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ON PATH CONTINUITY

Abstract

Topological properties of family path continuous functions are investigated. We consider weak path continuity, path limit set and properties of σ-systems in metric spaces.

1 Notation

We apply standard symbols and notation. By $\mathbb{R}$ we denote the set of real numbers, by $\mathbb{Q}$ ($\mathbb{N}$) we denote the set of rational numbers (positive integers). For a metric space Z, $z \in Z$ and $R > 0$ let $K_Z(z, R)$ (or simply $K(z, R)$) be the open ball with center z and radius R.

If there is no $y \in Z$ and $r > 0$ such that $K(y, r) \subset K(z, R) \setminus A$, then let $\gamma(z, R, A) = 0$. Otherwise let $\gamma(z, R, A) = \sup\{r > 0 : \exists_{y \in X} K(y, r) \subset K(z, R) \setminus A\}$. If $\limsup\limits_{R \to 0^+} \frac{\gamma(z,R,A)}{R} > 0$, then we say that A is *porous* at z. We say that A is *superporous* at z, if for every set $B \subset X$ porous at z the set $A \cup B$ is porous at z.

Throughout the sequel let X, Y be metric spaces. We denote by d_X, d_Y, τ and ν the metric on X, on Y, family of open subsets of X, of Y respectively. If $x \in X$, $y \in Y$, then by $\tau(x)$ and $\nu(y)$ denote the family of open neighborhoods of x and y respectively. For $A \subset X$ let $\mathrm{cl}(A), \mathrm{int}(A), I(A)$ denote the closure of A, interior of A, the set of isolated points of A respectively. A set $A \subset X$ is *boundary* if $\mathrm{int}(A) = \emptyset$. For $A, B \subset X$ we denote $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

An *ideal* on X is a collection of subsets of X containing $\emptyset$ and closed under subsets and finite unions.

If $\mathcal{A}$ is the family of functions from X to Y, then $b\mathcal{A}$ denotes the space of bounded elements of $\mathcal{A}$ with the metric of uniform convergence d_s. By $\mathcal{C}(X, Y)$ we denote the family of all continuous functions from X to Y. If $y \in Y$, then

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by $const_y$ we denote the constant function with value y. If $f : X \to Y$ is a function, then by C_f we denote the set of all continuity points of the function f. No distinction is made between a function and its graph.

Let $x \in X$. If $x \notin I(X)$, then a *path leading to* x is a set E such that $x \in E$ and x is a point of accumulation of E [1, 3]. If $x \in I(X)$, then a *path leading to* x is a set E such that $x \in E$. *A system of paths at* $x \in X$ is a nonempty family $\mathcal{F}$ of subsets of X such that for each $E \in \mathcal{F}$, E is a path leading to x. *A system of paths on* X is an arbitrary mapping on X such that $\mathcal{E}(x)$ is a system of paths at x, for every $x \in X$.

Let $\mathcal{E}$ be a system of paths on X and let $f : X \to Y$. We shall say that f is $\mathcal{E}$*-continuous* at x if there exists a path $E \in \mathcal{E}(x)$ such that $f \restriction E$ is a continuous function at x. By $\mathcal{C}_{\mathcal{E}}(X,Y)$ we denote the family of all $\mathcal{E}$-continuous functions from X to Y. The set of all $\mathcal{E}$-continuity points of f we shall denote by $C_f^{\mathcal{E}}$.

Let $\mathcal{E}$ be a system of paths on X. For $x \in X$ let

$$\mathcal{F}_{\mathcal{E}}(x) = \{A \subset X :\ \exists_{U \in \tau(x)} \exists_{E \in \mathcal{E}(x)} E \cap U \subset A\}.$$

Then $\mathcal{F}_{\mathcal{E}}$ is a system of paths too. Notice that $\mathcal{C}_{\mathcal{E}}(X,Y) = \mathcal{C}_{\mathcal{F}_{\mathcal{E}}}(X,Y)$.

