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PATH DERIVED NUMBERS AND PATH DERIVATIVES OF CONTINUOUS FUNCTIONS WITH RESPECT TO CONTINUOUS SYSTEMS OF PATHS

Abstract

V. Jarnik showed that a typical continuous function on the unit interval $[0, 1]$ has every extended real number as a derived number at every point of $[0, 1]$. In this paper we classify the derived numbers of a continuous function and study the likelihood of Jarnik's Theorem for path derived numbers of a continuous system of paths. We also provide some results indicating that some of the nice behaviors of first return derivatives are shared by path derivatives of continuous functions when the path system is continuous.

Bruckner, O'Malley and Thomson [5] introduced the concept of Path derivatives and showed that many known derivatives fall in this frame work. They showed that most of the nice behavior of generalized derivatives is due to the thickness of the paths as well as the way paths R_x and R_y intersect when x and y are close. In [1] we introduced the notion of continuous systems of paths and showed that it is also another important factor for the good behavior of generalized derivatives.

The set of accumulation points of A is denoted by A' and the class of functions with the Darboux property is denoted by D. Let x belong to $[0, 1]$. A path leading to x is a set $R_x \subseteq [0, 1]$ such that $x \in R_x \cap (R_x)'$. A system of paths R is a collection $\{R_x : x \in [0, 1]\}$, where each R_x is a path at x. For a function $f : [0.1] \to \mathbb{R}$ the path derivative of f at point x with respect

to path system R is defined as $\lim_{y \in R_x, y \to x} \frac{f(y)-f(x)}{y-x}$ if it exists and is finite. The extreme path derivatives are defined in a usual way. It is known that many theorems about the differentiability of functions can be obtained from conditions on the thickness of paths, how they intersect each other, and the continuity of path systems (see [1,2,3,5].)

A path system R is said to have the external intersection condition denoted by $E.I.C.$ (intersection condition denoted by $I.C.$, internal intersection condition denoted by $I.I.C.$) if there is a positive function $\delta(x)$ on $[0,1]$ such that $R_x \cap R_y \cap (y, 2y-x) \neq \emptyset$ and $R_x \cap R_y \cap (2x-y, x) \neq \emptyset$ $(R_x \cap R_y \cap [x,y] \neq \emptyset,$ $R_x \cap R_y \cap (x,y) \neq \emptyset$, respectively), whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$. The thickness of paths is studied in terms of porosity. Let E be a set and let $a < b$. Then $\lambda(E, a, b)$ is the length of the largest open subinterval of (a, b) that contains no point of E. At any point x one defines the right porosity of E at x as $\rho^+(E,x) = \limsup_{h \to 0^+} \frac{\lambda(E,x,x+h)}{h}$, the left porosity of E at x as $\rho^-(E,x) = \limsup_{h \to 0^+} \frac{\lambda(E,x,x-h)}{h}$, and the bilateral porosity of E at x as $\rho(E,x) = \limsup_{h \to 0} \frac{\lambda(E,x,x+h)}{|h|}$. The latter of which is, of course the maximum of two unilateral porosities. Note that the porosity is always a number in the interval $[0,1]$ and that both extremes can occur. The smaller $\rho(E,x)$ is, the ticker is the set E at point x. In Definition 2, we define $\overline{\gamma}(\{h_n\}, t)$ for every monotone sequence $\{h_n\}_{n=1}^\infty$ with $\lim_{n \to \infty} h_n = t$. This concept, which is closely related to porosity, indicates how fast or slow the sequence $\{h_n\}$ converges.

In [1] we introduced the notion of continuous systems of paths and showed that aside from intersection conditions and porosity conditions, this notion is also another factor in the differentiability of real functions. Some known path systems like ordinary, sequential, and congruent systems of paths $R = \{R_x : R_x = x + Q\}$ when Q is a closed set with $0 \in Q'$ are examples of continuous systems of paths (see [4,6]). We used continuity of path systems for the study of extreme path derivatives. This concept was generalized in [7] by Milan Matejdes. Let $R = \{R_x : x \in [0,1]\}$ be a system of paths, with each R_x compact. We endow R with the Hausdorff metric d_H to form a metric space. If the function $P : x \to R_x$ is a continuous function, we say R is a continuous system of paths. The left continuous and right continuous systems of paths are defined similarly.

