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THE NON-UNIFORM RIEMANN APPROACH TO THE MUTLIPLE ITÔ-WIENER INTEGRAL

Abstract

The Riemann approach to integration is well-known for its explicitness and directness. In this paper we use the Non-Uniform Riemann Approach to give an alternative definition of the Multiple Itô-Wiener Integral and prove that our definition is in fact equivalent to the classical definition.

1 Introduction

The theory of Multiple Stochastic Integral was first studied by N. Wiener in 1938, see [11]. This study was followed up in greater details by K. Itô in the early 1950s, see [5]. Similar to his study of the stochastic integral in one-dimension, he gave a non-explicit L^2 procedure in defining what we call the Multiple Itô-Wiener integral.

A natural question that arises is whether it is possible to use an explicit procedure of defining Multiple Itô-Wiener integral by the Riemann approach, which is well known for its explicitness and directness. In fact, it was proved that by using Riemann approach with non-uniform mesh, stochastic integral can be seen as the limit of a sequence of Riemann sums, see [1], [10], [12].

In this paper, we shall show that in fact the non-uniform Riemann approach can also be used to give an alternative definition to the Multiple Itô-Wiener integral and that this definition, which we shall call Multiple Itô-McShane integral, is equivalent to the classical Multiple Itô-Wiener Integral.

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2 Classical Multiple Itô-Wiener Integral

In this section we shall review the construction and some basic results of the classical Multiple Itô-Wiener Integral. The details of the results can be obtained from [5], Section 2, Pp 183 - 186].

Let $T = [0, 1]$ and $T^m = [0, 1]^m$. We shall denote Lebesgue measure on T by λ. Let λ^m be the corresponding Lebesgue measure on T^m. For any interval I of T, we may use $|I|$ or $\lambda(I)$ to denote the length of I. The norm in $L^2(T^m, \lambda^m)$ is denoted by $|| \cdot ||_m$.

Definition 2.1. Let (Ω, P) be a probability space and $W = \{W_t(\omega) : t \in [0, 1]\}$ be a family of random variables. Then W is said to be a canonical Brownian motion if it satisfies the following properties:

1. $W_0(\omega) = 0$ for all $\omega \in \Omega$,
2. it has **Normal Increments**; that is, $W_t - W_s$ has a Normal distribution with mean 0 and variance $t - s$ for all $t > s$ (which naturally implies that W_t has a Normal distribution with mean 0 and variance t),
3. it has **Independent Increments**; that is, $W_t - W_s$ is independent of its past; that is, W_u, $0 \leq u < s < t$ and
4. its sample paths are continuous; i.e., for each $\omega \in \Omega$, $W_t(\omega)$ as a function of t is continuous on $[0, 1]$.

Definition 2.2. Let $\{I_1, I_2, I_3, \ldots, I_n\}$ be a collection of left-open subintervals which form a partition of $(0, 1]$; i.e., the intervals $I_1, I_2, \ldots I_n$ are disjoint and $\bigcup_{k=1}^n I_k = (0, 1]$. An *elementary function* on T^m is a function $g : T^m \to \mathbb{R}$ that can be expressed in the form

$$g = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1, i_2, \ldots, i_m} 1_{I_{i_1} \times I_{i_2} \times \cdots \times I_{i_m}} \tag{1}$$

where $\{I_{i_1}, I_{i_2}, \ldots, I_{i_m}\}$ is a subset from $\{I_1, I_2, I_3, \ldots, I_n\}$ and $a_{i, i_2, \ldots, i_m} = 0$ if any two of the indices $i_1, i_2, \ldots i_m$ are equal.

Note that the definition of an elementary function shows that g vanishes on the set of elements in T^m which have some equal components; that is, if $t = (t_1, t_2, t_3, \ldots, t_n) \in T^m$ such that $t_i = t_j$ for some $i \neq j$, then $g(t) = 0$.

The Multiple Itô-Wiener integral of an elementary function g is defined by

$$IW(g) = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1, i_2, \ldots, i_m} \prod_{j=1}^{m} W(I_{i_j}),$$

where W is Brownian motion (see Definition 2.1) and $W(I)$ denotes $W_b - W_a$ if $I = (a, b]$. It is known that $E(IW(g))^2 \leq m!||g||_m^2$, see [5], p. 162].

Let $f \in L^2(T^m, \lambda^m)$. Then there exists a sequence $\{f_n\}$ of elementary functions on T^m such that $\lim_{n\to\infty} ||f_n - f||_m = 0$. On the other hand, $E\left(IW(f_p - f_q)\right)^2 \leq m!||f_p - f_q||_m^2$. Hence $\{IW(f_n)\}$ is a Cauchy sequence in $L^2(\Omega)$. By completeness, the limit exists and hence the Multiple Itô-Wiener integral $IW(f)$ of f is defined by $\lim_{n\to\infty} E\left((IW(f_n) - IW(f)\right)^2 = 0$, see [5], p. 162].

Definition 2.3. Let $f : T^m \to \mathbb{R}$ be a real-valued function. For each $\pi \in S_m$ where S_m is the set of all permutations on m objects, let f_π denote the *permuted function* of f under π, which is the function

$$f_\pi(t_1, t_2, t_3, \ldots, t_m) = f(t_{\pi(1)}, t_{\pi(2)}, \ldots, t_{\pi(m)})$$

for each $(t_1, t_2, \ldots, t_m) \in T^m$.