2 Topological Properties of Family Path Continuous Functions.

Theorem 1. *Let X be a connected metric space,* $\mathrm{card}(X) \geq 2$*. Then there exists a system of path $\mathcal{E}$ such that family $b\mathcal{C}(X,\mathbb{R})$ is superporous at every point of space $b\mathcal{C}_{\mathcal{E}}(X,\mathbb{R})$.*

Proof. Let $x_0 \in X$. For $\varsigma > 0$ let $S_\varsigma = \{x \in X : d_X(x,x_0) = \varsigma\}$. Observe, that because X is connected and $\mathrm{card}(X) \geq 2$ there exists $\varsigma_0 > 0$ such that $S_\varsigma \neq \emptyset$, for $0 < \varsigma \leq \varsigma_0$. Let $(\delta_n)_{n=1}^{\infty}$ be such that $\lim\limits_{n\to\infty} \delta_n = 0$, $0 < \delta_{n+1} < \delta_n < \varsigma_0$, for $n = 1, 2 \dots$. Let $E = \{x_0\} \cup \bigcup\limits_{n=1}^{\infty} S_{\delta_n}$ and let $\mathcal{E}(x_0) = \{E\}$, $\mathcal{E}(x) = \{X\}$ if $x \neq x_0$. Let $f \in b\mathcal{C}_{\mathcal{E}}(X,\mathbb{R})$. We show that $b\mathcal{C}(X,\mathbb{R})$ is superporous at f. Let $B \subset b\mathcal{C}_{\mathcal{E}}(X,\mathbb{R})$ be porous at f. We shall prove that $b\mathcal{C}(X,\mathbb{R}) \cup B$ is porous at f. Because B is be porous at f

$$\limsup_{H\to 0^+} \frac{\gamma(f,H,B)}{H} > 0. \tag{1}$$

Let $R > 0$. We shall show that

$$\gamma(f,R,b\mathcal{C}(X,\mathbb{R}) \cup B) \geq \frac{\gamma(f,R,B)}{16}. \tag{2}$$

According to (1), there exists $g \in b\mathcal{C}_{\mathcal{E}}(X,\mathbb{R})$ and $r_1 > \frac{\gamma(f,R,B)}{2} > 0$ such, that $K(g,r_1) \subset K(f,R)$ and $K(g,r_1) \cap B = \emptyset$. Let $r = \frac{r_1}{8}$. Of course $r > \frac{\gamma(f,R,B)}{16}$. We shall show that there exists $h \in b\mathcal{C}_{\mathcal{E}}(X,\mathbb{R})$ such that

$$K(h,r) \subset K(f,R) \textit{ and } K(h,r) \cap (b\mathcal{C}(X,\mathbb{R}) \cup B) = \emptyset. \tag{3}$$

If $K(g,r) \cap b\mathcal{C}(X,\mathbb{R}) = \emptyset$, then we take $h = g$. Otherwise let $\xi \in K(g,r)$ be a continuous function. There exists $\delta > 0$ such that $\xi(K_X(x_0,\delta)) \subset (\xi(x_0) - r, \xi(x_0) + r)$. Let $(\sigma_n)_{n=1}^{\infty}$ be such that $\delta_{n+1} < \sigma_n < \delta_n$ (for $n = 1, 2, \dots$) and n_0 be such that $\delta_{n_0} < \delta$. Define $\xi_1 : \bigcup_{n=n_0}^{\infty} (S_{\sigma_n} \cup S_{\delta_n}) \to [\xi(x_0) - 3r, \xi(x_0) + 3r]$ by

$$\xi_1(x) = \begin{cases} \xi(x) & \text{if } x \in \bigcup_{n=n_0}^{\infty} S_{\delta_n}, \\ \xi(x_0) + 3r & \text{if } x \in \bigcup_{n=n_0}^{\infty} S_{\sigma_n}. \end{cases}$$

Observe that ξ_1 is continuous. The set $\bigcup_{n=n_0}^{\infty} (S_{\delta_n} \cup S_{\sigma_n})$ is closed in $X \setminus \{x_0\}$. Thus there exists $\xi_2 : X \setminus \{x_0\} \to [\xi(x_0) - 3r, \xi(x_0) + 3r]$, continuous and such that $\xi_2(x) = \xi_1(x)$ for $x \in \bigcup_{n=n_0}^{\infty} (S_{\delta_n} \cup S_{\sigma_n})$ Let $h : X \to \mathbb{R}$ be given by

$$h(x) = \begin{cases} \xi_2(x) & \text{for } x \in \operatorname{cl}(K_X(x_0,\delta_{n_0})) \setminus \{x_0\} \\ \xi(x) & \text{otherwise.} \end{cases}$$

Observe that $h \restriction (X \setminus \{x_0\})$ is continuous. Indeed, let $H = X \setminus K_X(x_0,\delta_{n_0})$, $F = \operatorname{cl}(K_X(x_0,\delta_{n_0})) \setminus \{x_0\}$. Then F, H are closed in $X \setminus \{x_0\}$, $F \cap H \subset S_{\delta_{n_0}}$ and $\xi \restriction S_{\delta_{n_0}} = \xi_2 \restriction S_{\delta_{n_0}}$. Thus $h \restriction (X \setminus \{x_0\}) = (\xi_2 \restriction F) \cup (\xi \restriction H)$ is continuous. Clearly h is $\mathcal{E}$-continuous at x_0. Hence $h \restriction (E \cap K_X(x_0,\delta_{n_0})) = \xi \restriction (E \cap K_X(x_0,\delta_{n_0}))$ and ξ is continuous. Thus $h \in b\mathcal{C}_{\mathcal{E}}(X,\mathbb{R})$.

