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Janusz Matkowski, Institute of Mathematics, University of Zielona Góra, PL-65-246 Zielona Góra, Poland. email: J.Matkowski@im.uz.zgora.pl

CONVEX FUNCTIONS WITH RESPECT TO A MEAN AND A CHARACTERIZATION OF QUASI-ARITHMETIC MEANS

Abstract

Let $M : (0,\infty)^2 \to (0,\infty)$ be a homogeneous strict mean such that the function $h := M(\cdot, 1)$ is twice differentiable and $0 \neq h'(1) \neq 1$. It is shown that if there exists an M-affine function, continuous at a point which is neither constant nor linear, then M must be a weighted power mean. Moreover the homogeneity condition of M can be replaced by M-convexity of two suitably chosen linear functions. With the aid of iteration groups, some generalizations characterizing the weighted quasi-arithmetic means are given. A geometrical aspect of these results is discussed.

1 Introduction

A real function M defined on the Cartesian product $J \times J$ of an interval $J \subset \mathbb{R}$ is said to be a *mean* if it is internal; that is, if $\min \leq M \leq \max$. A function φ mapping a subinterval I of J into J is called, M-affine, M-convex, and M-concave, if, respectively,

$$\varphi(M(x,y)) = M(\varphi(x), \varphi(y))$$
$$\varphi(M(x,y)) \leq M(\varphi(x), \varphi(y))$$
$$\varphi(M(x,y)) \leq M(\varphi(x), \varphi(y))$$

for all $x, y \in I$ (cf. G. Aumann [5] where even two different means are involved; also J. Aczél [1], and [12], [13]). For $M = A$ where A is the arithmetic mean, we obtain the classical notions of Jensen convexity, concavity and affinity. It

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is well known that every measurable, or one-sided bounded at a point, Jensen affine function is of the form $\varphi(x) = ax + b$ for some real a, b. The family of all A-affine functions is rich in the following sense. For any two distinct points from the domain of A there exists exactly one A-affine function the graph of which passes through these points. This fact allows the acquisition of the epigraph of an A-convex function as the intersection of all the epigraphs of its supporting A-affine functions. This property is also shared by functions convex with respect to the weighed quasi-arithmetic means. (In this connection, in the last section, we introduce a notion of "M-affinely convex function".) In [11] it is shown that the logarithmic mean L does not have this property, because every L-affine function is either constant or linear (that is, of the form $\varphi(x) = ax$).

The main result of Section 3 says that if a mean M is homogeneous, the function $M(\cdot, 1)$ is twice differentiable, and there is an M-affine function, continuous at least at one point, which is neither linear nor constant, then M must be a power mean. In Section 4 we generalize this result replacing the homogeneity of M by the assumption that two suitably chosen linear functions are M-convex. A mean M on $(0, \infty)$ is homogeneous iff for every $a > 0$ the linear function $\varphi(x) = ax$ is M-affine and, moreover, the family of these functions forms a (multiplicative) iteration group. In Section 5, replacing the homogeneity condition of M in the main result of Section 3 by the assumption that there is a family of M-affine functions which form an iteration group, we prove that M must be a weighted quasi-arithmetic mean, which is a new characterization of this kind of means. In the last section, to discuss some consequences of these results in relation to classically convex functions we define a function to be "M-affinely convex". Finally we mention a recent result by J. Aczél and R. Duncan Luce [3], motivated by some problems in utility theory and psychophysics, in which the functional equation $H[K(s,t)] = L[h(s), h(t)]$ is considered, and we formulate an open problem.

Note that some questions related to a characterization of L^p-norm [9] and the Euler gamma function [6], [7] in a natural way lead to the M-convexity with $M \neq A$.

2 Preliminaries

Let $J \subset \mathbb{R}$ be an interval. A function $M : J^2 \to \mathbb{R}$ is said to be a *mean on* J if $\min(x,y) \le M(x,y) \le \max(x,y),\ x, y \in J$. Moreover, if for all $x, y \in J$, $x \neq y$, these inequalities are strict, M is called a *strict* mean and if $M(x,y) = M(y,x)$ for all $x, y \in I$, M is called *symmetric*.

