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CHAOS AND THE RECURRENT SET

Abstract

Let f be an element of $C(I, I)$ with $R(f) = \{x \in I : x \in \omega(x, f)\}$ its recurrent set. We study the relationship between the structure of $R(f)$ and the chaotic nature of the function f . We show that $R(f)$ is always a G_δ set whenever f has zero topological entropy, although $R(f)$ is closed for the typical continuous function f with zero topological entropy. We also develop necessary and sufficient conditions on f for $R(f)$ to be closed.

1 Introduction

A recurring error made in the study of discrete dynamical systems has been the assertion that the recurrent point set of a continuous self-map of a compact interval is closed if and only if the function in question possesses zero topological entropy. Found in Sarkovskii's early work, this error has more recently appeared in [2] and [14]. The purpose of this paper is to study the relationship between the chaotic nature of a function f in $C(I, I)$ and the structure of its recurrent set $R(f) = \{x \in I : x \in \omega(x, f)\}$.

We begin in Section two with a presentation of the notation, definitions and previously known results we will need in the balance of the paper. There we also review the three forms of chaos we consider in the sequel: topological entropy, Li and Yorke's notion of a scrambled set, and the Baire class of Bruckner and Ceder's map $\omega_f : I \to K$ given by $x \mapsto \omega(x, f)$. In Section three we find that for a continuous function possessing zero topological entropy is a necessary but not a sufficient condition for its recurrent set to be closed; a construction found in [10] provides an example of a function $f \in C(I, I)$ with zero topological entropy for which $R(f)$ is not closed. We are able to show, however, that $R(f)$ is always a G_δ set whenever f possesses zero topological entropy, and $R(f)$ is closed whenever the map $\omega_f : I \to K$ given by $x \mapsto$

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$\omega(x, f)$ is in the first Baire class. This does not provide a necessary condition, however, as there are functions f in $C(I, I)$ for which ω_f is not in the first Baire class, yet $R(f)$ is still closed. We conclude in Section four by showing that the typical function f in $C(I, I)$ possessing zero topological entropy does have a closed recurrent set, so that in some sense functions like that constructed in [10] are exceptional.

2 Preliminaries

We shall be concerned with the class $C(I, I)$ of continuous functions mapping the unit interval $I = [0, 1]$ into itself, and the iterative properties this class of functions possesses. For f in $C(I, I)$ and any integer $n \geq 1, f^n$ denotes the n^{th} iterate of f. Let $P(f)$ represent those points $x \in I$ that are periodic under f. For each x in I, we call the set of all subsequential limits of the trajectory $\tau(x, f) = \{f^n(x)\}_{n=0}^{\infty}$ the ω-limit set of f generated by x, and write $\omega(x, f)$. Let $\Lambda(f) = \cup_{x \in I} \omega(x, f)$ represent the ω-limit points of f, while $\Omega(f) = \{\omega(x, f) : x \in I\}$ denotes the set composed of the ω-limit sets of f. If $x \in \omega(x, f)$, we call x a recurrent point of f; let $R(f)$ represent those points which are recurrent with respect to f.

A function $f : X \to Y$ is in the first Baire class if it is a pointwise limit of a sequence of continuous functions from X to Y. Let B_1 represent the class of functions in the first class of Baire.

We now turn our attention to the Baire Category Theorem. Let (X, ρ) be a metric space. A set is of the first category in (X, ρ) if it can be written as a countable union of nowhere dense sets; otherwise, the set is of the second category. A set is residual if it is the complement of a first category set; an element of a residual subset of (X, ρ) is called a typical element of X. With these definitions in mind, we recall Baire's theorem on category.