The *symmetrization* of the function f, denoted by $\tilde{f}$, is the function $\tilde{f} : T^m \to \mathbb{R}$ defined by

$$\tilde{f}(t_1, t_2, \ldots, t_m) = \frac{1}{m!} \sum_{\pi \in S_m} f_\pi(t_1, t_2, \ldots, t_m)$$

where the summation is over all $\pi \in S_m$.

Theorem 2.4. *[See, for example, [5]] Let $f : T^m \to \mathbb{R}$ and $g : T^m \to \mathbb{R}$ be Multiple Itô-Wiener integrable. Then*

(i) $\tilde{f}$ is Multiple Itô-Wiener integrable and $IW(f) = IW(\tilde{f})$,

(ii) $E[IW(f)] = 0$ and

(iii) $af + bg$ is Itô-Wiener integrable for any constants $a, b \in \mathbb{R}$, and further

$$IW(af + bg) = aIW(f) + bIW(g).$$

3 Multiple Itô-McShane Integral

In this section we shall use the non-uniform Riemann approach to define the Multiple Itô-McShane integral. First we shall define the non-uniform division of T^m that we shall consider. This type of division in one-dimension was considered by McShane. Hence we call our integral the Multiple Itô-McShane integral.

The intervals of T^m that we shall consider in the rest of this paper are of the form $I = \prod_{i=1}^{m} I_i$, where each I_i is a left-open interval of $[0,1]$ of the form $(a_i, b_i]$.

Definition 3.1. Let δ be a positive function defined on T^m, $\xi = (\xi_1, \xi_2, \ldots, \xi_m) \in T^m$ and $I = \prod_{i=1}^{m} I_i$ be an interval of T^m. An *interval-point pair* (I, ξ) is said to be δ-fine if $I_k \subset [\xi_k - \delta(\xi), \xi_k + \delta(\xi)]$ for each $k = 1, 2, 3, \ldots, m$.

Note that ξ_k may or may not be in I_k for each $k = 1, 2, 3, \ldots, m$. A finite collection D of interval-point pairs $\{(I^{(i)}, \xi^{(i)}) : i = 1, 2, 3, \ldots, n\}$ is said to be a δ-fine division of T^m if

(i) $I^{(i)}$, $i = 1, 2, 3, \ldots, n$, are disjoint left-open intervals of T;

(ii) $\bigcup_{i=1}^{n} I^{(i)} = (0,1]^m$.

We remark that for any given positive function δ on T^m, a δ-fine division of T^m exists, which is a direct consequence of the Heine-Borel open covering theorem or can be proved directly using continued bisection.

Notation. It can be seen that T^m consists of two parts; namely, the diagonal part of T^m

$$\mathcal{D} = \{(x_1, \ldots, x_m) \in T^m : x_i = x_j \text{ for some } i \neq j\},$$

and

$$\mathcal{D}^c = \{(x_1, \ldots, x_m) \in T^m : x_i \neq x_j \text{ for any } i \neq j\},$$

which is the non-diagonal part of T^m. The non-diagonal set $\mathcal{D}^c$ plays a basic role in the construction of the multiple Itô-Wiener integral, as can be seen in Definition 2.2 that the elementary function vanishes on the diagonal set $\mathcal{D}$. The non-diagonal set can be decomposed to $m!$ open connected sets in T^m.

For each $\pi \in S_m$, the group of all permutations of m objects, we let

$$G_\pi = \{(x_1, x_2, x_3, \ldots, x_m) \in T^m : x_{\pi(1)} < x_{\pi(2)} < x_{\pi(3)} < \cdots < x_{\pi(m)}\},$$

and there are $m!$ such sets. Each of these sets is said to be *contiguous* to the diagonal $\mathcal{D}$.

In this paper, we shall focus on the integral on the non-diagonal $\mathcal{D}^c$ of T^m. The classical treatment of the integral on the diagonal $\mathcal{D}$ can be found in [9]. Let f be a real-valued function on T^m. Define a function by

$$f_0(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{D}^c \\ 0 & \text{otherwise.} \end{cases}$$

Then f_0 is called the non-diagonal part of f.

Let f be a real-valued function on T^m. If $D = \{(I^{(i)}, x^{(i)})\}$ is a δ-fine division of T^m, then we let $S(f, \delta, D)$ denote the Riemann sum

$$S(f, \delta, D) = (D) \sum f(x^{(i)}) W(I^{(i)}),$$

where $W(I^{(i)}) = \prod_{j=1}^m W(I_j^{(i)})$, $I^{(i)} = \prod_{j=1}^m I_j^{(i)}$ and each $I_j^{(i)}$ is a left-open interval of T.

Definition 3.2. A function $f : T^m \to \mathbb{R}$ is said to be Multiple Itô-McShane integrable to a function $IM(f)$ on T^m if for every $\varepsilon > 0$, there exists a positive function δ such that $E\big(|S(f_0, \delta, D) - IM(f)|^2\big) < \varepsilon$ whenever $D = \{(I^{(i)}, x^{(i)}) : i = 1, 2, 3, \ldots, n\}$ is a δ-fine division of T^m.