We have $|h(x) - \xi(x)| < 4r$, for $x \in X$. Indeed, if $x \notin \operatorname{cl}(K_X(x_0,\delta_{n_0})) \setminus \{x_0\}$, then clearly $h(x) = \xi(x)$. If $x \in \operatorname{cl}(K_X(x_0,\delta_{n_0})) \setminus \{x_0\}$, then $h(x) = \xi_2(x)$ and $|\xi_2(x) - \xi(x)| \le |\xi_2(x) - \xi(x_0)| + |\xi(x_0) - \xi(x)| < 4r$. Let $\zeta \in K(h,r)$ and $x \in X$. Then $|\zeta(x) - g(x)| \le |\zeta(x) - h(x)| + |h(x) - \xi(x)| + |\xi(x) - g(x)| < 6r$. Hence $K(h,r) \subset K(g, \frac{3r_1}{4})$.

To finish the proof observe that $K(h,r) \cap (b\mathcal{C}(X,\mathbb{R}) \cup B) = \emptyset$. Indeed, $K(h,r) \cap B \subset K(g,r_1) \cap B = \emptyset$. Thus it suffices to prove that $K(h,r) \cap b\mathcal{C}(X,\mathbb{R}) = \emptyset$. Let $\psi \in K(h,r)$. Then $\psi(x_0) < h(x_0) + r = \xi(x_0) + r$, and $\psi(x) > h(x) - r = \xi(x_0) + 2r$ for $x \in \bigcup_{n=n_0}^{\infty} S_{\sigma_n}$. The point x_0 is a point of accumulation of $\bigcup_{n=n_0}^{\infty} S_{\sigma_n}$. Thus ψ is not continuous at x_0. We obtain that for $R > 0$, $\gamma(f, R, b\mathcal{C}(X, \mathbb{R}) \cup B) \geq \frac{\gamma(f,R,B)}{16}$. Consequently $\limsup_{R\to 0^+} \frac{\gamma(f,R,b\mathcal{C}(X,\mathbb{R})\cup B)}{R} > 0$. Thus $b\mathcal{C}(X, \mathbb{R})$ is superporous at every point of space $b\mathcal{C}_{\mathcal{E}}(X, \mathbb{R})$. □

Theorem 2. *For every system of paths $\mathcal{E}$ the space $b\mathcal{C}_{\mathcal{E}}(X, \mathbb{R})$ is connected.*

Proof. We shall prove that for every function $g \in b\mathcal{C}_{\mathcal{E}}(X, \mathbb{R})$ there exists connected set $H_g \subset b\mathcal{C}_{\mathcal{E}}(X, \mathbb{R})$ such that $const_0 \in H_g$ and $g \in H_g$. Fix $g \in b\mathcal{C}_{\mathcal{E}}(X, \mathbb{R})$. Define a function ϕ_g such that $\phi_g(t) = t \cdot g$ for $t \in [0, 1]$. Let $H_g = \phi_g([0, 1])$. Observe that ϕ_g is a continuous function. Hence H_g is connected subset of $b\mathcal{C}_{\mathcal{E}}(X, \mathbb{R})$. We have in addition $const_0 = \phi(0) \in H_g$ and $g = \phi(1) \in H_g$. □

3 The Set of Continuity Points of Path Continuous Functions.

Let $\mathcal{E}$ be a system of paths in metric space X. How big is the set of continuity points of a $\mathcal{E}$-continuous function? Consider the following system of paths. For $x, y \in \mathbb{R}$ let $x \sim y$ if $x - y \in \mathbb{Q}$, and for $x \in \mathbb{R}$, let $\mathcal{E}(x) = \{[x]_\sim\}$ for $x \in \mathbb{R}$ (where $[x]_\sim$ denote equivalence class of x). We have $\{[x]_\sim : x \in \mathbb{R}\} = \{A_\alpha\}_{\alpha<\mathfrak{c}}$ where $A_\beta \neq A_\delta$ for $\beta \neq \delta$. Let $(x_\alpha)_{\alpha<\mathfrak{c}}$ be a transfinite sequence of all reals and $f : \mathbb{R} \to \mathbb{R}$ be such that $f \restriction A_\alpha = const_\alpha$. Of course f is $\mathcal{E}$-continuous and Darboux, but $C_f = \emptyset$.

