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CONVERGENCE THEOREMS FOR THE HENSTOCK INTEGRAL INVOLVING SMALL RIEMANN SUMS

Abstract

We generalize the functionally-small-Riemann-sum (FSRS) property for the Henstock integral to the n-dimensional Euclidean space. We prove a convergence theorem and its connection with the equi-integrability condition.

1 Introduction

Darmawijaya [1] and others [2] gave the *functionally-small-Riemann-sum* (FSRS) property for the Henstock integral on the real line. Gong [2] used the FSRS property for the Henstock integral on the real line to give some convergence theorems. He used the dominated convergence theorem to deduce his theorems. In this paper, we extend Gong's convergence theorem to the n-dimensional space (Theorem 2) and establish its connection with the equi-Henstock integrability theorem (Theorem 5 and (Theorem 6).

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm

$$\|\mathbf{x}\|_\infty = \max\{|x_k| : 1 \le k \le n\}.$$

Let E be a cell (non-degenerate interval) of $\mathbb{R}^n$. We use $|E|$ to represent the Lebesgue measure of E, that is, the volume of a cell E. All functions in this paper are real-valued functions on a cell.

Key Words: Henstock Integral, δ-fine partition , Functionally Small Riemann Sum, Uniformly Functionally Small Riemann sum, Uniformly Strong Lusin Condition, Equi-Henstock integrability

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If E is a cell and δ is a positive function on E, for $\mathbf{x} \in E$

$$B(\mathbf{x}, \delta(\mathbf{x})) = \{\mathbf{y} : \|\mathbf{x} - \mathbf{y}\|_\infty < \delta(\mathbf{x})\}$$

is called an open ball with center at $\mathbf{x}$ and radius $\delta(\mathbf{x})$. A collection of cell-point pairs,

$$\mathcal{D} = \{(I, \mathbf{x})\} = \{(I_1, \mathbf{x}_1), (I_2, \mathbf{x}_2), ..., (I_p, \mathbf{x}_p)\},$$

is called δ-*fine partition of* a cell E, if $E = \cup_{i=1}^p I_i$ with $\mathbf{x}_i \in I_i \subseteq B(\mathbf{x}_i, \delta(\mathbf{x}_i))$, and $I_i^o \cap I_j^o = \emptyset$ for $i \neq j$ where $i = 1, 2, ..., p$. Furthermore, $(I, \mathbf{x}) \in \mathcal{D}$ is called δ-fine cell I with associated point $\mathbf{x}$. If $\cup_{i=1}^p I_i \subseteq E$ then the partition is called δ-*fine partial partition in* E.

A function f defined on a cell E is said to be *Henstock integrable* or H-integrable on a cell E, if there is a number A such that for any $\epsilon > 0$ there is a positive function δ on E such that for any δ-fine partition $\mathcal{D} = \{(I, \mathbf{x})\}$ of E we have

$$|(\mathcal{D}) \sum f(\mathbf{x})|I| - A| < \epsilon.$$

Here, $(\mathcal{D}) \sum f(\mathbf{x})|I|$ is taken to mean the sum over the δ-fine partition $\mathcal{D}$ of E. If a function f is H -integrable on a cell E, then the H-integral value of f on E is unique. Furthermore, the number A is called the H-integral value of f on E and will be written

$$A = (H) \int_E f.$$

If we only want to know whether a function f is Henstock integrable on a cell E without using its H-integral value, we may use Cauchy's Criterion. More precisely, *A function f is Henstock integrable on a cell E, if and only if for any $\epsilon > 0$ there is a positive function δ on E such that for any two δ-fine partitions $\mathcal{D}_1 = \{(I', \mathbf{x}')\}$ and $\mathcal{D}_2 = \{(I'', \mathbf{x}'')\}$ of E we have*

$$|(\mathcal{D}_1) \sum f(\mathbf{x}')|I'| - (\mathcal{D}_2) \sum f(\mathbf{x}'')|I''|| < \epsilon.$$