Lemma 3.3. *Let δ be a positive function on T^m and $\{D_k\}$ be a finite family of δ-fine divisions of T^m. Then there exists a partition $\{A_1, A_2, \ldots, A_q\}$ of $[0, 1]$ and a finite family of δ-fine divisions of T^m denoted by $\{D_k'\}$ such that each interval of any D_k' is of the form $A_{l_1} \times A_{l_2} \times \cdots \times A_{l_m}$ and each D_k' is a refinement of D_k. Furthermore, $S(f_0, \delta, D_k) = S(f_0, \delta, D_k')$ for all k.*

PROOF. The assertion follows from the following facts. First, if (I, ξ) is δ-fine in T^m, and if $I = J \cup K$, where J and K are two disjoint left-open subintervals, then (J, ξ) and (K, ξ) are δ-fine. Second, $f(\xi)W(I) = f(\xi)W(J) + f(\xi)W(K)$, and by taking $\{A_1 A_2, A_3, \ldots, A_q\}$ to be all the intervals formed by taking all the division points of D_k, the proof is easily completed. □

Definition 3.4. A finite collection of δ-fine division of T^m of the form D_k' (in Lemma 3.3) is said to be a *standard* δ-fine division of T^m; that is, all the partitions of $\{D_k'\}$ have the same division points on T.

In view of Lemma 3.3, we shall assume that all finite collections of δ-fine divisions of T^m that we consider in Definition 3.2 are all standard divisions.

Remark 3.5. From standard properties of Brownian motion, we know that

(A): if $I_i = (u_i, v_i]$ and $I_j = (u_j, v_j]$ are disjoint, then $E\big(W(I_i)W(I_j)\big) = 0$

while

(B): if $I_i = I_j = (u, v]$, then $E\big(W(I_i)W(I_j)\big)$ is a function of $|v-u|$.

By using a *standard* division (as in Definition 3.4) of T^m, we ensure that that (A) occurs. Thus we are able to have Lemma 4.9, which is crucial to prove the Equivalence Theorem. This is the rationale for using *standard* δ-fine divisions instead of the δ-fine divisions of Definition 3.1.

Proposition 3.6. *The Multiple Itô-McShane integral, if it exists, is unique.*

Proof. The proof is standard in the theory of Henstock integration. For details, see for example [7, p. 32, Theorem 2.4.6] for the proof. □

In view of Proposition 3.5, we may let $IM(f)$ denote the Multiple Itô-Wiener integral of f for our subsequent sections.

4 Basic Properties of Multiple Itô-McShane Integral

In this section we shall state and prove the basic results of the Multiple-Itô Stochastic Integral.

Proposition 4.1. *A function f is Multiple Itô-McShane integrable on T^m if and only if given $\varepsilon > 0$, there exists a positive function δ on T^m such that $E\big(|S(f_0, \delta, D_1) - S(f_0, \delta, D_2)|^2\big) < \varepsilon$ whenever D_1, D_2 are standard δ-fine divisions of T^m.*

Definition 4.2. A function $f : T^m \to \mathbb{R}$ is said to be Multiple Itô-McShane integrable on an interval I of T^m if $f1_I$ is Multiple Itô-McShane integrable on T^m.

Using Cauchy's Criteria (Proposition 4.1), we have the following two results on the integrability of subintervals. The proof is standard in the theory of Henstock integration and hence is omitted.

Proposition 4.3. *Let f be Multiple Itô-McShane integrable on T^m. Then f is Multiple Itô-McShane integrable on any interval I of T^m.*

Proposition 4.4. *Let f be Multiple Itô-McShane integrable on T^m. Suppose that $I = J \cup K$, where I, J and K are intervals from T^m, then f is Multiple Itô-McShane integrable on T^m and $IM(f, I) = IM(f, J) + IM(f, K)$ where $IM(f, I)$ denote the Multiple Itô-McShane integral of f over I in T^m.*

Lemma 4.5. *Let f be Multiple Itô-McShane integrable function on T^m. Then there exist positive functions δ_n on T^m, $n = 1, 2, 3, \ldots$, with $\delta_{n+1} < \delta_n$ for all $n = 1, 2, 3, \ldots$ such that $IM(f)$ is the limit of $S(f, \delta_n, D_n)$ in the L_2-norm.*

PROOF. For each $n = 1, 2, \ldots$, there exists a positive function δ_n on T^m such that the inequality in Definition 3.2 holds with $\varepsilon = \frac{1}{n}$. For each $n = 1, 2, 3, \ldots$, fix a δ_n-fine division D_n. We may assume that $\delta_{n+1}(\xi) < \delta_n(\xi)$ for each n and each $\xi \in T^m$. Hence we have $\lim_{n\to\infty} E\big(|S(f_0, \delta_n, D_n) - IM(f)|^2\big) = 0$. $\square$

Proposition 4.6. *Let f be Multiple Itô-McShane integrable on T^m with primitive $IM(f) \in L^2(\Omega)$. Then $E\,[IM(f)] = 0$.*