The next theorems show that for a path system with path $\mathcal{N}$-residual or $\mathcal{N}$-second category with $\mathcal{N}$-Baire property (where $\mathcal{N}$ is some ideal of boundary subsets of X) the set of continuity points of a path continuous functions must be somehow big.

Let $\mathcal{N} \subset 2^X$ be an ideal of boundary subsets of X and $A, B \subset X$. We say that A is *$\mathcal{N}$-residual in B* if $B \setminus A \in \mathcal{N}$.

Theorem 3. *Let $\mathcal{N}$ be an ideal of the boundary subsets of X and let $\mathcal{E}$ be a system of paths, such that for $x \in X$, $E \in \mathcal{E}(x)$, the set E is $\mathcal{N}$-residual in some neighborhood of x. If $f : X \to Y$ is $\mathcal{E}$-continuous, then $C_f = X$.*

Proof. Let $x \in X$. Fix $x \in X$ and $\epsilon > 0$. Let $E_x \in \mathcal{E}(x)$ be such that $f(U_x \cap E_x) \subset K(f(x), \frac{\epsilon}{2})$ and $U_x \setminus E_x \in \mathcal{N}$ where $U_x \in \tau(x)$. We shall show that $f(U_x) \subset K(f(x), \epsilon)$. Let $t \in U_x$. Then there exists $E_t \in \mathcal{E}(t)$, $U_t \in \tau(t)$ such that $U_t \subset U_x$, $f(U_t \cap E_t) \subset K(f(t), \frac{\epsilon}{2})$ and $U_t \setminus E_t \in \mathcal{N}$. The set U_t is nonempty and open. Thus $U_t \notin \mathcal{N}$. Observe that $U_t = (U_t \cap (E_x \cap E_t)) \cup (U_t \setminus (E_x \cap E_t))$. Hence $U_t \cap E_x \cap E_t \neq \emptyset$. Let $u \in U_t \cap E_x \cap E_t$. Then $f(u) \in K(f(x), \frac{\epsilon}{2}) \cap K(f(t), \frac{\epsilon}{2})$. Hence $f(t) \in K(f(x), \epsilon)$. □

We say that A is $\mathcal{N}$*-residual at point* $x \in X$ if for every neighborhood U of x there exists a nonempty open set $W \subset U$ such that $W \setminus A \in \mathcal{N}$.

Theorem 4. *Suppose X and Y are two complete metric spaces, $\mathcal{N}$ is an ideal of boundary subsets of X and $\mathcal{E}$ is a system of paths. If for every $x \in X$, $E \in \mathcal{E}(x)$, the set E is $\mathcal{N}$-residual at x and $f : X \to Y$ is $\mathcal{E}$-continuous, then C_f is residual in X.*

PROOF. Let $\mathcal{E}$ be such a system of paths. We shall show that every $\mathcal{E}$-continuous function is quasi continuous. Let $x \in X$, $U \in \tau(x)$ and $V \in \nu(f(x))$. There exists $\epsilon > 0$ such that $K(f(x), \epsilon) \subset V$, $U_x \in \tau(x)$, $U_x \subset U$ and $E \in \mathcal{E}(x)$ such that $f(U_x \cap E_x) \subset K(f(x), \frac{\epsilon}{2})$. Let $W_x \subset U_x$ be open and nonempty such that $W_x \setminus E_x \in \mathcal{N}$. Then $W_x \subset U$. We shall show that $f(W_x) \subset V$. For $t \in W_x$ there exists $U_t \in \tau(t)$ and $E_t \in \mathcal{E}(t)$ such that $U_t \subset W_x$, $f(U_t \cap E_t) \subset K(f(t), \frac{\epsilon}{2})$. Let W_t be a nonempty open subset of U_t, such that $W_t \setminus E_t \in \mathcal{N}$. Because $W_t \notin \mathcal{N}$ and $W_t \setminus (E_t \cap E_x) \subset (W_t \setminus E_t) \cup (W_x \setminus E_x) \in \mathcal{N}$. Hence there exists $u \in W_t \cap E_t \cap E_x$. We have $d_Y(f(t), f(x)) \leq d_Y(f(t), f(u)) + d_Y(f(u), f(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence $f(t) \in K(f(x), \epsilon) \subset V$. Because X, Y are complete and f is quasi continuous we obtain that C_f is a dense G_δ set; hence residual in X. $\square$