If $M : J^2 \to \mathbb{R}$ is a mean, then M is *reflexive*; that is, $M(x,x) = x,\ x \in J$.

It is easy to see that every reflexive function $M : J^2 \to \mathbb{R}$ which is (strictly) increasing with respect to each variable is a (strict) mean. The reflexivity of a mean M implies that $M(I^2) = I$ for every interval $I \subset J$, and $M|_{I\times I}$. is a mean on I. This property permits to generalize the classical notions of the convex, concave and affine functions in the following way (cf. [1], [5], [12], [13]).

Definition 1. Let $J \subset \mathbb{R}$ be an interval, $M : J^2 \to J$ a mean on J, and $I \subset J$ an interval. A function $\varphi : I \to J$ is said to be:

convex with respect to M on I, or simply, M-*convex* on I, if

$$\varphi\left(M(x,y)\right) \le M(\varphi(x), \varphi(y)),\ x, y \in I,$$

M-*concave* on I, if the inequality is reversed and

M-*affine* on I, if it is both M-convex and M-concave; i.e., if,

$$\varphi\left(M(x,y)\right) = M(\varphi(x), \varphi(y)),\ x, y \in I.$$

Remark 1. Suppose that $M : J^2 \to J$ is a mean. Then

1. every constant function $\varphi : J \to J$ and the identity function $\varphi = id|_J$ is M-affine,
2. for $M = \min$ or $M = \max$ every increasing function $\varphi : J \to J$ is M-affine. Thus, if M is not strict, then the class of M-affine functions is, in general, essentially lager,
3. if $\varphi : J \to J$ is M-convex, strictly increasing and onto, then the inverse function φ^{-1} is M-concave.

Note that taking in these definitions $M = A$, where $A : \mathbb{R}^2 \to \mathbb{R}$ denotes the arithmetic mean, $A(x, y) = \frac{x+y}{2}$, we obtain the classical Jensen affine and Jensen convex functions.

Remark 2. Suppose that a mean $M : (0,\infty)^2 \to (0,\infty)$ is a homogeneous function of an order $p \in \mathbb{R}$; that is, $M(tx, ty) = t^p M(x, y),\ t, x, y > 0$. Then

1. $p = 1$,
2. setting $h(t) := M(t, 1),\ t > 0$, we have

$$M(x,y) = yh\left(\frac{x}{y}\right),\ x, y > 0;\ h(1) = 1$$

$$0 \le \frac{h(x) - 1}{x - 1} \le 1,\ x > 0,\ x \ne 1,$$

and these inequalities are strict iff M is a strict mean. Moreover, if h is differentiable at the point 1, then $0 \leq h'(1) \leq 1$,

3. besides the constant functions, every linear function $\varphi(x) = \varphi(1)x$, $x \in \mathbb{R}$, is M-affine,
4. if $c \in (0, \infty)$ and $\varphi : (0, \infty) \to (0, \infty)$ is M-affine, then so is $c\varphi$.

Remark 3. Suppose that $M : J^2 \to J$ is a mean and $I_1, I_2 \subseteq J$ are intervals. If $\varphi_1 : I_1 \to I_2$, $\varphi_2 : I_2 \to J$ are M-affine, then clearly, the composition $\varphi_2 \circ \varphi_1$ is also M-affine.

Let us note the following.