PROOF. By using Lemma 4.5, we have $E[IM(f)] = \lim_{n\to\infty} E\,[S(f_0, \delta_n, D_n)]$ and it is clear that $E[S(f_0, \delta_n, D_n)] = 0$ for all n. $\square$

Proposition 4.7. *Let f be Multiple Itô-McShane integrable with value $IM(f)$ and let $\tilde{f}$ denote the symmetrization (see Definition 2.3) of the function f. Then $\tilde{f}$ is also Multiple Itô-McShane integrable and $IM(f) = IM(\tilde{f})$.*

PROOF. Given $\varepsilon > 0$ there exists a positive function δ such that

$$E\Big(\Big|(D)\sum_{i=1}^{n} f_0(\xi^i)\prod_{r=1}^{m} W(I_{i_r}) - IM(f)\Big|^2\Big) < \frac{\varepsilon}{(m!)^2}$$

for any δ-fine standard division $D = \big\{\big(\prod_{r=1}^m I_{i_r}, \xi^i\big) : i = 1, 2, 3, \ldots, n\big\}$ of T^m. For each $\pi \in S_m$ let $(f_0)_\pi$ denote the *permuted function of f under π* (see Definition 2.3); that is, $(f_0)_\pi(t_1, t_2, \ldots, t_m) = f_0(t_{\pi(1)}, t_{\pi(2)}, \ldots, t_{\pi(m)})$ for any $(t_1, t_2, \ldots, t_m) \in T^m$. Let δ_π be the permuted function of δ, so that considering any δ_π-fine standard division $D_\pi = \big\{\big(\prod_{i=1}^m I_{i_{\pi(r)}}, \xi_\pi^i\big)\big\}$ of T^m we have

$$E\Big(\Big|(D_\pi)\sum_{i=1}^{n} (f_0)_\pi(\xi_\pi^i)\prod_{r=1}^{m} W(I_{i_{\pi(r)}}) - IM(f)\Big|^2\Big) < \frac{\varepsilon}{(m!)^2}.$$

Note that $\displaystyle\prod_{r=1}^{m} W(I_{i_r}) = \prod_{r=1}^{m} W(I_{i_{\pi(r)}})$. Let $\delta(\xi) = \min_{\pi\in S_m} \delta_\pi(\xi)$ for all $\xi \in T^m$. Consider any δ-fine standard division $D = \Big\{\big(\prod_{r=1}^m I_{i_r}, \xi^i\big) : i =$

$1, 2, \dots, n\Big\}$ of T^m. We have

$$
\begin{aligned}
&E\Big(\big|(D)\sum_{i=1}^{n}(\tilde{f})_0(\xi^i)\prod_{r=1}^{m}W(I_{i_r}) - IM(f)\big|^2\Big)\\
=&E\Big(\big|(D)\sum_{i=1}^{n}\big(\frac{1}{m!}\sum_{\pi\in S_m}(f_\pi)_0(\xi^i)\big)\prod_{r=1}^{m}W(I_{i_r}) - IM(f)\big|^2\Big)\\
\le&\frac{1}{(m!)^2}E\Big(\big|(D)\sum_{\pi\in S_m}\Big\{\sum_{i=1}^{n}(f_\pi)_0(\xi^i)\prod_{r=1}^{m}W(I_{i_r}) - IM(f)\Big\}\big|^2\Big)\\
\le&\frac{1}{(m!)^2}(m!)\sum_{\pi\in S_m}E\Big(\big|(D)\sum_{i=1}^{n}(f_\pi)_0(\xi^i)\prod_{r=1}^{m}W(I_{i_r}) - IM(f)\big|^2\Big)\\
\le&\frac{1}{m!}\sum_{\pi\in S_m}\varepsilon = \varepsilon.
\end{aligned}
$$

□

Lemma 4.8. *Let f be Multiple Itô-McShane integrable on T^m, and let $F(I)$ be the Multiple Itô-McShane integral of f on the subinterval $I \subset T^m$. Let I and J be two disjoint intervals from T^m in the same contiguous set G_π for some $\pi \in S_m$ such that the components of I and J are either disjoint or equal; that is, if $I = \prod_{i=1}^{m} I_i$ and $J = \prod_{j=1}^{m} J_j$ and if $I_i \bigcap J_j \neq \phi$, then $I_i = J_j$. Then*

(i) *$E[W(I)W(J)] = 0$,*

(ii) *F has the orthogonal increment property; that is, $E[F(I)F(J)] = 0$,*

(iii) *$E[W(I)F(J)] = 0$ and*

(iv) *$E[(cW(I) - F(I))(cW(J) - F(J))] = 0$.*

PROOF. Let I and J be as defined as in the above statement of the lemma. Then it is clear that there exists an $I_i, i = 1, 2, 3, \dots, m$ such that $I_i \bigcap I_k$ is empty for all $k = 1, 2, 3, \dots, m$, since both I and J are from the same contiguous set G_π. By the orthogonal increment of the Brownian motion, (i) follows easily.