The assumption that X, Y are complete is essential. In fact let $X = \mathbb{Q}$, with natural metric and $\mathcal{N} = \{\emptyset\}$. For $x \in \mathbb{Q}$ let $\mathcal{E}(x) = \{E \subset \mathbb{Q} : x \in E \ \&\ \text{E is } \mathcal{N}\text{-residual at } x\}$ and let $f : \mathbb{Q} \to \mathbb{R}$ be a function that for every $x \in \mathbb{Q}$, f is continuous from right and not continuous from left at x. For $q \in \mathbb{Q}$ the set $[q, \infty) \cap \mathbb{Q}$ is $\mathcal{N}$-residual at the point q. Because f is continuous from the right at every point $q \in \mathbb{Q}$, f is $\mathcal{E}$-continuous, however $C_f = \emptyset$.

We call the set A $\mathcal{N}$-second category at x, if for every neighborhood U of x, $A \cap U \notin \mathcal{N}$.

We will say, that $A \subset X$ has the $\mathcal{N}$-Baire property if there exist open set G and $N \in \mathcal{N}$ such, that $A = G \triangle N$. The family of all sets having the $\mathcal{N}$-Baire property, where $\mathcal{N}$ is an ideal is denoted by $\Omega_{\mathcal{N}}$.

Theorem 5. *Let X, Y be complete metric space with $I(X) = \emptyset$ and let $\mathcal{N}$ be the ideal of boundary subsets of X such that for $x \in X$, $\{x\} \in \mathcal{N}$. If $\mathcal{E}$ is a system of paths such that for every $x \in X$ and $E \in \mathcal{E}(x)$, E has the $\mathcal{N}$-Baire property and is $\mathcal{N}$-second category at x, then the set of continuity points of f is residual in X.*

PROOF. We shall show that if E has the $\mathcal{N}$-Baire property and is $\mathcal{N}$-second category at x, then it is $\mathcal{N}$-residual at x. Indeed, let $x \in X$ and $E \in \mathcal{E}(x)$. Then there exist sets $G \in \tau$ and $N \in \mathcal{N}$ such, that $E = G \triangle N$. Let $U \in \tau(x)$. If $G \cap U = \emptyset$, then $U \cap E = U \cap (N \setminus G) \subset N \in \mathcal{N}$. This proves, that

$W = G \cap U \neq \emptyset$. Note that $W \in \tau$, $W \subset U$ and $W \setminus E \subset G \setminus (G \triangle N) \subset N \in \mathcal{N}$. Thus E is $\mathcal{N}$-residual at x. $\square$

The assumption of the $\mathcal{N}$-Baire property is essential. In fact let $\mathcal{N}$ be a family of meager subsets of $\mathbb{R}$. Let B be a Bernstein set (i.e., such set that for every perfect set F, $B \cap F \neq \emptyset \neq F \setminus B$). Define a system of paths $\mathcal{E}$ by $\mathcal{E}(x) = \{B\}$ if $x \in B$ and $\mathcal{E}(x) = \{\mathbb{R} \setminus B\}$ if $x \notin B$. Then for every $x \in X$ and $E \in \mathcal{E}(x)$, E is $\mathcal{N}$-second category at x. Note that the characteristic function of B is an everywhere discontinuous function which is $\mathcal{E}$-continuous.

4 Weak path continuity and path limit set.

Let $\mathcal{E}$ be a system of paths on X and let $f : X \to Y$. We shall say that f *is weak $\mathcal{E}$-continuous at x* if for all $U \in \nu(f(x))$, we have $f^{-1}(U) \in \mathcal{F}_{\mathcal{E}}(x)$. If f is weak $\mathcal{E}$-continuous at x for all $x \in X$, we say that f is *weak $\mathcal{E}$-continuous.*

Let $\mathcal{E}$ be a system of paths on X. For $f : X \to Y$ and $x \in X$ consider the $\mathcal{E}$-limit set

$$ls_{\mathcal{E}}(f, x) = \{y \in Y : \forall_{V \in \nu(y)} \exists_{E \in \mathcal{E}(x)} \exists_{U \in \tau(x)} f(U \cap E \setminus \{x\}) \subset V\}.$$

Note that f is weak $\mathcal{E}$-continuous at x iff $f(x) \in ls_{\mathcal{E}}(f, x)$,

Let us notice that for every system of path $\mathcal{E}$ and $f : X \to Y$ the set $ls_{\mathcal{E}}(f, x)$ is closed. Moreover the following assertion is true.