Lemma 1. *Let $J \subset \mathbb{R}$ be an interval and $M : J^2 \to \mathbb{R}$ a strict and continuous mean. Suppose that M is strictly monotonic with respect to one of the variables (in a neighborhood of the diagonal $\{(x, x) : x \in J\}$). If $I \subset J$ is an interval and $\varphi, \psi : I \to J$ are M-affine, continuous, and $\varphi(x_1) = \psi(x_1)$, $\varphi(x_2) = \psi(x_2)$ for some $x_1, x_2 \in I$, $x_1 \neq x_2$, then $\varphi = \psi$.*

PROOF. Assume that M is strictly monotonic with respect to the first variable. Put $I_0 := \{x \in I : \varphi(x) = \psi(x)\}$. By the continuity of φ and ψ the set I_0 is closed in I. Assume that $x_1 < x_2$. We shall show that $[x_1, x_2] \subset I_0$. Indeed, in the opposite case the set $[x_1, x_2] \setminus I_0$ would be at most countable sum of nonempty intervals. If (a, b) is one of such an intervals, then $\varphi(a) = \psi(a)$, $\varphi(b) = \psi(b)$. Hence we get

$$\varphi(M(a, b)) = M(\varphi(a), \varphi(b)) = M(\psi(a), \psi(b)) = \psi(M(a, b)).$$

Since M is a strict mean, we have $a < M(a, b) < b$ and consequently, $M(a, b) \in I_0$; that is, a desired contradiction. In particular we have proved that I_0 is an interval. Suppose that $I_0 \neq I$. Then at least one of the endpoints of the interval I_0 would be an interior point of I. Assume, for instance, that $c := \min I_0$ belongs to I. Let us take $x_0 \in I_0$, $x_0 > c$. Since M is strict, we have $c < M(c, x_0) < x_0$. The continuity of the function $I \ni x \to M(x, x_0)$ implies that there is a $\delta > 0$ such that $[c - \delta, x_0] \subset I$ and $M(x, x_0) \in [c, x_0]$ for all $x \in [c - \delta, x_0]$. Hence for $x \in [c - \delta, x_0]$ we have

$$\begin{aligned} M(\psi(x), \varphi(x_0)) &= M(\psi(x), \psi(x_0)) = \psi(M(x, x_0)) \\ &= \varphi(M(x, x_0)) = M(\varphi(x), \varphi(x_0)). \end{aligned}$$

Since M is strictly increasing with respect to the first variable, we infer that $\psi(x) = \varphi(x)$ for all $x \in [c - \delta, x_0]$, which contradicts to the definition of c. (Choosing x_0 close enough to c, we can argue similarly in the case when M is increasing with respect to the first variable in a neighborhood of the diagonal.) □

3 A Basic Result for Homogeneous Means

The main result of this section reads as follows.

Theorem 1. *Let $M : (0,\infty)^2 \to (0,\infty)$ be a strict and homogeneous mean. Suppose that the function $h : (0,\infty) \to (0,\infty)$ defined by $h(x) := M(x,1)$, $x > 0$, is twice differentiable, and $0 \neq h'(1) \neq 1$. If there exists an M-affine function, continuous at a point which is neither constant nor linear, then there is a $p \in \mathbb{R}$ such that*

$$M(x,y) = \begin{cases} (wx^p + (1-w)y^p)^{1/p} & \text{for } p \neq 0 \\ x^w y^{1-w} & \text{for } p = 0 \end{cases}, \ x,y > 0,$$

where $w := h'(1)$.

Proof. Let $\varphi : (0,\infty) \to (0,\infty)$ be continuous at a point x_0, and M-affine function; i.e.,

$$\varphi\left(M(x,y)\right) = M(\varphi(x),\varphi(y)),\ x,y > 0. \tag{1}$$

Suppose that φ is nontrivial; that is, it is neither linear nor constant in $(0,\infty)$. By Remark 2 we have $0 < h'(1) < 1$. The continuity of h' implies that h is strictly monotonic in a neighborhood of 1. It follows that in a neighborhood of the diagonal M is locally strictly increasing with respect to both variables. To show it note that there is an $\varepsilon > 0$ such that $0 < h'(t) < 1,\ t \in (1-\varepsilon, 1+\varepsilon)$. Let us fix an arbitrary $y > 0$. Since, by the homogeneity of M,