The proofs of parts (ii), (iii) and (iv) follow from Part (i) using Lemma 4.5 and hence are omitted. □

Lemma 4.9. *Let f be Multiple Itô-McShane integrable on T^m with $F(I)$ denoting the integral on the interval I. Let a positive function δ on T^m be given and $D = \{(I, \xi)\}$ be a standard δ-fine partial division of T^m (recall Definition 3.4). Then*

(i) $E\left(\left|\sum f1_{G_\pi}(\xi)(W(I))\right|^2\right) = \sum f^2 1_{G\pi}(\xi)|I|$

(ii) $E\left(\left|\sum f1_{G_\pi}(\xi)W(I) - F(I)\right|^2\right) = E\left(\sum |f1_{G_\pi}(\xi)W(I) - F(I)|^2\right)$.

PROOF. First, we remark that the condition that the components of I and J are either disjoint or equal (see the statement of Lemma 4.8) is guaranteed by using *standard* δ-fine partial division (see Definition 3.4) since all the components have the same division points.

To prove (i), observe that

$$\begin{aligned} E\Big(\Big|\sum f(\xi)W(I)\Big|^2\Big) &= \sum E\left(f^2(\xi)W(I)^2\right) \text{ by Lemma 4.8(i)} \\ &= \sum f^2(\xi)|I|. \end{aligned}$$

To prove (ii), by Lemma 4.8 (iv) we have

$$E\left(\left|\sum (f(\xi)W(I) - F(I)\right|^2\right) = E\left(\sum |(f(\xi)W(I) - F(I)|^2\right). \qquad \square$$

In other words, Lemma 4.9 says that we have orthogonal increment property if all the intervals from T^m that we consider are from the same set G contiguous to the diagonal. The next theorem we are going to prove is Itô's isometric inequality. Before we proceed further, we quote one result of classical Henstock integration theory for Euclidean space about the equivalence of McShane and Lebesgue Integrals on T^m. We remark that by Lemma 3.3, we may replace a δ-fine division by a *standard* δ-fine division.

Proposition 4.10. *A function $f : T^m \to \mathbb{R}$ is Lebesgue integrable to $A \in \mathbb{R}$ if and only if for every $\varepsilon > 0$, there exists a positive function δ on T^m such that for every standard δ-fine division of T^m, denoted by $D = \{(I, x)\}$, we have $|(D)\sum f(x)|I| - A| < \varepsilon$.*

Lemma 4.11. *Let f be a function defined on T^m. Then $f1_{G_\pi} \in L^2(T^m, \lambda^m)$ if and only if there exists a real number B, a decreasing sequence $\{\delta_n\}$ of positive functions on T^m such that for any sequence of δ_n-fine division D_n of T^m, we have $\lim_{n\to\infty} \left|\hat{S}(f, \delta_n D_n) - B\right| = 0$.*

The proof of Lemma 4.11 is similar to that of Lemma 4.5; hence we shall not repeat the proof here.

Proposition 4.12. *Let $f : T^m \to \mathbb{R}$. Suppose that f is Multiple Wiener-Itô integrable. Then f^2 is Lebesgue integrable there and*

$$E\left(IM(f^2\right) \le m! \int_{T^m} f^2 \, d\lambda^m \tag{2}$$

where the integral on the right hand side is the Lebesgue integral on T^m.

PROOF. We just need to prove that $E\left(IM(f1_{G_\pi})^2\right) = (L)\int_{T^m} f^2 1_{G_\pi}(t)\,dt$ for any contiguous set G_π of T^m. Thus

$$\begin{aligned} E(IM(f1_{G_\pi}))^2 &= \lim_{n\to\infty} E\left(|S(f1_{G_\pi}, D_n, \delta_n)|^2\right) = \lim_{n\to\infty} E\left(\sum f(x)1_{G_\pi} W(I)\right)^2 \\ &= \lim_{n\to\infty} \sum f^2(x) 1_{G_\pi} |I| \text{ by Lemma 4.9.} \end{aligned}$$

The Riemann sum $\sum f^2 1_{G_\pi}|I|$ converges to $\int_{T^m} f^2 1_{G_\pi}\,d\lambda^m$ by Lemma 4.11. Hence $\int_{T^m} f^2 1_{G_\pi}\,d\lambda^m = E\left(IM(f1_{G_\pi})^2\right)$. $\square$

Lemma 4.13. *Let A be a set of λ^m-measure zero in T^m. Suppose that $A \cap B = \phi$, where B is the diagonal set in T^m. Then 1_A is both McShane and Itô-McShane integrable on T^m and $IM(1_A) = 0$.*

PROOF. We remark that the condition $A \cap B$ is empty is necessary. It is sufficient to consider the case where A lies entirely in one of the sets G contiguous to the diagonal. Let $\varepsilon > 0$. Then there exists an open set $O \subset G$ such that $A \subset O$ and $O \cap B = \phi$ with $\lambda^m(O) < \varepsilon$. Now we shall define $\delta(\xi) > 0$ on T^m. If $\xi \in A$, define $\delta(\xi) > 0$ such that $I \subset O$ whenever (I, ξ) is δ-fine. It is possible since O is open. If $\xi \notin A$, then $\delta(\xi)$ can take any positive value. Let $D = \{(I^{(i)}, \xi^{(i)}) : i = 1, 2, \ldots, n\}$ be any standard δ-fine division of T^m. By Lemma 4.8,

$$E\Big(\Big|(D)\sum_{i=1}^{n} 1_A(\xi^{(i)}) \prod_{k=1}^{m} W(I_k^{(i)})\Big|^2\Big) = \sum_{\xi^{(i)}\in A} \prod_{k=1}^{m} \lambda(I_k^{(i)}) < \lambda^m(O) < \varepsilon.$$

Hence $IM(1_A) = 0$. $\square$

5 Equivalence Theorem

The objective of this section is to establish that the Multiple Itô-McShane integral defined in this paper is in fact equivalent to the classical Multiple Itô-Wiener integral. The idea of the proof is a generalization of that from [1].