Theorem 6. *Let X be uncountable, locally connected separable metric space and Y be connected space with* $\operatorname{card}(Y) \geq 2$. *Then $F \subset Y$ is closed in Y iff there exists a system of paths $\mathcal{E}$ and a function $f : X \to Y$ such that $ls_{\mathcal{E}}(f, x) = F$ for $x \in X \setminus I(X)$*

PROOF. First notice that if $0 < \kappa \leq \mathfrak{c}$, then there is a family $\{K_\alpha : \alpha < \kappa\}$ of disjoint sets dense in $X \setminus I(X)$ such that $X = \bigcup_{\alpha<\kappa} K_\alpha$. Let $0 < \kappa \leq \mathfrak{c}$, $\mathcal{B}$ be a countable base of X such that every $B \in \mathcal{B}$ is a connected set and $\mathcal{B}_0 = \{U \in \mathcal{B} : \operatorname{card}(U) > 1\}$. Hence X is uncountable. Then $\mathcal{B}_0 \neq \emptyset$. The family $\mathcal{B}_0$ is a base of $X \setminus I(X)$. Let $\mathcal{B}_0 = \{U_n : n = 1, 2, \ldots\}$. Because X is a separable metric space and U_n is connected, $\operatorname{card}(U_n) = \mathfrak{c}$, (for $n = 1, 2, \ldots$). Of course $X \setminus I(X)$ is separable. Therefore there exists a set H_0 countable and dense in $X \setminus I(X)$. If $\alpha < \mathfrak{c}$ and $\{H_\beta : \beta < \alpha\}$ is the family of disjoint, countable dense sets in $X \setminus I(X)$, then $\operatorname{card}(U_n) = \mathfrak{c}$. Hence $U_n \setminus \bigcup_{\beta<\alpha} H_\beta \neq \emptyset$, for $n = 1, 2, \ldots$. Let (for $n = 1, 2, \ldots$) $x_n \in U_n \setminus \bigcup_{\beta<\alpha} H_\beta$ and $H_\alpha = \{x_n : n = 1, 2, \ldots\}$. Clearly H_α is countable, dense in $X \setminus I(X)$ and $H_\beta \cap H_\alpha = \emptyset$ for $\beta < \alpha$. Define $K_\beta = H_\beta$, for $0 < \beta < \kappa$ and $K_0 = X \setminus \bigcup_{0<\alpha<\kappa} H_\alpha$. Let $F \subset Y$ be a closed set. If $F = \emptyset$, let A and B be disjoint sets dense in $X \setminus I(X)$ such that

$A \cup B = X$, $s_1, s_2 \in Y$, $s_1 \neq s_2$, $f : X \to Y$ be such that $f(x) = s_1$ for $x \in A$ and $f(x) = s_2$ for $x \in B$ and $\mathcal{E}(x) = \{X\}$, for $x \in X$. Then $ls_{\mathcal{E}}(f, x) = \emptyset$, for $x \in X \setminus I(X)$.

Suppose now that $F \neq \emptyset$. Let $\kappa = \operatorname{card}(F)$. Hence Y is separable and $\operatorname{card}(Y) \leq \mathfrak{c}$. Therefore $\kappa \leq \mathfrak{c}$. Let $\{K_\alpha : \alpha < \kappa\}$ be a family of disjoint sets each dense in $X \setminus I(X)$ and such that $X = \bigcup_{\alpha<\kappa} K_\alpha$ and $\mathcal{E}(x) = \{K_\alpha \cup \{x\} : \alpha < \kappa\}$, $(x \in X)$. Let $(y_\alpha)_{\alpha<\kappa}$ be sequence of all elements of F. We define $f : X \to Y$ by $f(x) = y_\alpha$ for $x \in K_\alpha$, $\alpha < \kappa$. Clearly $ls_{\mathcal{E}}(f, x) = F$ for $x \in X \setminus I(X)$. $\square$

5 σ-Systems of Paths.

In paper [2] K. Banaszewski considered bilateral σ-systems of paths in $\mathbb{R}$. We now define analogous notion; namely, σ-systems of paths in metric spaces and give some characterization. We also prove the notion of weak path continuity is the same as the notion of path continuity for such systems.