V. Jarnik showed that a typical continuous function on the unit interval $[0, 1]$ has every extended real number as a derived number at every point of $[0, 1]$. By typical we mean that all continuous functions except for those in some first category subset of the complete metric space $C[0, 1]$. We will first classify the derived numbers of a continuous function and we will study the likelihood of Jarnik's Theorem for path derived numbers when the path system

is a continuous system of paths. Sierpinski proved the following theorem.

Theorem 1. *Let $\{h_n\}$ be any sequence of nonzero numbers converging to zero. Let for all $x \in \mathbb{R}$, $R_x = \{x + h_n : n = 0, 1, \dots\} \cup \{x\}$. Then for any finite function $f : \mathbb{R} \to \mathbb{R}$, there is a function F such that $F'_R = f$ for all x.*

Sierpinski's Theorem indicates that any finite function g could be the path derivative of some function f with respect to a continuous system of paths. Thus we cannot expect such path derivatives or path derived numbers have interesting properties, unless some extra assumptions on the function f are imposed. The following example indicates that the existence of f'_R does not imply the nice behavior of f even when R is continuous.

Example 1. There exists $f \in B_2 \setminus B_1$ and a nonporous continuous system of paths $R = \{R_x : x \in \mathbb{R}\}$ so that $f'_R(x) = 0$ exists for all x.

PROOF. Let $f(x) = \chi_{\mathbb{Q}}(x)$ where $\mathbb{Q}$ is the set of rational numbers and let $\{h_n\}$ be a sequence in $\mathbb{Q}$ converging to zero with $\rho(\{h_n\}_{n=1}^{\infty}, 0) = 0$. Then it is clear that $f'_R = 0$. □

In [8] it is shown that a first return differentiable function is an element of DB_1^*. Example 1, indicates that even for a continuous system of paths, a path differentiable function need not be an element of B_1 and may not have the Darboux property. It also shows that the path derivatives could be zero with f not being a constant function. Theorem 2 provides some positive results under the extra assumption that f is continuous and the system of path is bilateral. However, Example 2 indicates that even for a continuous function and a continuous system of paths, the path derivative might not have Darboux property.

Theorem 2. *Let $f \in C([0,1])$ and let $R = \{R_x : x \in [0,1]\}$ be a bilateral system of paths.*

   (i) *If $f'_R(x)$ exists, then $f'_R(x) \in D$.*

   (ii) *If $-M \le \underline{f'_R}(x) \le \overline{f'_R}(x) \le M$ for all $x \in [0,1]$, then f is a Lipschitz one function.*

PROOF. The proof of this theorem is straight forward and thus is omitted. □

When the path system R is not bilateral, the results of Theorem 2 do not hold.

Example 2. There exists a continuous function f and a continuous system of paths $R = \{R_x : x \in \mathbb{R}\}$ so that $f'_R(x) \notin D$.

Proof. Let $g(x) = x$, for $0 \leq x < \frac{1}{2}$, $g(x) = 2x$, for $\frac{1}{2} \leq x \leq 1$, $f(x) = \int_0^x g$, and $R_x = x + \frac{1}{n}$, for $n = 1, 2, \dots$. Then f is continuous and $f'_R = g$. However $g \notin D$. $\qquad\square$

In [3] we showed that the right (left) first return system of paths are right (left) continuous system of paths, respectively. Thus one could ask what kind of nice behavior of first return derivatives is shared by path derivatives when the path system is continuous. In [3] we also showed that for a continuous function some of the nice behavior of first return derivatives also hold for path derivatives when the path system is continuous. Here we provide some other results which illustrates other properties of first return derivatives shared by path derivatives when the path system is continuous.

Definition 1. Let R be a system of paths. A number α is called an R-derived number of the function f at a point $x \in [0, 1]$ if there exists a sequence $\{x_n\} \subseteq R_x$ so that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} \frac{f(x_n) - f(x)}{x_n - x} = \alpha$. A number α is called an R-derived number of the function f, if it is an R-derived number of f at some point $x \in [0, 1]$. The set of R-derived numbers of f at x and the set of R-derived numbers of f are denoted by $D(R, f, x)$ and $D(R, f)$, respectively.

