{\infty} E_{s_p}$.

But, for every $p \geq p_0 \geq 2$, we can write

$$\begin{aligned} E_{s_p} &= D_{i_{j(s_p)}(n_{s_p}),n_{s_p}} \subset \Big(A_{n_{s_p}-1}(\varepsilon)\backslash A_{n_{s_p}}(\varepsilon)\Big) \subset A_{n_{s_p}-1}(\varepsilon) \\ &\subset A_{n_{s_{p_0}}-1}(\varepsilon) \subset A_{n_1}(\varepsilon). \end{aligned}$$

It follows $B_S \subset \coprod_{p=1}^{p_0-1} E_{s_p} \coprod A_{n_1}(\varepsilon) =: U.$

Note that we really have a disjoint union, because $E_{s_p} = D_{i_{j(s_p)}(n_{s_p}),n_{s_p}} \subset (A_{n_{s_p}}(\varepsilon))^c$. Now, for every $1 \leq p \leq p_0 - 1$, we have $n_{s_p} \leq n_1$, and therefore $A_{n_1}(\varepsilon) \subset A_{n_{s_p}}(\varepsilon)$; so that

$$E_{s_p} \bigcap A_{n_1}(\varepsilon) \subset \Big((A_{n_{s_p}}(\varepsilon))^c \bigcap A_{n_1}(\varepsilon)\Big) \subset \Big((A_{n_1}(\varepsilon))^c \bigcap A_{n_1}(\varepsilon)\Big) = \emptyset.$$

As $A_{n_1}(\varepsilon) = \coprod_{j \in J_{n_1}(\varepsilon)} D_{j,n_1}$, it follows that $U = \coprod_{p=1}^{p_0-1} E_{s_p} \coprod \coprod_{j \in J_{n_1}(\varepsilon)} D_{j,n_1}$.

If $U = \emptyset$, then $B_S = \emptyset \in \mathcal{B}$. Suppose $U \neq \emptyset$. From Remark 1.7, there exists $\pi = \{C_1, \ldots, C_r\} \in \Pi_X$ such that the non-empty elements of $\mathcal{D}$ which constitute U appear among the C_j.

Suppose that $C_j \bigcap B_S \neq \emptyset$ and $C_j \bigcap (B_S)^c \neq \emptyset$; then, C_j cannot be one of the E_{s_p}, $1 \leq p \leq p_0 - 1$, because $E_{s_p} \subset B_S$. As C_j cannot be in U^c, the only possibility is that C_j is one of the D_{j,n_1} for a $j \in J_{n_1}(\varepsilon)$. Hence, $\coprod_{j \in \Delta_{\pi,B_S}} C_j \subset A_{n_1}(\varepsilon)$, and so $0 \leq \sum_{j \in \Delta_{\pi,B_S}} \mu(C_j) \leq \mu(A_{n_1}(\varepsilon)) < \eta$. It follows that $B_S \in \mathcal{B}$ for every $S \subset \mathbb{N}^*, S \neq \emptyset$.

Recall that $f_3 = \sum_{l=1}^{\infty} \alpha_l \cdot 1_{E_l}$ and $f(X) \subset \coprod_{j=1}^{q} V_j$, where $V_j \neq \emptyset$ and $||y-z|| < 2\varepsilon$ if $y, z \in V_j$, for $1 \leq j \leq q$. We observe that $f_3(X) \subset \coprod_{j=1}^{q} V_j$ (because $0 \in V_{j_0}$ and if $f_3(x) \neq 0$, then $f_3(x) = \alpha_{l(x)} = f(\tilde{x}_{l(x)}) \in \coprod_{j=1}^{q} V_j$). For every $1 \leq j \leq q$, let $\tilde{B}_j = f_3^{-1}(V_j)$. Then $\tilde{B}_j \in \mathcal{B}, 1 \leq j \leq q$. This is true because $\emptyset \in \mathcal{B}$, and if $j \neq j_0$ with $f_3^{-1}(V_j) \neq \emptyset$, then $f_3^{-1}(V_j) = \coprod_{l:\alpha_l \in V_j} E_l \in \mathcal{B}$ (from what precedes), and finally, we have $f_3^{-1}(V_{j_0}) = \coprod_{l:\alpha_l \in V_{j_0}} E_l \coprod \Big(\coprod_{l=1}^{\infty} E_l\Big)^c \in \mathcal{B}$.