$$M(x,y) = yh\left(\frac{x}{y}\right),\ x,y > 0, \tag{2}$$

we have

$$\frac{\partial M}{\partial x}(x,y) = h'\left(\frac{x}{y}\right),\ x,y > 0,$$

and, consequently, there is an $\varepsilon > 0$ such that $\frac{\partial M}{\partial x}(x,y) > 0$ for all $x,y > 0$ such that $1-\varepsilon < \frac{x}{y} < 1+\varepsilon$. which proves that $M(\cdot,y)$ is increasing in a neighborhood of y for every $y > 0$. Similarly, since

$$\frac{\partial M}{\partial y}(x,y) = h\left(\frac{x}{y}\right) - \frac{x}{y}h'\left(\frac{x}{y}\right),\ x,y > 0,$$

and, $h(1) = 1$, we infer that, there is an $\varepsilon > 0$ such that $\frac{\partial M}{\partial y}(x,y) > 0$ for all $x,y > 0$ such that $1-\varepsilon < \frac{x}{y} < 1+\varepsilon$. This proves that our mean M is strictly increasing with respect to both variables in a neighborhood of the diagonal.

Suppose that φ is continuous at a point $x_0 > 0$. Choose $y > 0$, $y \neq x_0$, such that M is strictly increasing with respect to both variables in a joint neighborhood of the points $(x_0, x_0), (x_0, y), (y, y)$. Assume, for instance, that $x_0 < y$. Then $x_0 < M(x_0, y) < y$. Take an arbitrary point $z_0 \in (x_0, M(x_0, y))$. By the continuity and the strict increasing monotonicity of the function $M(x_0, \cdot)$, there is a unique $y_0 \in (x_0, y)$ such that $z_0 = M(x_0, y_0)$ and the function $M(\cdot, y_0)$ is strictly increasing in a neighborhood of x_0. Let (z_n) be an arbitrary sequence such that $z_n \to z_0$ as $n \to \infty$ and $z_n \in (x_0, M(x_0, y))$ for all $n \in \mathbb{N}$. Hence, for every n there is a unique $x_n \in (x_0, y)$ such that $M(x_n, y_0) = z_n$. Moreover we have $z_n \to z_0$ as $n \to \infty$. In fact, in the opposite case, for a subsequence of (x_{n_k}), by the continuity of M, we would get

$$\lim_{k\to\infty} M(x_{n_k}, y_0) = M(\bar{x}, y_0) = z_0,$$

for some $\bar{x} \neq x_0$, which contradicts to the strict monotonicity of $M(\cdot, y_0)$ in $[x_0, y]$. Now, making use of the M-affinity of φ, the continuity of M, and the continuity of φ at x_0, we get

$$\begin{aligned}\lim_{k\to\infty} \varphi(z_n) = \lim_{k\to\infty} \varphi(M(x_n, y_0)) &= \lim_{k\to\infty} M(\varphi(x_n), \varphi(y_0)) \\ &= M(\varphi(x_0), \varphi(y_0)) = \varphi(M(x_0, y_0)) = \varphi(z_0)\end{aligned}$$

which proves that φ is right-continuous at z_0. Assuming that $y < M(x_0, y) < x_0$ in the same way we can show that φ is left-continuous at z_0. Thus we have shown that φ is continuous in a neighborhood of the point x_0. (The argument used in the proof of the continuity is similar to that applied in [10].)

Let (a, b) denote the maximal open interval of the continuity of φ such that $x_0 \in (a, b)$. Suppose that $b < \infty$. Since M is strictly increasing in a neighborhood of (b, b), choosing z_0 sufficiently close to b, and the numbers x_0, y_0, $x_0 < b \leq z_0 < y_0$, we can argue as in the previous step to show that φ is continuous in a right neighborhood of b. This contradicts the definition of b and proves that $b = \infty$. A similar argument shows that $a = 0$. Thus φ is continuous on $(0, \infty)$ is completed.