Definition 2. Let $\{x_n\}$ be a monotone sequence with $\lim_{n \to \infty} x_n = x_0$. If it is decreasing, we define $\overline{\gamma}(\{x_n\}, x_0) = \limsup_{n \to \infty} \frac{x_n - x_{n+1}}{x_n - x_0}$ and if it is increasing, we define $\overline{\gamma}(\{x_n\}, x_0) = \limsup_{n \to \infty} \frac{x_{n+1} - x_n}{x_0 - x_n}$.

Remark 1. For every $\alpha \in D(R, f, x)$ there exists a monotone sequence in R_x converging to x with $\lim_{n \to \infty} \frac{f(x_n) - f(x)}{x_n - x} = \alpha$. Thus in Definition 1 we can take $\{x_n\}$ to be monotone.

Definition 3. Let $f : [0, 1] \to \mathbb{R}$ be a function, R be a system of paths on $[0, 1]$. For every $\alpha \in D(R, f, x)$, let

$$S(\alpha, R, f, x) = \{\{x_n\}_{n=1}^{\infty} \subseteq R_x : \{x_n\} \text{ is a monotone sequence converging to } x$$

$$\text{with} \quad \lim_{n \to \infty} \frac{f(x_n) - f(x)}{x_n - x} = \alpha\},$$

$S(\alpha, R, f) = \cup_{x \in [0,1]} S(\alpha, R, f, x)$ and $C(\alpha, R, f, x) = \inf\{\overline{\gamma}(\{x_n\}, x) : \{x_n\} \in S(\alpha, R, f, x)\}$. We call $C(\alpha, R, f, x)$ the index of the path derived number α of f with respect to the path system R at x. The index of path derived number α with respect to the path system R is defined as $C(\alpha, R, f) = \inf_{x \in [0,1]} C(\alpha, R, f, x)$. It is clear that for each $\alpha \in D(R, f)$ we have $0 \leq C(\alpha, R, f) \leq 1$.

Example 3. For each $r, 0 \le r \le 1$ and real number α there exists a continuous function f defined on $[0,1]$ and a continuous systems of paths R so that $\alpha \in D(R,f)$ with $C(\alpha, R, f) = r$.

Proof. For each r, $0 \le r \le 1$, let $\{h_n\}$ be a decreasing sequence with $h_1 = 1$, $\lim_{n\to\infty} h_n = 0$ and $\overline{\gamma}(\{h_n\}, 0) = r$. For each $x \in [0,1]$, define $R_x = (\{x\} \cup \{x+h_n, x-h_n\}_{n=1}^\infty) \cap [0,1]$. It is clear that R is a bilateral continuous system of paths. Let $f(x) = \alpha x$ for $x \in \{h_n\}_{n=1}^\infty$, $f(x) = 0$ for $x \in \left\{ \frac{h_n + h_{n+1}}{2} \right\}_{n=1}^\infty$, and f linear elsewhere. It is clear that f is continuous and $\alpha \in D(R, f, 0)$, and $S(\alpha, R, f, 0) = \{\{Z_n\}_{n=1}^\infty : \{Z_n\} \text{ is a decreasing subsequence of } \{h_n\}\}$. Since $\{h_n\} \in S(\alpha, R, f, 0)$, we have $C(\alpha, R, f, 0) \le r$. Let $\{Z_m\}$ be an arbitrary decreasing subsequence of $\{h_n\}$ and $\overline{\gamma}(\{h_n\}, 0) = r = \lim_{k\to\infty} \frac{h_{n_k} - h_{n_{k+1}}}{h_{n_k}}$. Then for each k, there exists m_k such that $Z_{m_{k+1}} \le h_{n_{k+1}} < h_{n_k} \le Z_{m_k}$. Thus we have $\frac{Z_{m_k} - Z_{m_{k+1}}}{Z_{m_k}} \ge \frac{h_{n_k} - h_{n_{k+1}}}{h_{n_k}}$, implying

$$\overline{\gamma}(\{Z_n\}, 0) \ge \limsup_{k\to\infty} \frac{Z_{m_k} - Z_{m_{k+1}}}{Z_{m_k}} \ge \lim_{k\to\infty} \frac{h_{n_k} - h_{n_{k+1}}}{h_{n_k}}.$$

