Real Analysis Exchange
Vol. 29(1), 2003/2004, pp. 497–502

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CATEGORY OF DENSITY POINTS OF FAT CANTOR SETS

Abstract

Denote by $D_\gamma(P)$ the set of those points where the lower Lebesgue density of $P \subset \mathbb{R}$ is bigger or equal than γ. We show that if $\gamma > 0.5$ then $D_\gamma(P) \cap P$ is always of first category in any nowhere dense perfect set P. On the other hand, there exists a fat Cantor set Q which is a subset of $D_{0.5}(Q)$ while for other fat Cantor sets P it is possible that $D_+(P) = \cup_{\gamma>0} D_\gamma(P)$ is of first category in Q.

1 Introduction

In this note we answer a question asked from us by A. Danielyan during the Sixteenth Spring Miniconference on Real Analysis. First it seemed that the not too difficult answer to this question is known and published but we could not find any reference to it. Danielyan's question was motivated by his research in Complex Analysis, namely, study of convergence properties of polynomials bounded on certain sets, see [4], [5], and [6]. So this argument might be of interest for people outside of Real Analysis.

Let us start with the original question. We will use the Lebesgue measure λ on $\mathbb{R}$. We call a nowhere dense perfect set $P \subset \mathbb{R}$ a fat Cantor set if each of its portions is of positive Lebesgue measure. The lower density of P at $x \in \mathbb{R}$, denoted by $\underline{D}(x, P)$, is defined as

$$\liminf_{h\to 0} \frac{\lambda((x-h, x+h) \cap P)}{2h}.$$

Key Words: Baire category, density point, nowhere dense perfect set, Cantor set
Mathematical Reviews subject classification: Primary 28A15; Secondary 28A75, 54E52
Received by the editors April 23, 2003
Communicated by: R. Daniel Mauldin
*This work was completed while this author visited Department of Mathematics of University of North Texas.

Given $\gamma \in (0,1]$ we denote by $D_\gamma(P)$ the set of those points in $\mathbb{R}$ where $\underline{D}(x,P) \geq \gamma$. Of course, $D_1(P)$ is the set of Lebesgue density points of P. For these points one could use lim instead of lim inf in the definition of $\underline{D}(x,P)$ and by Lebesgue's density theorem almost all points of P are in $D_1(P)$, that is, $\lambda(P \setminus D_1(P)) = 0$. The points of positive lower density of P, namely, those points where $\underline{D}(x,P) > 0$ will be denoted by $D_+(P)$. Since $P \setminus D_+(P) \subset P \setminus D_1(P)$, we also have $\lambda(P \setminus D_+(P)) = 0$. So $D_1(P) \cap P$ and $D_+(P) \cap P$ are large. Danielyan's questions were concerning what can be said about the Baire category of $D_1(P)$ and $D_+(P)$. The answers are the following.

In Theorem 1 we show that for any $\gamma > 0.5$ and any nowhere dense perfect set, $D_\gamma(P)$ is always of first category. So for fat Cantor sets $P \setminus D_1(P)$ and $D_1(P) \cap P$ give a natural decomposition into two subsets, one having measure zero, the other one being of first category. This "unpleasant relationship" of measure and category is the topic of Oxtoby's book [8]. In the proof of Theorem 4.1 of our joint paper with W. F. Pfeffer, [3], the interested reader can find an argument how can one use the category theorem in some difficult situations.

In Theorem 2 we show that there exists a fat Cantor set, P, such that $D_{0.5}(P) \supset P$. On the other hand, in Theorem 3 we show that there are fat Cantor sets for which $D_+(P) \cap P$ is of first category in P. So if the lower density is less than or equal $1/2$ then the geometry of the set determines how large is $D_\gamma(P)$.

In the metric case one can ask that apart from having Lebesgue measure zero how small the set $P \setminus D_1(P)$ should be, if one considers Hausdorff measure. For results of this flavor see the classical papers [1] and [2] by Besicovitch.

In our search of the literature about results related to the topic of this paper we also found interesting the sequence of results by O'Malley, Malý, Preiss and Zajíček, [9], [7], and [10] related to the O'Malley density property which states that if $A \subset \mathbb{R}$ is a bounded F_σ set with left density one at all of its points then its complement contains a point with right density one.