The system of path $\mathcal{E}$ is called a *σ-system of paths* iff for every $x \in X$, and every sequence $(x_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = x$ if $H_n \in \mathcal{F}_{\mathcal{E}}(x_n)$, $(n = 1, 2, \dots)$, then $\bigcup_{n=1}^{\infty} H_n \cup \{x\}$ belong to $\mathcal{F}_{\mathcal{E}}(x)$.

Theorem 7. *The system of paths $\mathcal{E}$ is a σ-system of paths iff for every function $f : X \to Y$ (where Y is metric , $\operatorname{card}(Y) \geq 2$) the set $f \restriction C_f^{\mathcal{E}}$ is closed in f.*

Proof. Suppose that $\mathcal{E}$ is a σ-system, $f : X \to Y$, $\lim_{n\to\infty} z_n = z$ where $z_n \in f \restriction C_f^{\mathcal{E}}$ (for $n = 1, 2, \dots$) and $z \in f$. We shall show that $z \in f \restriction C_f^{\mathcal{E}}$. Let $z_n = (x_n, f(x_n))$, $z = (x, f(x))$, where $x \in X$, $x_n \in C_f^{\mathcal{E}}$ (for $n = 1, 2, \dots$), $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} f(x_n) = f(x)$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x$, then of course $x \in C_f^{\mathcal{E}}$ and consequently $z \in f \restriction C_f^{\mathcal{E}}$. Thus we can assume that $x_n \neq x$ and $f(x_n) \in K(f(x), \frac{1}{2n})$, for $n = 1, 2, \dots$. Fix $n \in \mathbb{N}$. Because f is $\mathcal{E}$-continuous at x_n, there are $U_n \in \tau(x_n)$ and $E_n \in \mathcal{E}(x_n)$ such that $f(U_n \cap E_n) \subset K(f(x_n), \frac{1}{2n})$. We can assume that (for $n = 1, 2, \dots$) $x \notin \operatorname{cl}(U_n)$. There are $E \in \mathcal{E}(x)$ and $G \in \tau(x)$ such that $E \cap G \subset \bigcup_{n=1}^{\infty} (E_n \cap U_n) \cup \{x\}$. We shall show that $f \restriction E$ is continuous at x.

Fix $\epsilon > 0$. Let $n_0 > 1$ be such that $\frac{1}{n_0} < \epsilon$. Let (for $n < n_0$) W_n be an open neighborhood of x, such that $W_n \cap U_n = \emptyset$. Let $U = G \cap \bigcap_{n<n_0} W_n$. Then U is an open neighborhood of x and $E \cap U \subset \bigcup_{n=n_0}^{\infty} (E_n \cap U_n) \cup \{x\}$. Hence

$f(E_n \cap U_n) \subset K(f(x_n), \frac{1}{2n})$ and $(x_n) \in K(f(x), \frac{1}{2n})$. Then $f(E_n \cap U_n) \subset K(f(x), \frac{1}{n})$ for $n = 1, 2, \ldots$. Thus $f(E \cap U) \subset f(\bigcup_{n=n_0}^{\infty}(E_n \cap U_n) \cup \{x\}) \subset \bigcup_{n=n_0}^{\infty} K(f(x), \frac{1}{n}) = K(f(x), \frac{1}{n_0}) \subset K(f(x), \epsilon)$. We obtain that $x \in C_f^{\mathcal{E}}$ and $z \in f \restriction C_f^{\mathcal{E}}$.

Let Y be metric, $\mathrm{card}(Y) \geq 2$. Suppose that for every function $f : X \to Y$ the set $f \restriction C_f^{\mathcal{E}}$ is closed in f. Let $x \in X$, $(x_n)_{n=1}^{\infty}$, $(H_n)_{n=1}^{\infty}$ be a sequence such that $\lim_{n\to\infty} x_n = x$ and $H_n \in \mathcal{F}_{\mathcal{E}}(x_n)$, for $n = 1, 2, \ldots$. We shall show that there exist $E \in \mathcal{E}(x)$ and $U \in \tau(x)$ such, that $E \cap U \subset \bigcup_{n=1}^{\infty} H_n \cup \{x\}$. Let $y_0, y_1 \in Y$, $y_0 \neq y_1$. We define $f : X \to Y$ by

$$f(t) = \begin{cases} y_0 & \text{if } t \in \bigcup_{n=1}^{\infty} H_n \cup \{x\}, \\ y_1 & \text{otherwise.} \end{cases}$$