Thus $C(\alpha, R, f, 0) \ge r$, implying $C(\alpha, R, f, 0) = r$. $\qquad\square$

Lemma 1. *Let $R = \{R_x : x \in [0,1]\}$ be a continuous system of paths and $m > 1$ be an integer. Then for each m, there exists a sequence $\{h_n^m\}_{n=1}^\infty \subseteq [0,1]$ such that $0 < h_{n+1}^m < h_n^m$ for all n, $\lim_{n\to\infty} h_n^m = 0$, $R_x \cap ([x+h_{n+1}^m, x+h_n^m] \cup [x - h_n^m, x - h_{n+1}^m]) \ne \emptyset$ for all $x \in [0,1]$, and $\liminf_{n\to\infty} \frac{h_n^m - h_{n+1}^m}{h_n^m} \ge 1 - \frac{1}{m}$.*

Proof. Let $h_1^m = 1$. Define $h_2^m = \frac{1}{m} \inf_{x \in [0,1]} \sup\{|\ y\ |: y \in (R_x - x) \cap [-h_1^m, h_1^m]\}$. Obviously $h_2^m \le \frac{1}{m} h_1^m < h_1^m$. Let h_n^m be defined. Inductively define $h_{n+1}^m = \frac{1}{m} \inf_{x \in [0,1]} \sup\{|\ y\ |: y \in (R_x - x) \cap [-h_n^m, h_n^m]\}$. Then we have $h_{n+1}^m \le \frac{1}{m} h_n^m < h_n^m$ for all n. We claim for each n, $h_n^m > 0$. If not, there exists a natural number n_0 so that $h_{n_0}^m = 0$. But $h_{n_0-1}^m > 0$. Since $0 \in (R_x - x)'$, we have $(R_x - x) \cap [-h_{n_0-1}^m, h_{n_0-1}^m] \ne \emptyset$. Thus we have

$$r_x = \sup\{|\ y\ |: y \in (R_x - x) \cap [-h_{n_0-1}^m, h_{n_0-1}^m]\} > 0$$

for every $x \in [0,1]$ and $\inf_{x \in [0,1]} r_x = h_{n_0}^m = 0$. This implies there is a sequence $\{x_n\}_{n=1}^\infty \subset [0,1]$, so that $\lim_{n\to\infty} r_{x_n} = 0$. Since $\{x_n\}$ is bounded, without loss of generality we may assume that $\lim_{n\to\infty} x_n = x_0$. From the continuity of the path system R, it follows that $\lim_{n\to\infty} d_H(R_{x_n}, R_{x_0}) = 0$. This implies $\lim_{n\to\infty} r_{x_n} = r_{x_0} = 0$. But R_{x_0} is the path leading to x_0 and thus r_{x_0} can not be zero; so $h_n^m > 0$ for all n. Also we have

$$0 < h_n^m \le \frac{1}{m} h_{n-1}^m \le \frac{1}{m^2} h_{n-2}^m \le \cdots \le \frac{1}{m^{n-1}} h_1^m = \frac{1}{m^{n-1}}.$$

Thus $0 \leq \lim_{n \to \infty} h_n^m \leq \lim_{n \to \infty} \frac{1}{m^{n-1}} = 0$. It is clear that

$$R_x \cap ([x + h_{n+1}^m, x + h_n^m] \cup [x - h_n^m, x - h_{n+1}^m]) \neq \emptyset \text{ for all } x \in [0,1],$$

and $\frac{h_n^m - h_{n+1}^m}{h_n^m} \geq \frac{h_n^m - \frac{1}{m} h_n^m}{h_n^m} = 1 - \frac{1}{m}$ for each n. Thus $\liminf_{n \to \infty} \frac{h_n^m - h_{n+1}^m}{h_n^m} \geq 1 - \frac{1}{m}$. $\square$

Using a proof similar to that of Lemma 1, when the path system is bilateral and continuous, we will be able to prove the following.