2 Main Results

Theorem 1. *If $\gamma > 0.5$ and $P \subset \mathbb{R}$ is any nowhere dense perfect set then $D_\gamma(P)$ is always of first category in P.*

PROOF. Proceeding towards a contradiction assume that $D_\gamma(P)$ is of second category. Choose γ' such that $0.5 < \gamma' < \gamma$. Set

$$H_n = \left\{ x \in P : \frac{\lambda((x-h, x+h) \cap P)}{2h} \geq \gamma', \ \forall h \in (0, \frac{1}{n}) \right\}.$$

Then $D_\gamma(P) \subset \cup_n H_n$. So there exists an n such that H_n is of second category in P. Choose and fix such an n. Then there exists a portion of P such that H_n is dense in it, that is, there is an open interval J such that $J \cap P \neq \emptyset$ and H_n is dense in $J \cap P$. Since P is nowhere dense one can choose an interval $I = (a, b)$ which is contiguous to P, $[a, b] \subset J$, and $b - a < \frac{1}{n}$. Then by the density of H_n in $J \cap P$ one can choose a point $p \in H_n \subset P$, close to $a \in P$ such that letting $h = b - p$ we have $h < 1/n$ and $\lambda((p-h, p+h) \cap P)/2h < \gamma'$ contradicting the definition of H_n. □

Theorem 2. *There exists a fat Cantor set $P \subset \mathbb{R}$ such that $P \subset D_{0.5}(P)$.*

PROOF. Set

$$G_n = \bigcup_{l \in \mathbb{Z}} \left(\frac{l}{2^{n^2}}, \frac{l}{2^{n^2}} + \frac{1}{2^{n^2+n}} \right),$$

and $P = \mathbb{R} \setminus \cup_{n=1}^{\infty} G_n$. If

$$I = \left(\frac{j}{2^{m^2}}, \frac{j+1}{2^{m^2}} \right) \not\subset \bigcup_{n=1}^{m-1} G_n$$

then one can easily see that

$$\lambda(I \setminus \bigcup_{n=m}^{\infty} G_n) \geq \prod_{n=m}^{\infty} (1 - \frac{1}{2^n}) \lambda(I) = \exp\left(\sum_{n=m}^{\infty} \log(1 - \frac{1}{2^n}) \right) \lambda(I) >$$

$$> \exp\left(2 \sum_{n=m}^{\infty} -\frac{1}{2^n} \right) \lambda(I) = \exp(-\frac{1}{2^{m-2}}) \lambda(I).$$

Therefore,

$$\lambda(I \cap \bigcup_{n=m}^{\infty} G_n) \leq (1 - \exp(-\frac{1}{2^{m-2}})) \lambda(I).$$

Assume $x \in P$, $0 < h < 1/2$ and choose m such that

$$\frac{1}{2^{(m+1)^2}} < h \leq \frac{1}{2^{m^2}}.$$

We separate two cases.

Case I. First we also assume that

$$\frac{1}{2 \cdot 2^{m^2}} \leq h \leq \frac{1}{2^{m^2}}. \tag{1}$$

Since $x \in P$ there is an interval I_1 of the form $[j/2^{m^2}, (j+1)/2^{m^2}]$ which is not a subset of the closure of $\cup_{n=1}^{m-1} G_n$ and contains x. Then (1) implies that there is a subinterval $I_2 \subset I_1 \cap (x-h, x+h)$ which is of length h. Now,

$$\lambda((x-h,x+h)\cap P) \geq \lambda(I_2 \cap P) = \lambda(I_2 \setminus \bigcup_{n=m}^{\infty} G_n) \geq$$
$$\geq \lambda(I_2) - \lambda(I_1 \cap \bigcup_{n=m}^{\infty} G_n) \geq \lambda(I_2) - \lambda(I_1)(1-\exp(-\frac{1}{2^{m-2}})) \geq$$
$$\geq h - 2h(1-\exp(-\frac{1}{2^{m-2}})) = h(1-2(1-\exp(-\frac{1}{2^{m-2}}))) = h\nu_m,$$

where $\nu_m \to 1$ as $m \to \infty$.