Let $E_n \in \mathcal{E}(x_n)$, $U_n \in \tau(x_n)$ be such that $E_n \cap U_n \subset H_n$ (for $n \in \mathbb{N}$). Then f is constant on $E_n \cap U_n$. Thus $x_n \in C_f^{\mathcal{E}}$ (for $n \in \mathbb{N}$). Hence $f(x_n) = y_0 = f(x)$ (for $n \in \mathbb{N}$) and $\lim_{n\to\infty} x_n = x$. Then $\lim_{n\to\infty}(x_n, f(x_n)) = (x, f(x))$. Because $f \restriction C_f^{\mathcal{E}}$ is closed in f, we obtain that $x \in C_f^{\mathcal{E}}$. Thus there is $E \in \mathcal{E}(x)$ such that $f \restriction E$ is continuous at x. Let $\epsilon = d_Y(y_0, y_1)$ and U be an open neighborhood of x such that $f(U \cap E) \subset K(f(x), \epsilon) = K(y_0, \epsilon)$. Then $E \cap U \subset f^{-1}(y_0) = \bigcup_{n=1}^{\infty} H_n \cup \{x\}$. □

Corollary 1. *Let $\mathcal{E}$ be σ-system on X and $f : X \to Y$. If $f \restriction C_f^{\mathcal{E}}$ is dense in f, then f is $\mathcal{E}$-continuous.*

Theorem 8. *Let X, Y be a metric spaces $f : X \to Y$ and $\mathcal{E}$ be a σ-system of paths on X. Then f is $\mathcal{E}$-continuous iff it is weak $\mathcal{E}$-continuous.*

PROOF. If f is $\mathcal{E}$-continuous and $x \in X$, then

$$f(x) \in \{y \in Y : \exists_{E\in\mathcal{E}(x)} \forall_{V\in\nu(y)} \exists_{U\in\tau(x)} f(U \cap E \setminus \{x\}) \subset V\} \subset ls_{\mathcal{E}}(f, x).$$

Hence f is weak $\mathcal{E}$-continuous at x.

Suppose that $f : X \to Y$ is weak $\mathcal{E}$-continuous and let $x \in X$. We can assume that $x \notin I(X)$. For every $n \in \mathbb{N}$ there exists $F_n \in \mathcal{E}(x)$ and $G_n \in \tau(x)$ such that $f(F_n \cap G_n) \subset K(f(x), \frac{1}{2n})$. Because x is a point of accumulation of F_n, there exists $x_n \in K(x, \frac{1}{n}) \cap G_n \cap F_n \setminus \{x\}$ (for $n = 1, 2, \ldots$). Clearly $\lim_{n\to\infty} x_n = x$. We can assume that $d_X(x_{n+1}, x) < d_X(x_n, x)$ for $n = 1, 2, \ldots$.

Let (for $n = 1, 2, \dots$) $r_n = d_X(x_n, x)$, $\delta_n = r_n - r_{n+1}$. Observe that $r_n > 0, \delta_n > 0$ and

$$K(x, r_n) \cap \bigcup_{k=1}^{\infty} K(x_k, \delta_k) \subset \bigcup_{k=n}^{\infty} K(x_k, \delta_k), \text{ for } n = 1, 2, \dots . \tag{4}$$

Because f is weak $\mathcal{E}$-continuous, (for every $k \in \mathbb{N}$) there exists $E_k \in \mathcal{E}(x_k)$ and $U_k \in \tau(x_k)$ such that $U_k \subset K(x_k, \delta_k)$, $f(E_k \cap U_k) \subset K(f(x_k), \frac{1}{2k})$. Observe that

$$E_k \cap U_k \subset f^{-1}(K(f(x), \frac{1}{k})), \text{ for } k = 1, 2, \dots . \tag{5}$$

Because $\mathcal{E}$ is σ-system of paths, there exist a set $E \in \mathcal{E}(x)$ and $U \in \tau(x)$ such that $E \cap U \subset \bigcup_{k=1}^{\infty} (E_k \cap U_k) \cup \{x\}$. For every $k \in \mathbb{N}$, $U_k \subset K(x_k, \delta_k)$ from (4) and (5) we obtain $E \cap U \cap K(x, r_n) \setminus \{x\} \subset \bigcup_{k=n}^{\infty} (E_k \cap U_k) \subset f^{-1}(K(f(x), \frac{1}{n}))$. Because $\lim_{n\to\infty} r_n = 0$, $f \restriction E$ is continuous in x. $\square$

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