Moreover, we have $X = \coprod_{j=1}^{q} \tilde{B}_j$. For each $1 \leq j \leq q$, let $\tilde{b}_j \in V_j$ and consider $\psi_\varepsilon = \sum_{j=1}^{q} \tilde{b}_j \cdot 1_{\tilde{B}_j} \in \mathcal{S}_F(\mathcal{B})$. So, we have $||f_3 - \psi_\varepsilon||_\infty \leq 2\varepsilon$. Let

$\xi_\varepsilon = \varphi_\varepsilon + \psi_\varepsilon \in \mathcal{S}_F(\mathcal{B})$; we can write

$$\begin{aligned}||f - \xi_\varepsilon||_\infty &= ||f - \varphi_\varepsilon - \psi_\varepsilon||_\infty = ||f_1 + f_2 - \varphi_\varepsilon - \psi_\varepsilon||_\infty \\ &\leq ||f_1 - \varphi_\varepsilon||_\infty + ||f_2 - f_3||_\infty + ||f_3 - \psi_\varepsilon||_\infty \leq 5\varepsilon\,.\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we deduce that f is the uniform limit of functions of $\mathcal{S}_F(\mathcal{B})$. □

2.6 Remarks

(1) If $\mathcal{C}$ is an algebra of subsets of X such that for every $C \in \mathcal{C}$, the function $\varphi = a{\cdot}1_C$ is $\mathfrak{D}$-F-integrable for an $a \in F\backslash\{0\}$ (F is supposed to be non-reduced to $\{0\}$), then $\mathcal{C} \subset \mathcal{B}$. (Indeed, for every $\varepsilon > 0$, there exists $\pi_\varepsilon = \{D_1, \ldots, D_n\}$ a finite $\mathcal{D}$-partition of X verifying

$$\varepsilon \cdot ||a|| > \sum_{i=1}^{n} \mathrm{diam}(\varphi(D_i)) \cdot \mu(D_i) = \sum_{i \in \Delta_{\pi_\varepsilon, C}} ||a|| \cdot \mu(D_i),$$

therefore $\sum_{i \in \Delta_{\pi_\varepsilon, C}} \mu(D_i) < \varepsilon$. It follows that $C \in \mathcal{B}$, and then we have the assertion.)

(2) Note that $\mathcal{B}$ is independent of F. As a matter of fact, $\mathcal{B}$ depends only on $(X, \mathcal{D}, \mu)$.

The following corollary corresponds to exercise 116 from § 7 of [2] adapted to the case of a basic space.

2.7 Corollary

Let $(X, \mathcal{D}, \mu)$ be a basic space, F a Banach space of finite dimension and $f : X \mapsto F$ a (bounded) function. Then, f is $\mathfrak{D}$-F-integrable if and only if $f \in \overline{\mathcal{S}(\mathcal{B})}^{||\cdot||_\infty}$.

PROOF. As $\overline{f(X)}$ is compact, the result follows from Theorem 2.5. □

2.8 Examples

(1) Consider $X = [a, b]$ with

$$\mathcal{D} = \{\,[a, \beta] : a \leq \beta \leq b\} \bigcup \{\,]\alpha, \beta] : a \leq \alpha \leq \beta \leq b\}\ ,$$

$\mu([a, \beta]) = (\beta - a)\,,\ \mu(]\alpha, \beta]) = (\beta - \alpha)\,$, where X is provided with the usual topology Θ, and let $F = \mathbb{R}$.

With reference to [6], let $\mathcal{B} = \mathcal{A}$, the algebra of subsets A of $[a, b]$ such that $\Lambda(\mathrm{Fr}(A)) = 0$, where Λ denotes the Lebesgue measure, which is complete.