Case II. Now assume $\frac{1}{2^{(m+1)^2}} < h < \frac{1}{2 \cdot 2^{m^2}}$. Choose I_1 as above and $I_2 \subset I_1 \cap (x-h, x+h)$ such that it is of length h and $I_2 \cap \bigcup_{n=1}^{m} G_n = \emptyset$. Our assumption about h, $\lambda(I_2) = h$ and the way G_n's are defined now imply that

$$\lambda(I_2 \setminus \bigcup_{n=m+1}^{\infty} G_n) \geq \prod_{n=m+1}^{\infty} \left(1 - \frac{2}{2^n}\right) \lambda(I_2) \geq h \exp(-\frac{1}{2^{m-2}}).$$

Then,

$$\lambda((x-h,x+h)\cap P) \geq \lambda(I_2 \cap P) = \lambda(I_2 \setminus \bigcup_{n=m+1}^{\infty} G_n) \geq$$
$$\geq h \exp(-\frac{1}{2^{m-2}}) = h\nu'_m,$$

where $\nu'_m \to 1$ as $m \to \infty$.

From the above two cases it follows that $\underline{D}(x, P) \geq 0.5$. □

Theorem 3. *There exists a fat Cantor set $P \subset \mathbb{R}$ such that $D_+(P)$ is of first category in P.*

Proof. We will choose the sequence δ_n by induction so that, apart from one more later assumption, we have $0 < \delta_n < 1/n$, $\delta_n < \delta_{n-1}$, and $\sum_{n=1}^{\infty} \delta_n < 0.1$.

We will define our fat Cantor set P as the intersection of the nested sets E_n, $n = 0, 1, \dots$. We choose $E_0 = [0,1]$, $\delta_1 = 0.05$. We define our sets E_n by induction so that E_n will consist of 3^n many disjoint nondegenerate closed intervals, so called base intervals of E_n. We also assume that δ_{n+1} is chosen so small that it is less than any of the gaps between the intervals comprising E_n.

Assume n is fixed, $I=[a,b]$ is a base interval of E_{n-1}. Set $I_1=[a,a+(b-a)\frac{\delta_n}{n}]$, $I_2=[a+(b-a)(\frac{\delta_n}{n}+\delta_n),\frac{a+b}{2}]$ and $I_3=[\frac{a+b}{2}+(b-a)\delta_n,b]$. We will define E_n so that $E_n\cap[a,b]=I_1\cup I_2\cup I_3$. Then $\lambda(I_1\cup I_2\cup I_3)=(1-2\delta_n)\lambda(I)$ and adding these equalities together for all base intervals of E_{n-1} we have $\lambda(E_n)=(1-2\delta_n)\lambda(E_{n-1})$. If E_{n-1} consists of 3^{n-1} intervals then E_n will consist of 3^n.

If $x\in I_1$ then

$$\lambda((x-(b-a)\delta_n,x+(b-a)\delta_n)\cap E_n)=(b-a)\delta_n/n. \tag{2}$$

By induction one can see that $\lambda(I\cap E_m)=\prod_{k=n}^{m}(1-2\delta_k)\lambda(I)$ for $m>n$, and $\lambda(I\cap P)=\prod_{k=n}^{\infty}(1-2\delta_k)\lambda(I)>0$. So P is fat.

Now we can argue as in the proof of Theorem 1. Proceeding towards a contradiction assume that $D_+(P)$ is of second category.

Set

$$H_n=\left\{x\in P:\frac{\lambda((x-h,x+h)\cap P)}{2h}\geq\frac{1}{n},\ \forall h\in(0,\frac{1}{n})\right\}.$$

If $x\in D_+(P)$ then there exists an n for which $x\in H_n$. Hence we can choose an n for which H_n is of second category. Choose a portion $(\alpha,\beta)\cap P\neq\emptyset$ such that H_n is dense in $(\alpha,\beta)\cap P$ and $\beta-\alpha<1/n$. Select an $x_0\in(\alpha,\beta)\cap P$ and an $m>n$ so large that the component, $I=[a,b]$, of E_{m-1} containing x_0 is in (α,β). Since H_n is dense in $(\alpha,\beta)\cap P$ there is $x_1\in H_n\cap I_1=H_n\cap[a,a+(b-a)\frac{\delta_m}{m}]$. Then using $h=(b-a)\delta_m$ and (2) we obtain

$$\frac{1}{n}\leq\frac{\lambda((x-h,x+h)\cap P)}{2h}\leq\frac{\lambda((x-h,x+h)\cap E_m)}{2h}=\frac{1}{2m},$$

which is a contradiction. □

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