Lemma 5.1. *Let g be an elementary function on T^m in the form (1). Then g is Multiple Itô-McShane integrable on T^m and*

$$IM(g) = \sum_{i_1,\ldots,i_m=1}^{n} a_{i_1,i_2,\ldots,i_m} \prod_{k=1}^{m} W(I_{i_k}).$$

PROOF. It is sufficient to prove the special case when $g = 1_{I_1 \times I_2 \times \cdots \times I_m}$, where $I_i, i = 1,2,\ldots,m$, are disjoint left-open subintervals of $[0,1]$. Let $I = I_1 \times I_2 \times \cdots \times I_m$ and ∂I be the boundary of I. It is clear by definition that g does not vanish only on one of the set G contiguous to the diagonal. Denote $\prod_{k=1}^{m} W(I_k)$ by $W(I)$. By Lemma 4.9, we need only consider $g = 1_{I \setminus \partial I}$ since ∂I is a set of measure zero. Let $\varepsilon > 0$. There exists an open set $O \supset \partial I$ and $G \supset O$ such that $\lambda^m(O) < \varepsilon$. Now we shall define $\delta(\xi) > 0$ on T^m. If $\xi \in I \setminus \partial I$, define the positive function δ to be such that $J \subset I \setminus \partial I$ whenever (J, ξ) is δ-fine. If $\xi \in \partial I$, define $\delta(\xi) > 0$ such that $J \subset O$ whenever (J, ξ) is δ-fine. If $\xi \notin I$, then $\delta(\xi)$ can take any positive value. Let $D = \{(J^{(i)}, \xi^{(i)}) : i = 1,2,\ldots,n\}$ be a δ-fine division of T^m, where $J^{(i)} = J_1^{(i)} \times J_2^{(i)} \times \cdots \times J_m^{(i)}$. Then

$$\begin{aligned}
&E\left(\left|(D)\sum g(\xi^{(i)})W(J^{(i)}) - W(I)\right|^2\right) \\
=&E\Big(\big|(D)\sum_{\xi^{(i)} \in I \setminus \partial I} g(\xi^{(i)})W(J^{(i)}) - W(I)\big|^2\Big) \\
=&E\Big(\big|(D)\sum_{\xi^{(i)} \in I \setminus \partial I} W(J^{(i)}) - W(I)\big|^2\Big) \\
\le&2E\Big(\big|(D)\sum_{\xi^{(i)} \in I} W(J^{(i)}) - W(I)\big|^2\Big) + 2E\Big(\big|(D)\sum_{\xi^{(i)} \in \partial I} W(J^{(i)})\big|^2\Big) \\
\le&2E\Big(\big|(D)\sum_{\xi^{(i)} \in I} W(J^{(i)}) - W(I)\big|^2\Big) + 2\varepsilon \text{ (since } \lambda^m(O) < \varepsilon \text{)} \\
\le&4E\Big(\big|(D)\sum_{\xi^{(i)} \in I} \Big[W(J^{(i)}) - W(I \cap J^{(i)})\Big]\Big|^2\Big) \\
&+ 4E\Big(\big|(D)\sum_{\xi^{(i)} \in I} W(I \setminus J^{(i)})\big|^2\Big) + 2\varepsilon \\
\le&4\varepsilon + 0 + 2\varepsilon, \text{ (since } \lambda^m(O) < \varepsilon) = 6\varepsilon.
\end{aligned}$$

Hence $IM(f) = \prod_{k=1}^{m} W(I_k)$. From the classical definition, $IW(f) = \prod_{k=1}^{m} W(I_k)$. Therefore f is Multiple Itô-McShane integrable with $IM(f) = IW(f)$. □

Lemma 5.2. *Let f be Multiple Itô-McShane integrable on T^m and $IM(f1_I) = A(I)$ for every subinterval I of T^m. Then for every $\varepsilon > 0$, there exists a positive function δ on T^m such that for any partial standard δ-fine division $D = \{(I^{(i)}, \xi^{(i)}) : i = 1, 2, \ldots, n\}$ of T^m we have*

$$E\Big(\big|(D)\sum_{i=1}^{n}\{f_o(\xi^{(i)})W(I^{(i)}) - A(I^{(i)})\}\big|^2\Big) < \varepsilon.$$

The proof of Lemma 5.2 is standard in the theory of Henstock integration, see for example [6, p. 11, Theorem 3.7] or [7, pp. 81–82, Theorem 3.2.1].