Let $B \in \mathcal{B}$. For every $\varepsilon > 0$, there exists $\pi_\varepsilon = \{D_1, \ldots, D_l\} \in \Pi_X$ such that $\sum_{i \in \Delta_{\pi_\varepsilon, B}} \mu(D_i) < \varepsilon$. Considering if necessary $\{a\}$ and $]a, \beta]$, we can suppose that $D_1 = \{a\}$ and for $2 \le i \le l$, $D_i =]\alpha_i = \beta_{i-1}, \beta_i]$ with $\beta_{i-1} < \beta_i$ and $\alpha_2 = a$. Suppose that $x \in \mathrm{Fr}(B)$. There exists $1 \le i_x \le l$ such that $x \in D_{i_x}$. If $x \in]\alpha_{i_x}, \beta_{i_x}[$, then $i_x \in \Delta_{\pi_\varepsilon, B}$, so that $\mathrm{Fr}(B) \subset \left(\coprod_{i \in \Delta_{\pi_\varepsilon, B}} D_i \right) \bigcup \{a; \beta_i : 2 \le i \le l\}$. We deduce $0 \le \Lambda(\mathrm{Fr}(B)) \le \sum_{i \in \Delta_{\pi_\varepsilon, B}} \mu(D_i) + \Lambda(\{a; \beta_i : 2 \le i \le l\}) < \varepsilon + 0 = \varepsilon$, for every $\varepsilon > 0$. Consequently, $\Lambda(\mathrm{Fr}(B)) = 0$. We conclude $B \in \mathcal{A}$. It follows $\mathcal{B} \subset \mathcal{A}$.

But, for every $A \in \mathcal{A}$, the function $f = 1_A : [a, b] \mapsto \mathbb{R}$ is Riemann-integrable by the article [6]; therefore, f is $\mathfrak{D}$-$\mathbb{R}$-integrable, as we have seen in Application 1.20. From the remark (1) of 2.6, we obtain $A \in \mathcal{B}$. It follows $\mathcal{A} \subset \mathcal{B}$, and finally $\mathcal{B} = \mathcal{A}$.

(2) Consider $X = \mathbb{N}$, $\mathcal{D} = \mathcal{A}(\mathcal{D}) = \{D \subset X : D \text{ or } D^c \text{ is finite}\}$, $\mu(D) = 0$ if D is finite, and $\mu(D) = 1$ if D^c is finite. Let $E \subset X$ such that E and E^c are infinite. Then, given $D \in \mathcal{D}$ with D^c finite, we deduce that $D \cap E \neq \emptyset$ and $D \cap E^c \neq \emptyset$. So, as $\mu(D) = 1$, E cannot be an element of $\mathcal{B}$. We conclude from the definition of $\mathcal{D}$ that $\mathcal{B} = \mathcal{D}$.

3 Darboux-Integrability and Semi-Norm $||\cdot||_\mu$

In this paragraph, we only cite some results which are related to the Darboux-integrability and a semi-norm defined on $B(X, F)$. This semi-norm allows, especially, to consider the sequences of $\mathfrak{D}$-F-integrable functions, and also to characterize the $\mathfrak{D}$-F-integrable functions by the $\mathcal{D}$-simple functions.

3.1 Definition

Given a basic space $(X, \mathcal{D}, \mu)$ and F a Banach space, for every function $f \in B(X; F)$, let

$$||f||_\mu = \inf_{\substack{\gamma \in \mathcal{S}_{\mathbb{R}_+}(\mathcal{D}) \\ \text{and } \gamma \ge ||f||}} I_{\mathfrak{D}}(\gamma)\,,$$

where $I_{\mathfrak{D}}(\gamma) = \sum_{i=1}^{n} r_i \cdot \mu(D_i)$ (if $\gamma = \sum_{i=1}^{n} r_i \cdot 1_{D_i}$), and $\gamma \ge ||f||$ means $\gamma(x) \ge ||f(x)||, x \in X$.

We note that this definition extends to the case of a basic space a notion (of superior Riemann-integral) introduced in [5].