Lemma 2. *Let $R = \{R_x : x \in [0,1]\}$ be a bilateral continuous system of paths and $m > 1$ be an integer. Then for each m, there exists a sequence $\{h_n^m\}_{n=1}^{\infty} \subseteq [0,1]$ such that $0 < h_{n+1}^m < h_n^m$ for all n, $\lim_{n \to \infty} h_n^m = 0$, $R_x \cap ([x + h_{n+1}^m, x + h_n^m] \neq \emptyset$ and $R_x \cap [x - h_n^m, x - h_{n+1}^m]) \neq \emptyset$ for all $x \in [0,1]$, and $\liminf_{n \to \infty} \frac{h_n^m - h_{n+1}^m}{h_n^m} \geq 1 - \frac{1}{m}$.*

Theorem 3. *Let $m > 1$ be an integer, R be a continuous system of paths, h_n^m be the sequence obtained in Lemma 1 and x_1 be an arbitrary point of $[0,1]$. Then for each R-derived number $\alpha \in D(R, f, x_1)$ with $C(\alpha, R, f, x_1) < 1 - \frac{1}{m}$ there exists a monotone sequence $\{t_n\}$ so that for sufficiently large n, $t_n \in R_{x_1} \cap ([x_1 + h_{n+1}^m, x_1 + h_n^m] \cup [x_1 - h_n^m, x_1 - h_{n+1}^m])$ and $\lim_{n \to \infty} \frac{f(t_n) - f(x_1)}{t_n - x_1} = \alpha$.*

PROOF. Since $C(\alpha, R, f, x_1) < 1 - \frac{1}{m}$, there exists a monotone sequence $\{y_j\} \subseteq R_{x_1}$, so that $\lim_{j \to \infty} y_j = x_1$, $\lim_{j \to \infty} \frac{f(y_j) - f(x_1)}{y_j - x_1} = \alpha$, and $\overline{\gamma}(\{y_j\}, x_1) < 1 - \frac{1}{m}$. We claim that there exists a positive integer n_1 so that for all $n \geq n_1$, $([x_1 + h_{n+1}^m, x_1 + h_n^m] \cup [x_1 - h_n^m, x_1 - h_{n+1}^m]) \cap \{y_j\}_{j=1}^{\infty} \neq \emptyset$. If this is not true, then for some subsequence $\{n_k\}$ of natural numbers we have

$$([x_1 + h_{n_k+1}^m, x_1 + h_{n_k}^m] \cup [x_1 - h_{n_k}^m, x_1 - h_{n_k+1}^m]) \cap \{y_j\}_{j=1}^{\infty} = \emptyset.$$

Without loss of generality we may assume that $\{y_j\}$ is a decreasing sequence and $[x_1 + h_{n_k+1}^m, x_1 + h_{n_k}^m] \cap \{y_j\}_{j=1}^{\infty} = \emptyset$ for each k. Thus we have a subsequence $\{y_{j_k}\}$ of the sequence $\{y_j\}$ so that $y_{j_{k+1}} < h_{n_{k+1}}^m + x_1 < h_{n_k}^m + x_1 < y_{j_k}$. Let $a = y_{j_{k+1}} - x_1$, $b = h_{n_{k+1}}^m$, $c = h_{n_k}^m$, $d = y_{j_k} - x_1$. Then $a < b < c < d$. Thus we have $\frac{a}{d} < \frac{b}{c}$ implying $ac < bd$; hence $\frac{c-b}{c} < \frac{d-a}{d}$. This shows that $\frac{h_{n_k}^m - h_{n_k+1}^m}{h_{n_k}^m} \leq \frac{(y_{j_k} - x_1) - (y_{j_{k+1}} - x_1)}{y_{j_k} - x_1} = \frac{y_{j_k} - y_{j_{k+1}}}{y_{j_k} - x_1}$ for all k. Thus $\overline{\gamma}(\{y_j\}, x_1) \geq \limsup_{k \to \infty} \frac{y_{j_k} - y_{j_{k+1}}}{y_{j_k} - x_1} \geq \liminf_{k \to \infty} \frac{h_{n_k}^m - h_{n_k+1}^m}{h_{n_k}^m} \geq \liminf_{n \to \infty} \frac{h_n^m - h_{n+1}^m}{h_n^m} \geq 1 - \frac{1}{m}$. This is a contradiction to $\overline{\gamma}(\{y_j\}, x_1) < 1 - \frac{1}{m}$. Hence for each $n \geq n_1$ we may choose $s_n \in R_{x_1} \cap ([x_1 + h_{n+1}^m, x_1 + h_n^m] \cup [x_1 - h_n^m, x_1 - h_{n+1}^m]) \cap \{y_j\}_{j=1}^{\infty}$. Let $\{t_n\}$ be the sequence defined by $t_n = s_{n_1} + \frac{1 - s_{n_1}}{n}$ or $t_n = s_{n_1} - \frac{s_{n_1}}{n}$, depending