Theorem 2.1. *Let (X, ρ) be a complete metric space, with S a first category subset of X. Then $X \setminus S$ is dense in X.*

In addition to the usual, Euclidean metric d on $I = [0, 1]$, we will be working in two metric spaces. Within $C(I, I)$ we will use the supremum metric given by $\|f - g\| = \sup\{| f(x) - g(x) | : x \in I\}$. Our second metric space (K, H) is composed of the class of nonempty closed sets K in I endowed with the Hausdorff metric H given by $H(E, F) = \inf\{\delta > 0 : E \subset B_\delta(F), F \subset B_\delta(E)\}$, where $B_\delta(F) = \{x \in I : d(x, y) < \delta, y \in F\}$. This space is compact [4] . Our interest in the space (K, H) stems from the following two theorems from [1] and [3] respectively.

Theorem 2.2. *For any f in $C(I, I)$, the set $\Lambda(f)$ is closed in I.*

Theorem 2.3. *For any f in $C(I, I)$, the set $\Omega(f)$ is closed in (K, H).*

We also make use of three different formulations of chaos.

Throughout the sequel we will restrict our attention to a closed subset E of $C(I, I)$ composed of those functions f having zero topological entropy, denoted by $\mathbf{h}(f) = 0$. The reader is referred to Theorem A of [8] for an extensive list of equivalent formulations of topological entropy zero. For our purposes, it suffices to note that every periodic orbit of a continuous function with zero topological entropy has cardinality of a power of two. The following theorem, due to Smital [12], sheds considerable light on the structure of infinite ω-limit sets for functions with zero topological entropy.

Theorem 2.4. *If ω is an infinite ω-limit set of $f \in C(I, I)$ possessing zero topological entropy, then there exists a sequence of closed intervals $\{J_k\}_{k=1}^{\infty}$ in $[0, 1]$ such that*

1. *for each $k, \{f^i(J_k)\}_{i=1}^{2^k}$ are pairwise disjoint, and $J_k = f^{2^k}(J_k)$;*
2. *for each $k, J_{k+1} \cup f^{2^k}(J_{k+1}) \subset J_k$;*
3. *for each $k, \omega \subset \cup_{i=1}^{2^k} f^i(J_k)$;*
4. *and for each k and $i, \omega \cap f^i(J_k) \neq \emptyset$.*

We make the following definitions with Smital's Theorem in mind. Let ω be an infinite compact subset of I, and let f map ω into itself. We call f a simple map on ω if ω has a decomposition $S \cup T$ into compact portions that f exchanges, and f^2 is simple on each of these portions. From Smital's Theorem, one sees that every map f with zero topological entropy is simple on each of its infinite ω-limit sets. Let $\{J_k\}_{k=1}^{\infty}$ be a nested sequence of compact periodic intervals with respect to ω and f as described in Smital's Theorem. Every set of the form $\omega \cap f^i(J_k)$ is periodic of period 2^k, and we call each such set a periodic portion of rank k. This system of periodic portions of ω, or of the corresponding periodic intervals, is called the simple system of ω with respect to f. Since $J_{k+1} \cup f^{2^k}(J_{k+1}) \subset J_k$ for all $k \geq 1$, the sequence $\{\cup_{i=1}^{2^k} f^i(J_k)\}_{k=1}^{\infty}\}$ is descending, so that $\cap_{k \geq 1} \cup_{i=1}^{2^k} f^i(J_k)$ is nonempty. Set $L = \cap_{k \geq 1} \cup_{i=1}^{2^k} f^i(J_k)$.

Given the very specific behavior that functions of zero topological entropy must demonstrate on their infinite ω-limit sets, it may not be too surprising that Bruckner and Smital have been able to characterize these sets [6].

Theorem 2.5. *An infinite compact set $W \subset (0, 1)$ is an ω-limit set of a map $f \in C(I, I)$ with zero topological entropy if and only if $W = Q \cup P$ where*

Q is a Cantor set and P is empty or countably infinite, disjoint with Q, and satisfies the following conditions:

1. *every interval contiguous to Q contains at most two points of P;*
2. *each of the intervals $[0, \min Q), (\max Q, 1]$ contains at most one point of P;*
3. *and $\overline{P} = Q \cup P$.*

We now define chaos in the sense of Li and Yorke [11] .