Definition 5.3. Let $A, A^{(n)}$, $n = 1, 2, 3, \ldots,$ be real-valued functions defined on the set of all left-open intervals of T^m. Then $A^{(n)}$ is said to be *variationally convergent* to A if for every $\varepsilon > 0$ there exists a positive integer N such that for any finite collection of disjoint left-open intervals $\{I^{(i)} : i = 1, 2, 3, \ldots, q\}$ we have $E\Big(\big|\sum_{i=1}^{q}\{A^{(n)}(I^{(i)}) - A(I^{(i)})\}\big|^2\Big) < \varepsilon$ whenever $n \geq N$.

Theorem 5.4. *Let $f, f^{(n)}$, $n = 1, 2, \ldots$ be real-valued functions defined on T^m such that $f^{(n)}$ converges to f a.s. on T^m. Suppose that each $f^{(n)}$ is Multiple Itô-McShane integrable to $A^{(n)}$. Let $A^{(n)}(I)$ be the Multiple Itô-McShane integral of $f^{(n)}$ on subinterval I. If $A^{(n)}$ variationally converges to A, then f is Multiple Itô-McShane integrable to $A(T^m)$ on T^m.*

PROOF. We may assume that f and $f^{(n)}$ vanishes on T^m except over a particular set G contiguous to the diagonal. Given $\varepsilon > 0$, by the variational convergence property, we may assume that for each $n = 1, 2, 3, 4, \ldots,$

$$E\Big(\big|(D)\sum\{A^{(n)}(I) - A(I)\}\big|^2\Big) < \frac{\varepsilon}{2^{2n}}$$

for any finite collection of disjoint intervals $\{I\}$ of T^m. By Lemma 5.2, for each positive integer n, there exists a positive function δ^n on T^m such that for any δ^n-fine partial McShane division of T^m, denoted by $D_n = \{(I, \xi)\}$ we have $E\Big(\big|(D_n)\sum\{f_0^{(n)}(\xi)W(I) - IM(f^{(n)})(I)\}\big|^2\Big) < \dfrac{\varepsilon}{2^{2n}}$. By Lemma 4.9, we may assume that $f^{(n)} \to f$ everywhere on T^m. So, for each $\xi \in T^m$, there exists a positive integer $n(\xi)$ such that $|f_0^{n(\xi)}(\xi) - f(\xi)| < \sqrt{\varepsilon}$. Take $\delta(\xi) = \delta^{n(\xi)}(\xi)$ and let $D = \{(I, \xi)\}$ be a δ-fine McShane full division of T^m. Then

$$\begin{aligned}
&E\Big(\big|(D)\sum f_0(\xi)W(I) - A\big|^2\Big) \leq 3E\Big(\big|(D)\sum[f_0(\xi) - f_0^{n(\xi)}(\xi)]W(I)\big|^2\Big)\\
&+ 3E\Big(\big|(D)\sum f_0^{n(\xi)}(\xi)W(I) - A^{n(\xi)}(I)\big|^2\Big)\\
&+ 3E\Big(\big|(D)\sum\Big\{A^{n(\xi)}(I) - A(I)\Big\}\big|^2\Big) = 3(I_1 + I_2 + I_3).
\end{aligned}$$

$$\begin{aligned}I_1 =&E\Big(\big|(D)\sum[f_0^{n(\xi)}(\xi)-f_0(\xi)]W(I)\big|^2\Big)\\ \le&\varepsilon E\Big(\big|(D)\sum|W^2(I)|\big|^2\Big)\le 3\varepsilon\sum|I|^2\le 3\varepsilon.\end{aligned}$$

$$\begin{aligned}I_2 =&||(D)\sum\left\{f_0^{n(\xi)}(\xi)W(I)-A^{n(\xi)}(I)\right\}||^2\\ \le&\Big(||(D)\sum\left\{f_0^k(\xi)W(I)-A^k(I)\right\}||\Big)^2\le\Big(\sum_{k=1}^{\infty}\frac{\varepsilon}{2^k}\Big)^2\le\varepsilon.\end{aligned}$$

$$\begin{aligned}I_3 =&E\Big(\big|(D)\sum\left\{A^{n(\xi)}(I)-A(I)\right\}\big|^2\Big)=||\big|(D)\sum\left\{A^{n(\xi)}(I)-A(I)\right\}\big|^2||^2\\ \le&\Big(\sum_{k=1}^{\infty}||(D_k)\sum\left\{A^k(I)-A(I)\right\}||\Big)^2\le\Big(\sum_{k=1}^{\infty}\frac{\varepsilon}{2^k}\Big)^2\le\varepsilon\end{aligned}$$

□

We are now ready to prove the equivalence Theorem.

Theorem 5.5. *(Equivalence Theorem). Let $f\in L^2(T^m,\lambda^m)$. Then f is Multiple Itô-McShane integrable on T^m and $IM(f)=IW(f)$.*