If f is H-integrable on E and I is a subcell of E then f is H-integrable on I. Let $F(I)$ denote the H-integral of f on $I \subseteq E$. Then F is called the *primitive* of f on E and Henstock's Lemma holds. More precisely, *a function f defined on E is H -integrable with primitive F if and only if for every $\epsilon > 0$ there is a positive function δ on E such that for any δ-fine partial partition $\mathcal{D}$ in E we have*

$$|(\mathcal{D}) \sum (f(\mathbf{x})|I| - F(I))| < \epsilon.$$

Henstock's Lemma is a powerful tool in proving theorems that only need partial partitions in E.

2 The FSRS Property

A measurable function f has *functionally small Riemann sums* or the FSRS property on a cell $E \subset \mathbb{R}^n$, if for every $\epsilon > 0$ there exist a non-negative Lebesgue integrable function g and a positive function δ on E such that for any δ-fine partition $\mathcal{D} = \{(I, \mathbf{x})\}$ of E, we have

$$|(\mathcal{D}) \sum_{|f(\mathbf{x})| > g(\mathbf{x})} f(\mathbf{x})|I|| < \epsilon,$$

where the sum is taken over $\mathcal{D} = \{(I, \mathbf{x})\}$ for which $|f(\mathbf{x})| > g(\mathbf{x})$.

Theorem 1. *A measurable function f has the FSRS property on a cell E if and only if f is Henstock integrable on E.*

For a proof see [2] or [5].

A sequence $\{f_k\}$ of measurable functions has *uniformly functionally small Riemann sums* or the UFSRS property, if the conditions for FSRS hold with f replaced by f_k and both g and δ independent of k.

The proof of the convergence theorem involving the UFSRS property and the rest of the theorems need the uniformly strong Lusin condition and Egoroff's Lemma [3].

A sequence $\{F_k\}$ is said to satisfy *uniformly strong Lusin* or the USL condition if for every $\epsilon > 0$ and every set S of measure zero there exists a positive function δ, independent of k, such that for any δ-fine partial partition $\mathcal{D} = \{(I, \mathbf{x})\}$, with $\mathbf{x} \in S$, and for all k we have

$$(\mathcal{D}) \sum |F_k(I)| < \epsilon.$$

If $F_k = F$ for all k, then F is said to satisfy the *strong Lusin* condition.

Theorem 2. *Let f_k, $k = 1, 2, \ldots$, be Henstock integrable on a cell E with the primitive F_k, $k = 1, 2, \ldots$, respectively. If $\{f_k\}$ has the UFSRS property, and $f_k \to f$ almost everywhere in E, then f is Henstock integrable on E and*

$$\lim_{k\to\infty} (H) \int_E f_k = (H) \int_E f, \quad \text{as } k \to \infty.$$

The proof follows that of Gong [2]. We sketch as follows. From the hypothesis, f has the FSRS property. Then, from Theorem 1, f is H-integrable on E. For the rest of the proof, we use the dominated convergence theorem and the UFSRS property. □

3 Equivalence with Equi-Henstock Integrability

The UFSRS condition plays a role in the above convergence theorem similar to that of equi-Henstock integrability in an early result [6]. In what follows, we give the relationship between these two conditions (Theorem 5 and Theorem 6).

A sequence of functions $\{f_k\}$ is said to be *equi-Henstock integrable* on a cell E, if for every $\epsilon > 0$ there is a positive function δ on E, independent of k, such that for any δ-fine partition $\mathcal{D} = \{(I, \mathbf{x})\}$ of E and for every k

$$|(\mathcal{D}) \sum f_k(\mathbf{x})|I| - (H) \int_E f_k| < \epsilon.$$

By the equi-Henstock integrability we have a convergence theorem for Henstock integral in the n-dimensional Euclidean space (Theorem 3). However, this theorem is well-known. For a proof, see [4] or Wang Pujie [6].