Take $\delta \geq 0$, with f in $C(I, I)$. Suppose $S \subseteq I$ such that for any $x, y \in S$ with $x \neq y$ we have $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > \delta$ and $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$. We call S a scrambled set of f, and if f possesses an uncountable scrambled set, then f is said to be chaotic in the sense of Li and Yorke. While not immediately apparent, a function f is chaotic in the sense of Li and Yorke if and only if there is a point $x \in I$ which is not approximately periodic with respect to f.

Our third notion of chaos comes from Bruckner and Ceder [5] .

To each function $f \in C(I, I)$ associate the map $\omega_f : I \to (K, H)$ given by $x \mapsto \omega(x, f)$. Bruckner and Ceder show that the Baire class of the map $\omega_f : I \to (K, H)$ well reflects the chaotic nature of the function f. In fact, those functions f for which ω_f is in the first Baire class exhibit a form of nonchaos that allows scrambled sets but not positive topological entropy. That is, f not chaotic in the sense of Li and Yorke $\Rightarrow \omega_f : I \to (K, H)$ is in the first Baire class $\Rightarrow f$ possesses zero topological entropy, but none of the reverse implications is true.

3 $R(f)$ and Chaos

We begin with a brief description of the function found in [10] as it provides an example of a function with zero topological entropy for which the recurrent set is not closed.

Let g be the function constructed in [10]. There exists an ω-limit set $\omega(x_0, g)$ of g such that $\omega(x_0, g) = Q \cup P$ in accordance with Theorem 2.5, and P is nonempty. Moreover, there is a sequence of periodic points $\{p_n\} \subset P(g)$ such that $\lim_{n\to\infty} p_n = y$, where $y = \min\{x : x \in \omega(x_0, g)\}$ and $y \in P$. Since $P \cap R(g) = \emptyset$, it follows that $R(g)$ is not closed. If we set $\omega_n = \omega(p_n, g)$, we can use the fact that $\Omega(g)$ is closed and (K, H) is compact to find a subsequence $\{\omega_{n_k}\} \subset \{\omega_n\}$ such that $\lim_{n\to\infty} \omega_{n_k} = \omega(x_0, g) = Q \cup P$ in (K, H). Thus, $\omega(x_0, g) = Q \cup P$ can be approximated by periodic ω-limit sets in (K, H) even though the generating trajectory $\tau(x_0, g)$ is not approximately periodic.

Let us now turn our attention to the relationship between $R(f)$ and the chaotic nature of the function f. We begin with a well known result.

Lemma 3.1. *If $f \in C(I,I)$ and $R(f)$ is closed, then $\mathbf{h}(f) = 0$.*

Proof. Suppose $f \in C(I,I)$ and $\mathbf{h}(f) > 0$. Let $X \subseteq I$ so that $f^n(X) = X$ and $f^n \mid X$ is semiconjugate to the shift operator σ on two symbols, for some natural number n [7] . Then $X \subseteq \overline{R(f)}$, but $X - R(f) \neq \emptyset$, so that $R(f)$ is not closed. $\square$

The example of [10] shows us that the converse of Lemma 3.1 is false, so that there are functions f with zero topological entropy for which $R(f)$ is not closed. Our next proposition does show, however, that the set structure of $R(f)$ cannot be too complicated for a function with zero topological entropy. Whenever $f \in C(I,I)$ such that $\mathbf{h}(f) = 0$, the set $R(f)$ is always a G_δ set; that is, $R(f)$ can be expressed as the intersection of a countable collection of open sets.

Proposition 3.2. *If $f \in C(I,I)$ for which $\mathbf{h}(f) = 0$, then $R(f)$ is a G_δ set.*

Proof. From the Main Theorem of [15] we know that $\overline{R(f)} \setminus R(f)$ is countable whenever $\mathbf{h}(f) = 0$. This allows us to express $\overline{R(f)} \setminus R(f)$ as a countable union of closed sets, each of which is a singleton; say $\overline{R(f)} \setminus R(f) = \cup_{n=1}^{\infty} F_n$. Then $R(f) = \overline{R(f)} \setminus \cup_{n=1}^{\infty} F_n = \cap_{n=1}^{\infty} (\overline{R(f)} \setminus F_n)$. Since the difference of closed sets is a G_δ set it follows that $\overline{R(f)} \setminus F_n$ is a G_δ set for any n, so that $R(f) = \cap_{n=1}^{\infty}$ $(\overline{R(f)} - F_n)$ is also a G_δ set. $\square$

Our next result provides a sufficient condition on $f \in C(I,I)$ for $R(f)$ to be closed.