PROOF. By Theorem 5.1, the result of Theorem 5.5 holds for any elementary function of the form (1). Let $f\in L^2(T^m,\lambda^m)$. By the definition of the Multiple Itô-Wiener integral, there exists a sequence of elementary functions $\{f^{(n)}\}$ such that $f^{(n)}$ converges to f a.s. on T^m and $\lim_{n\to\infty}||f^{(n)}-f||_m=0$. Furthermore $E\left(IW(f^{(n)}-f)\right)^2\le m!||f^{(n)}-f||_m$ for all n, and $\lim_{n\to\infty}E\left(IW(f^{(n)})-IW(f)\right)^2=0$. Let $A^{(n)}(I)$ and $A(I)$ be the Itô-Weiner integrals of $f^{(n)}$ and f on a subinterval I respectively. Then

$$E\left(IW(f^{(n)}1_E-f1_E)\right)^2\le m!||(f^{(n)}-f)1_E||_m^2\le m!||f^{(n)}-f||_m^2$$

where E is the union of finite disjoint left-open subintervals. Hence $A^{(n)}$ variationally converges to A. Note that $A^{(n)}(I)$ is also the Multiple Itô-McShane integral of $f^{(n)}$ on I by Theorem 5.3. By Theorem 5.4, f is Multiple Itô-McShane integrable on T^m and $IM(f)=IW(f)$. □

6 Characterization of Integrable Functions

We have shown that if f is L^2-integrable on T^m, then f is Multiple Itô-McShane integrable on T^m. We shall next characterize the class of all Multiple Itô-McShane integrable functions.

It can be seen from classical integration theory that $f \in L^2(T^m, \lambda^m)$ if and only if given any $\varepsilon > 0$, there exists a positive function δ such that for any δ-fine belated division of T^m, denoted by $D = \{(I, x)\}$, we have

$$\left|(D) \sum f^2(x)\lambda^m(I) - \int_{T^m} f^2 \, d\lambda^m\right| < \varepsilon.$$

Also since the diagonal is a set of Lebesgue-measure zero, f is integrable if and only if f_0 is integrable there.

Given a positive function δ, we shall let $\hat{S}(f, \delta, D) = (D) \sum f^2(x)\lambda^m(I)$ where D is a δ-fine division of T^m. Let $f1_{G_\pi}$ be a function on T^m. Then, by Lemma 4.9(i), we have $E\left(|S(f1_{G_\pi}, \delta, D)|^2\right) = \hat{S}(f1_{G_\pi}, \delta, D)$.

Theorem 6.1. *Let f be a function on T^m. Then $f1_{G_\pi}$ is Multiple Itô-McShane integrable on T^m if and only if $f1_{G_\pi} \in L^2(T^m, \lambda^m)$. Furthermore,*

$$E(IM(f1_{G_\pi})^2) = (L) \int_{T^m} f^2 1_{G_\pi} \, d\lambda^m.$$

PROOF. By Proposition 4.10, if $f1_{G_\pi}$ is Multiple Itô-McShane integrable, then $f1_{G_\pi}$ is square-Lebesgue integrable there. We just need to prove the converse. Suppose $f1_{G_\pi}$ is square-Lebesgue integrable. By Theorem 5.5, $f1_{G_\pi}$ is Multiple Itô-McShane integrable. Then by Proposition 4.10,

$$\begin{aligned}(L) \int_{T^m} f^2 1_{G_\pi} \, d\lambda^m &= \lim_{n\to\infty} \hat{S}(f, \delta_n, D_n) = \lim_{n\to\infty} E\left((D_n) \sum f(x) 1_{G_\pi} W(I)\right)^2 \\ &= \lim_{n\to\infty} E\left(|S(f1_{G_\pi}, D_n, \delta_n)|^2\right) = IM(f1_{G_\pi})^2. \qquad \square\end{aligned}$$

7 Conclusion

We have used the Non-Uniform Riemann approach to give an equivalent definition to the classical Multiple Wiener integral. We remark that the Non-Uniform Riemann approach can also be used to give an alternative definition to the Stratonovich integral and Fubini's Theorem and the classical Hu-Meyer Theorem can be derived. This will appear as a paper elsewhere.

References

[1] T. S. Chew, J. Y. Tay, T. L. Toh, *The non-uniform Riemann approach to Itô's integral*, Real Analysis Exchange, **27** (2) (2001/2002), 495–514.

[2] R. Henstock, *Lectures on the theory of integration*, World Scientific, Singapore, 1988.

[3] R. Henstock, *The general theory on integration*, Oxford University Press, 1991.

[4] R. Henstock, *Stochastic and other functional integrals*, Real Analysis Exchange, **16** (1990/1991), 460–470.

[5] K. Itô, *Multiple Wiener integral*, Journal of Mathematical Society of Japan, (1951), 157–169.

[6] P. Y. Lee, *Lanzhou lectures on Henstock integration*, World Scientific, Singapore, 1989.

[7] P. Y. Lee, R. Vyborny, *The integral, an easy approach after Kurzweil and Henstock*, Cambridge University Press, Melbourne, 2000.

[8] N. V. Norin, *The extended stochastic integral in Linear spaces with differential measures and related topics*, World Scientific, 1996.

[9] J. LL. Sole, F. Utzet, *Stratronovich integral and Trace*, Stochastics & Stochastics Reports, **29** (1990), 203–220.

[10] T. L. Toh, T. S. Chew, *A variational approach to Itô's integral*, Proceedings of SAP's **98**, Taiwan, 291–299, World Scientific, Singapore, 1999.

[11] N. Wiener, *The homogeneous chaos*, Amer. Journ. Math., **60** (1938), 897–936.

[12] J. G. Xu, P. Y. Lee, *Stochastic integrals of Itô and Henstock*, Real Analysis Exchange, **18** (1992/1993), 352–366.