Theorem 3. *Let f_k, $k = 1, 2, \ldots$, be Henstock integrable on a cell E with the primitives F_k, $k = 1, 2, \ldots$, respectively. If the sequence $\{f_k\}$ is equi-Henstock integrable on the cell E, and $f_k \to f$ almost everywhere in E as $k \to \infty$, and $\{F_k\}$ satisfies the USL condition on E, then f is Henstock integrable on E. Furthermore,*

$$\lim_{k\to\infty} (H) \int_E f_k = (H) \int_E f, \quad \text{as } k \to \infty.$$

We need the following lemma.

Lemma 4. *Let f_k, $k = 1, 2, \ldots$, be Henstock integrable on a cell E with the primitives F_k, $k = 1, 2, \ldots$, respectively. If there is a non-negative Lebesgue integrable function g on E such that $|f_k(\mathbf{x})| \le g(\mathbf{x})$ almost everywhere for every k, and $f_k \to f$ almost everywhere in E as $k \to \infty$, then $\{F_k\}$ satisfies the USL condition on E and $\{f_k\}$ is equi-Henstock integrable on E .*

The first part follows from the fact that the primitive of g satisfies the strong Lusin condition. The second part of proof follows from Egoroff's Lemma and the absolute continuity of G, the primitive of g. See [4]. $\Box$

Now we prove the two main theorems.

Theorem 5. *Let f_k, $k = 1, 2, \ldots$, be Henstock integrable on a cell E with the primitives F_k, $k = 1, 2, \ldots$, respectively. If the sequence $\{f_k\}$ is equi-Henstock integrable on the cell E, $f_k \to f$ almost everywhere in E as $k \to \infty$, and $\{F_k\}$ satisfies the USL condition on E, then $\{f_k\}$ is a sequence of measurable functions which has the UFSRS property on E.*

PROOF. Let $\epsilon > 0$ be given. It follows from Theorem 3, that f is Henstock integrable on E. There is a positive function δ_* on E such that for any δ_*-fine partition $\mathcal{D}$ of E and for every k, we have

$$|(\mathcal{D})\sum f_k(\mathbf{x})|I| - F_k(E)| < \epsilon,$$

and

$$|(\mathcal{D})\sum f(\mathbf{x})|I| - F(E)| < \epsilon.$$

Also, there is a positive integer K such that for every $k \geq K$

$$|F_k(E) - F(E)| < \epsilon.$$

Consequently, for any δ_*-fine partition $\mathcal{D}$ of E and for every $k \geq K$

$$|(\mathcal{D})\sum f_k(\mathbf{x})|I| - (\mathcal{D})\sum f(\mathbf{x})|I|| < 3\epsilon.$$

Similarly, it follows from Lemma 4, that there is a positive function δ_{**} such that for any δ_{**} -fine partition $\mathcal{D}$ of E and for every $k \geq K$, we have

$$|(\mathcal{D})\sum_{|f_k(\mathbf{x})|\leq g(\mathbf{x})} f_k(\mathbf{x})|I| - (\mathcal{D})\sum_{|f(\mathbf{x})|\leq g(\mathbf{x})} f(\mathbf{x})|I|| < \epsilon.$$

Further, by Theorem 1, there is a non-negative Lebesgue integrable function g and a positive function δ_{***} such that for any δ_{***}-fine partition $\mathcal{D}$ of E we have

$$|(\mathcal{D})\sum_{|f(\mathbf{x})|>g(\mathbf{x})} f(\mathbf{x})|I|| < \epsilon.$$