Proposition 3.3. *If $f \in C(I,I)$ and the map $\omega_f : I \to (K,H)$ given by $x \mapsto \omega(x,f)$ is in the first Baire class, then $R(f)$ is closed.*

Proof. If $f \in C(I,I)$ and the map $\omega_f : I \to (K,H)$ given by $x \mapsto \omega(x,f)$ is in the first Baire class, then all the ω-limit sets of f are either finite sets or perfect sets [5]. It follows that $\Lambda(f) \subseteq R(f)$, so that $R(f) = \Lambda(f)$ is closed. $\square$

From Proposition 3.3 one gets the following pair of corollaries.

Corollary 3.4. *If $f \in C(I,I)$ is not chaotic in the sense of Li and Yorke, then $R(f)$ is closed.*

Proof. We need only recall that the map $\omega_f : I \to (K,H)$ given by $x \mapsto \omega(x,f)$ is in the first Baire class whenever $f \in C(I,I)$ is not chaotic in the sense of Li and Yorke [5]. $\square$

Corollary 3.5. *If $f \in C(I,I)$ and the map $\omega_f : I \to (K,H)$ given by $x \mapsto \omega(x,f)$ is in the first Baire class, then $\overline{P(f)} = \Lambda(f)$.*

Proof. If $f \in C(I,I)$ and the map $\omega_f : I \to (K,H)$ given by $x \mapsto \omega(x,f)$ is in the first Baire class, then from the proof of Proposition 3.3 one sees that $R(f) = \Lambda(f)$. Since $R(f) \subset \overline{P(f)}$ for all $f \in C(I,I)$, it follows that $\overline{P(f)} = \Lambda(f)$ [1]. □

It follows from Proposition 3.3 and its Corollary 3.5 that there is a class of functions chaotic in the sense of Li and Yorke for which the recurrent sets are closed and the periodic points are dense in the collection of ω-limit points.

The next two examples show that, in some sense, Proposition 3.3 is the best we can do in relating the closed nature of the recurrent set to the chaotic nature of the generating function.

Example 3.6. There exists $f \in C(I,I)$ such that the map $\omega_f : I \to (K,H)$ given by $x \mapsto \omega(x,f)$ is not in the first Baire class, but $R(f)$ is closed.

Construction: Bruckner and Ceder create this type of function in the proof of Theorem 4.3 [5].

Example 3.7. There exists $f \in C(I,I)$ such that the map $\omega_f : I \to (K,H)$ given by $x \mapsto \omega(x,f)$ is not in the first Baire class, and $R(f)$ is not closed.

Construction: The function created by Hsin and Xiong in [10] is an example of such a function.

We note that for the function f found in Example 3.6, one clearly has $\overline{P(f)} \neq \Lambda(f)$. Surprisingly, however, the example of [10] may provide an example of a function f for which the map $\omega_f : I \to (K,H)$ given by $x \mapsto \omega(x,f)$ is not in the first Baire class and $R(f) \neq \overline{R(f)}$, yet $\overline{P(f)} = \Lambda(f)$.

Our final result of the section shows that whenever a function has zero topological entropy and is piecewise monotonic, its recurrent set is closed.

Proposition 3.8. *If $f \in C(I,I)$ is piecewise monotonic and $\mathbf{h}(f) = 0$, then $R(f)$ is closed.*

Proof. With Theorem 1.4 of [9] , Gedeon shows that every piecewise monotonic function f with zero topological entropy possesses only perfect infinite ω-limit sets. Since the map $\omega_f : I \to (K,H)$ given by $x \mapsto \omega(x,f)$ is in the first Baire class for such a function, our conclusion follows from Proposition 3.3. □

4 $R(f)$ and Typical Elements of E

The main result of this section is the following theorem which shows that the typical function possessing zero topological entropy has a set of recurrent points which is closed.