on whether $\{y_n\}$ is monotone decreasing or monotone increasing, for $n < n_1$ and $t_n = s_n$ for $n \geq n_1$. Then $\{t_n\}$ has all the desired properties. $\square$

Theorem 4. *Let R be a continuous system of paths. Then a typical continuous function in $C([0,1])$ has no finite R-derived number α with $C(\alpha, R, f) < 1$.*

PROOF. Let m, k, and p be positive integers, $A_{m,p} = \{f \in C([0,1]) : \exists \, x_f \in [0,1]$ so that f has an $R-$derived number α such that $|\alpha|$ is less than p with $C(\alpha, R, f, x_f) < 1 - \frac{1}{m}\}$, and $A = \{f \in C([0,1]) : f$ has a finite R-derived number α with $C(\alpha, R, f) < 1\}$. It is clear that $A \subseteq \cup_{m=1}^{\infty} \cup_{p=1}^{\infty} A_{m,p}$. From Theorem 3, it follows that for each $f \in A_{m,p}$, there exist a point $x_f \in [0,1]$, sequences $\{h_n^m\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ and positive integers $N_1(p)$ and $N_2(p)$ so that $y_n \in R_{x_f} \cap [x_f + h_{n+1}^m, x_f + h_n^m]$ or $y_n \in R_{x_f} \cap [x_f - h_n^m, x_f - h_{n+1}^m]$ for all $n \geq N_1(p)$, and $|\frac{f(y_n)-f(x_f)}{y_n-x_f}| \leq p$ for all $n \geq N_2(p)$. Now let

$$A_1(m,k,p) = \{f \in C([0,1]) : \exists \, x_f \in [0,1] \text{ and } y_k \in R_{x_f} \cap [x_f - h_k^m, x_f - h_{k+1}^m]$$

$$\text{such that } \left|\frac{f(y_k)-f(x_f)}{y_k-x_f}\right| \leq p\}$$

and

$$A_2(m,k,p) = \{f \in C([0,1]) : \exists \, x_f \in [0,1] \text{ and } y_k \in R_{x_f} \cap [x_f + h_{k+1}^m, x_f + h_k^m]$$

such that $|\frac{f(y_k)-f(x_f)}{y_k-x_f}| \leq p\}$. Then $A_{m,p} = \cap_{k=N(p)}^{\infty}[A_1(m,k,p) \cup A_2(m,k,p)]$ when $N(p) = \max\{N_1(p), N_2(p)\}$. Thus we have

$$A \subseteq \cup_{m=1}^{\infty} \cup_{p=1}^{\infty} \cap_{k=N(p)}^{\infty}[A_1(m,k,p) \cup A_2(m,k,p)].$$