Put $0 < \delta(\mathbf{x}) \leq \min\{\delta_*(\mathbf{x}), \delta_{**}(\mathbf{x}), \delta_{***}(\mathbf{x})\}$. Then, for any δ-fine partition $\mathcal{D}$ of E and for $k \geq K$, we have

$$\begin{aligned}|(\mathcal{D})\sum_{|f_k(\mathbf{x})|>g(\mathbf{x})} f_k(\mathbf{x})|I|| \leq & |(\mathcal{D})\sum f_k(\mathbf{x})|I| - (\mathcal{D})\sum f(\mathbf{x})|I|| \\ & + |(\mathcal{D})\sum_{|f_k(\mathbf{x})|\leq g(\mathbf{x})} f_k(\mathbf{x})|I| - (\mathcal{D})\sum_{|f(\mathbf{x})|\leq g(\mathbf{x})} f(\mathbf{x})|I|| \\ & + |(\mathcal{D})\sum_{|f(\mathbf{x})|>g(\mathbf{x})} f(\mathbf{x})|I|| < 5\epsilon.\end{aligned}$$

Modify δ and g, if necessary, so that the above inequality holds for all k. Hence $\{f_k\}$ has the UFSRS property on E. $\square$

Theorem 6. *Let f_k, $k = 1, 2, \ldots$, be Henstock integrable function on a cell E with the primitives F_k, $k = 1, 2, \ldots$, respectively. If the sequence $\{f_k\}$ has the UFSRS property on the cell E, and $f_k \to f$ almost everywhere in E as $k \to \infty$, then $\{f_k\}$ is equi-Henstock integrable on E .*

Proof. Let $\epsilon > 0$ be given. Since $\{f_k\}$ has the UFSRS property, then there exist a non-negative Lebesgue integrable function g on E and a positive function δ^*, both independently of k, such that for any δ^*-fine partition $\mathcal{D}$ of E and for every k, we have

$$|(\mathcal{D}) \sum_{|f_k(\mathbf{x})| > g(\mathbf{x})} f_k(\mathbf{x})|I|| < \epsilon.$$

It follows from Lemma 4, that there is a positive function δ_* such that for any two δ_*-fine partitions $\mathcal{D}_1 = \{(I', \mathbf{x}')\}$ and $\mathcal{D}_2 = \{(I'', \mathbf{x}'')\}$ of E and for every k, we have

$$|(\mathcal{D}_1) \sum_{|f_k(\mathbf{x}')| \leq g(\mathbf{x}')} f_k(\mathbf{x}')|I'| - (\mathcal{D}_2) \sum_{|f_k(\mathbf{x}'')| \leq g(\mathbf{x}'')} f_k(\mathbf{x}'')|I''|| < \epsilon.$$

Put $0 < \delta(\mathbf{x}) \leq \min\{\delta_*(\mathbf{x}), \delta^*(\mathbf{x})\}$. Then, for any two δ-fine partitions $\mathcal{D}_1 = \{(I', \mathbf{x}')\}$ and $\mathcal{D}_2 = \{(I'', \mathbf{x}'')\}$ of E and for every k

$$\begin{aligned}
&|(\mathcal{D}_1) \sum f_k(\mathbf{x}')|I'| - (\mathcal{D}_2) \sum f_k(\mathbf{x}'')|I''|| \\
&\quad \leq |(\mathcal{D}_1) \sum_{|f_k(\mathbf{x}')| \leq g(\mathbf{x}')} f_k(\mathbf{x}')|I'| - (\mathcal{D}_2) \sum_{|f_k(\mathbf{x}")| \leq g(\mathbf{x}")} f_k(\mathbf{x}'')|I''|| \\
&\qquad + |(\mathcal{D}_1) \sum_{|f_k(\mathbf{x}')| > g(\mathbf{x}')} f_k(\mathbf{x}')|I'| - (\mathcal{D}_2) \sum_{|f_k(\mathbf{x}")| > g(\mathbf{x}")} f_k(\mathbf{x}'')|I''|| \\
&\quad < 3\epsilon.
\end{aligned}$$

That is, $\{f_k\}$ is equi-Henstock integrable on E. □

As a corollary, if $f_k \to f$ everywhere in E as $k \to \infty$, then the UFSRS property of $\{f_k\}$ is equivalent to its equi-Henstock integrability.

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