Theorem 4.1. *The set* $\mathbf{R} = \{f \in E : R(f) = \overline{R(f)}\}$ *is a residual subset of* E.

Our theorem follows easily from the following proposition.

Proposition 4.2. *The set* $\mathbf{S} = \{f \in E : f$ *possesses a simple system for which* $\operatorname{int} L \neq \emptyset\}$ *is a first category subset of* E.

Proof. Let $\{(a_n, b_n)\}_{n=1}^{\infty}$ be an enumeration of the open subintervals of I with rational endpoints, and set $\mathbf{S}_n = \{f \in E : (a_n, b_n) \subset L$ for a simple system of $f\}$. Since $\mathbf{S} = \cup_{n=1}^{\infty} \mathbf{S}_n$, our goal is to show that $\mathbf{S}_n$ is nowhere dense for all n. It suffices to show that $\mathbf{S}_n$ is closed and that $E \setminus \mathbf{S}_n$ is dense.

$E \setminus \mathbf{S}_n$ is dense: From Theorem 4 of [13], functions generating only finite ω-limit sets are dense in E; it follows that $E \setminus \mathbf{S}_n$ is dense.

$\mathbf{S}_n$ is closed: Let $\{f_k\} \subset \mathbf{S}_n$ for which $f_k \to f$ in $C(I, I)$, and to each f_k is associated a simple system with $(a_n, b_n) \subset L_k$. Since (K, H) is a compact metric space, by restricting our attention to a subsequence of $\{f_k\}$ if necessary, we may presume that $L_k \to L$ in (K, H) while $f_k \to f$ in $C(I, I)$. Since $(a_n, b_n) \subset L_k$ for all k, and $L_k \to L$, we know that $(a_n, b_n) \subset L$; f is strongly invariant on L, too, as $f_k(L_k) = L_k$ for all k. We now show that f is a simple map on L . As f_k is a simple map on L_k, each L_k has a decomposition $S_k \cup T_k$ into disjoint compact portions that f_k exchanges, and f_k^2 is simple on each of these. Let us suppose that $\max S_k < \min T_k$, and by again restricting our attention to a subsequence of $\{f_k\}$ if necessary, we may assume that for each k, (a_n, b_n) is always an element of either S_k or T_k; say $(a_n, b_n) \subset S_k$. Since $f_k \to f$ and $L_k \to L$, it follows that $\lim_{n\to\infty} S_k = S$ and $\lim_{n\to\infty} T_k = T$ both exist in (K, H), with $S \cup T = L$, $f(S) = T$ and $f(T) = S$. Now, there exists a fixed point x_k for each f_k such that $\max S_k < x_k < \min T_k$, so that there exists a fixed point x for f where $\max S \leq x \leq \min T$; moreover, since f_k^2 is simple on both S_k and T_k, we conclude that $maxS \lneqq x \lneqq minT$. Since $(a_n, b_n) \subset S_k$ for each k, $(a_n, b_n) \subset S$, and as $f(T) = S$, T must be a nondegenerate closed interval, too. Thus, L can be decomposed into nondegenerate compact portions S and T that f exchanges. In a similar manner one shows that both S and T can be decomposed into nondegenerate compact portions that f^2 exchanges. We conclude that f possesses a simple system for which $(a_n, b_n) \subset L$, so that $\mathbf{S}_n$ is closed. This completes our proof. $\square$

We now prove Theorem 4.1.

Proof. Suppose f is an element of the residual set $C(I,I)-\mathbf{S}$. Then $\operatorname{int} L=\emptyset$ for any simple system of f, so that all the elements of $\Omega(f)$ are either finite or perfect. Thus, the map $\omega_f : I \to (K,H)$ given by $x \mapsto \omega(x,f)$ is in the first Baire class [5] . Our conclusion follows from Proposition 3.3. $\square$

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