3.2 Lemma

(a) $||\cdot||_\mu$ *is a semi-norm on* $B(X;F)$.
Moreover, $||f||_\mu \leq ||f||_\infty \cdot \mu(X)$ *for every* $f \in B(X;F)$.
(b) *For every* $f \in \mathcal{S}_F(\mathcal{D})$, $||f||_\mu = I_{\mathfrak{D}}(||f||)$.
(c) *Let* $f : X \mapsto F$ *be a bounded function such that* $||f|| : X \mapsto \mathbb{R}$ *is* $\mathfrak{D}$-$\mathbb{R}$-*integrable. Then,* $||f||_\mu = I_{\mathfrak{D}}(||f||)$.

3.3 Proposition

Let $(X, \mathcal{D}, \mu)$ *be a basic space and* F *a Banach space. Consider* $(f_n)_{n\geq 1}$ *a sequence of* $\mathfrak{D}$-F-*integrable functions,* $f_n : X \mapsto F$, *and let* $f \in B(X;F)$.
Suppose that $\lim\limits_{n\to\infty} ||f - f_n||_\mu = 0$. *Then,* f *is* $\mathfrak{D}$-F-*integrable and* $I_{\mathfrak{D}}(f) = \lim\limits_{n\to\infty} I_{\mathfrak{D}}(f_n)$.

3.4 Corollary

Let $f_n : X \mapsto F$ $(n \geq 1)$ *be a sequence of* $\mathfrak{D}$-F-*integrable functions and* $f \in B(X;F)$ *such that* $f_n \xrightarrow[n\to\infty]{} f$ *uniformly. Then,* f *is* $\mathfrak{D}$-F-*integrable and* $I_{\mathfrak{D}}(f) = \lim\limits_{n\to\infty} I_{\mathfrak{D}}(f_n)$.

3.5 Proposition

Let $(X, \mathcal{D}, \mu)$ *be a basic space,* F *a Banach space, and* $f \in B(X;F)$. *Then,*

$$f \text{ is } \mathfrak{D}\text{-}F\text{-integrable if and only if } f \in \overline{\mathcal{S}_F(\mathcal{D})}^{||\cdot||_\mu},$$

that is, there exists a sequence $(f_n)_{n\geq 1}$ *of functions of* $\mathcal{S}_F(\mathcal{D})$ *with* $\lim\limits_{n\to\infty} ||f - f_n||_\mu = 0$.

Remark: From Proposition 3.3, we have $I_{\mathfrak{D}}(f) = \lim\limits_{n\to\infty} I_{\mathfrak{D}}(f_n)$.

3.6 Remarks

(1) Precise however that, even if the notations and the approach used are different, the essential of Proposition 3.5 is in exercise 99 of § 7 of [2].

(2) Note that if the function f is defined on $\mathcal{D}$ instead of X, it is possible to consider an interesting type of integral (similar, but different, to those presented in [1]) as it is suggested by the article [3], where the author establishes, under the continuum hypothesis, an *integral* representation of the second dual of $C([0,1])$. Add that in [4], the author extends the integral representation in a more general context and in relation with the axioms of the set theory.

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References

[1] Y. Bolle, *Intégrales de Riemann dans les espaces abstraits*, Thèse **2107** (2000), Ecole Polytechnique Fédérale de Lausanne, 1999.

[2] H. Günzler, *Integration*, Bibliographisches Institut, Mannheim/Wien/Zürich, 1985.

[3] R. D. Mauldin, *A representation theorem for the second dual of* $C[0,1]$, Studia Mathematica, T. **XLVI** (1973), 197–200.

[4] R. D. Mauldin, *Some Effects of Set-Theoretical Assumptions in Measure Theory*, Advances In Mathematics, **27** (1978), 45–62.

[5] L. Schwartz, *Cours d'Analyse*, Vol. 1, Chapitre 4, Hermann, Paris, 1981.

[6] H. Šikić, *Riemann Integral vs. Lebesgue Integral*, Real Analysis Exchange, **17**(2) (1991-92), 622–632.