First we claim that $A_1(m,k,p)$ is a closed subset of $C([0,1])$ for any positive integers m, p, and $k \geq N(p)$. To prove our claim let $\{f_i\} \subset C([0,1]$ be a Cauchy sequence in $A_1(m,k,p)$ and $\lim_{i\to\infty} f_i = f$. Then for each i, there exist $x_{f_i} \in [0,1]$ and $t_k^{f_i} \in R_{x_{f_i}} \cap [x_{f_i} + h_{k+1}^m, x_{f_i} + h_k^m]$ so that $\left|\frac{f_i(t_k^{f_i})-f_i(x_{f_i})}{t_k^{f_i}-x_{f_i}}\right| \leq p$. By replacing a sequence with one of its subsequence, if necessary, we may assume that there exist $x_f \in [0,1]$ and subsequences $\{x_{f_{i_j}}\}$ of $\{x_{f_i}\}$, $\{t_k^{f_{i_j}}\}$ of $\{t_k^{f_i}\}$, such that $\lim_{j\to\infty} x_{f_{i_j}} = x_f$, $\lim_{j\to\infty} t_k^{f_{i_j}} = t_k^f$. Since $t_k^{f_{i_j}} \in R_{x_{f_{i_j}}} \cap [x_{f_{i_j}} + h_{k+1}^m, x_{f_{i_j}} + h_k^m]$ and R is a continuous system of paths, we have $t_k^f \in R_{x_f} \cap [x_f + h_{k+1}^m, x_f + h_k^m]$. From uniform convergence of f_{i_j} to f we also have

$$\left|\frac{f(t_k^f)-f(x_f)}{t_k^f-x_f}\right| = \lim_{j\to\infty} \left|\frac{f_{i_j}(t_k^{f_{i_j}})-f_{i_j}(x_{f_{i_j}})}{t_k^{f_{i_j}}-x_{f_{i_j}}}\right| \leq p.$$

That is $f \in A_1(m, k, p)$; so $A_1(m, k, p)$ is closed. Similarly we can show that $A_2(m, k, p)$ is also closed. Hence $\cap_{k=N(p)}^{\infty}[A_1(m, k, p) \cup A_2(m, k, p)]$ is a closed subset of $C([0, 1])$ for each positive integer p. To complete the proof we will show that $B_p = \cap_{k=N(p)}^{\infty}[A_1(m, k, p) \cup A_2(m, k, p)]$ is nowhere dense for every positive integer p. This implies that A is a subset of a first category set. Thus it is a first category set. To show B_p is nowhere dense, let $f \in C([0, 1])$, $\epsilon > 0$ be an arbitrary real number, and let $B(f, \epsilon)$ be the open ball about f with radius ϵ. Suppose $h \in B(f, \epsilon)$ is a saw tooth function so that the absolute value of all the slopes of its linear segments are greater than p. Then at each point $x \in [0, 1]$ for any sequence $\{z_k\} \subset [0, 1]$ so that $z_k \in [x - h_k^m, x - h_{k+1}^m]$ or $z_k \in [x + h_{k+1}^m, x + h_k^m]$ we have $\left|\frac{h(z_k) - h(x)}{z_k - x}\right| > p$ for sufficiently large k. Thus $h \notin B_p$ for any positive integer p. Thus B_p is a closed subset of $C([0, 1])$ that does not contain any open ball, hence it is nowhere dense. Therefore A is first category. $\square$

Theorem 5. *Let $f \in C([0, 1])$ and let R be a bilateral continuous system of paths so that $\overline{f'_R}(x) < \infty$ for all $x \in [0, 1]$ or $\underline{f'_R}(x) > -\infty$ for all $x \in [0, 1]$. Then there exists a dense open set $O \subseteq [0, 1]$ on which f is differentiable for almost all $x \in O$.*

PROOF. We prove the theorem for the case where $\underline{f'_R}(x) > -\infty$ for all $x \in [0, 1]$. The case where $\overline{f'_R}(x) < \infty$ for all $x \in [0, 1]$ follows from this case by interchanging the roles of f and $-f$. To this end, let $\{a_n\}$ be the sequence obtained in Lemma 1 for $m = 2$. Take

$$\underline{F_n}(x) = \inf\left\{\frac{f(y) - f(x)}{y - x} : y \in R_x \cap ([x + a_{n+1}, x + a_n] \cup [x - a_n, x - a_{n+1}])\right\}$$

and $A_{m,n} = \{x \in [0, 1] : \underline{F_n}(x) > m\}$. We have

$$[0, 1] = \cup_{m=-\infty}^{\infty}\{x : \underline{f'_R}(x) > m\}$$
$$= \cup_{m=-\infty}^{\infty}\{x \in [0, 1] : \lim_{n \to \infty}\inf \underline{F_n}(x) > m\}$$
$$\subseteq \cup_{m=-\infty}^{\infty} \cup_{l=1}^{\infty} \cap_{n=l}^{\infty}\{x \in [0, 1] : \underline{F_n}(x) > m\}$$
$$= \cup_{m=-\infty}^{\infty} \cup_{l=1}^{\infty} \cap_{n=l}^{\infty} A_{m,n} \subseteq \cup_{m=-\infty}^{\infty} \cup_{l=1}^{\infty} \overline{\cap_{n=l}^{\infty} A_{m,n}} \subseteq [0, 1].$$

Let $t \in \overline{\cap_{n=l}^{\infty} A_{m,n}}$. Then for each $n > l$, there exists a sequence $x_{n,i} \in A_{m,n}$ such that $\lim_{i \to \infty} x_{n,i} = t$. Thus $\underline{F_n}(x_{n,i}) > m$ for all $n \geq l$ and all $i \geq 1$. Let $n \geq l$ be a fixed natural number and $z \in R_t \cap ([t + a_{n+1}, t + a_n] \cup [t - a_n, t - a_{n+1}])$ be an arbitrary point. Because

$$\lim_{i \to \infty} d_H \Big(R_{x_{n,i}} \cap ([x_{n,i} + a_{n+1}, x_{n,i} + a_n] \cup [x_{n,i} - a_n, x_{n,i} - a_{n+1}]),$$

$$R_t \cap ([t + a_{n+1}, t + a_n] \cup [t - a_n, t - a_{n+1}])\Big) = 0,$$

there exists a sequence

$$t_{n,i} \in R_{x_{n,i}} \cap ([x_{n,i} + a_{n+1}, x_{n,i} + a_n] \cup [x_{n,i} - a_n, x_{n,i} - a_{n+1}])$$

with $\lim_{i \to \infty} t_{n,i} = z$. This implies

$$\frac{f(z) - f(t)}{z - t} = \lim_{i \to \infty} \frac{f(t_{n,i}) - f(x_{n,i})}{t_{n,i} - x_{n,i}} \geq \lim \inf_{i \to \infty} \underline{F_n}(x_{n,i}) \geq m.$$

Hence $\underline{F_n}(t) \geq m$, implying $\underline{f'_R}(t) \geq m$. Consequently for each $x \in \overline{\cap_{n=l}^{\infty} A_{m,n}}$ we have $D^+(f)(x) \geq \underline{f'_R}(x) \geq m$.

Finally let $[a, b]$ be a closed subinterval of $[0, 1]$. Then we have $[a, b] = \cup_{m=-\infty}^{\infty} \cup_{l=1}^{\infty} (\overline{\cap_{n=l}^{\infty} A_{m,n}} \cap [a, b])$; so by Baire Category Theorem there exist an open interval J and integers M and L so that $J \subseteq \overline{\cap_{n=L}^{\infty} A_{M,n}} \cap [a, b])$. Hence for every $x \in J$, $D^+(f)(x) \geq M$. Therefore the function $f(x) - Mx$ is increasing on J and hence differentiable at almost all points of J. This implies that for any arbitrary closed subinterval of $[0, 1]$ there is an open interval contained in the subinterval so that f is differentiable at almost all points of that open interval. Now let $\mathcal{A} = \{\mathcal{J} : \mathcal{J}$ is an open subinterval of $[0, 1]$ so that f is differentiable almost everywhere on $J\}$ and let $O = \cup_{J \in \mathcal{A}} J$. It is clear that O is an open dense subset of $[0, 1]$ and f is differentiable at almost all points of O. $\square$

The following theorem is an immediate consequence of Theorem 5.

Theorem 6. *Let $f \in C([0, 1])$ and let R be a bilateral continuous system of paths so that f'_R exists on $[0, 1]$. Then there is a dense open set $O \subset [0, 1]$ on which f is differentiable for almost all $x \in O$.*

Remark 2. We know that typically continuous functions defined on $[0, 1]$ are nowhere differentiable. Theorem 6 indicates that typically such functions are not R-differentiable on any nondegenerate closed subinterval of $[0, 1]$ when R is a bilateral continuous system of paths.

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