Since the constant and linear functions are M-affine, Lemma 1 implies that φ is strictly monotonic and there is no interval $I \subset (0, \infty)$ such that $\varphi|_I$ is constant or linear. Moreover equation (1) can be written in the form

$$\varphi\left(yh\left(\frac{x}{y}\right)\right) = \varphi(y)h\left(\frac{\varphi(x)}{\varphi(y)}\right), \ x, y > 0. \tag{3}$$

The function φ, being monotonic, is differentiable almost everywhere. Let $x > 0$ be a differentiability point of φ. Relation (3) and the assumed differentiability of h imply that, for arbitrarily fixed $y>0$, the function φ is differentiable at a point $yh\left(\frac{x}{y}\right)$. Consequently, φ is differentiable everywhere.

Differentiation of both sides with respect to x and y gives, respectively,

$$\varphi'\left(yh\left(\frac{x}{y}\right)\right)h'\left(\frac{x}{y}\right)=\varphi'(x)h'\left(\frac{\varphi(x)}{\varphi(y)}\right),\ x,y>0 \tag{4}$$

and

$$\begin{aligned}&\varphi'\left(yh\left(\frac{x}{y}\right)\right)\left[h\left(\frac{x}{y}\right)-h'\left(\frac{x}{y}\right)\frac{x}{y}\right]\\&=\varphi'(y)h\left(\frac{\varphi(x)}{\varphi(y)}\right)-h'\left(\frac{\varphi(x)}{\varphi(y)}\right)\frac{\varphi(x)\varphi'(y)}{\varphi(y)},\ x,y>0.\end{aligned} \tag{5}$$

(Note that the continuity of the right-hand side of (4) with respect to y implies the continuity of $\varphi'\left(yh\left(\frac{x}{y}\right)\right)$ with respect to y and, consequently, the continuity of φ'.) Suppose that $\varphi'(x_0)=0$ for some $x_0>0$. Since h' is continuous at 1 and $h'(1)\neq 0$, relation (4) implies that $\varphi'\left(yh\left(\frac{x_0}{y}\right)\right)=0$ for all y from a neighborhood of the point x_0. Moreover, the function $y\to yh\left(\frac{x_0}{y}\right)$ maps every interval neighborhood of x_0 on a nontrivial interval. In fact, in the opposite case, this function would be constant on some neighborhood of x_0; i.e., $h\left(\frac{x_0}{y}\right)=\frac{c}{y}$. Since $h(1)=1$, we infer that $c=x_0$ and $h(t)=t$ in a neighborhood of the point 1. Consequently, $M(x,y)=x$ in a neighborhood of the point (x_0,x_0). This is a contradiction because M is a strict mean. Hence $\varphi'(x)=0$ in a neighborhood of x_0, and φ would be constant in this neighborhood. By Lemma 1, φ would be constant on $(0,\infty)$. This contradicts the assumption that φ is nontrivial. Thus we have shown that $\varphi'\neq 0$ in $(0,\infty)$.

Let $(\alpha,\beta)\subset(0,\infty)$ be the maximal interval such that $1\in(\alpha,\beta)$ and $h'(t)\neq 0$ for all $t\in(\alpha,\beta)$. Take arbitrary $t\in(\alpha,\beta)$ and $x,y>0$ such that $\frac{x}{y}=t$. Since $\varphi'\neq 0$, from (4) we infer that $\frac{\varphi(x)}{\varphi(y)}\in(\alpha,\beta)$. Now from (5) and (4) we obtain

$$\frac{h\left(\frac{x}{y}\right)-h'\left(\frac{x}{y}\right)\frac{x}{y}}{h'\left(\frac{x}{y}\right)}=\frac{\varphi'(y)}{\varphi'(x)}\left(\frac{h\left(\frac{\varphi(x)}{\varphi(y)}\right)}{h'\left(\frac{\varphi(x)}{\varphi(y)}\right)}-\frac{\varphi(x)}{\varphi(y)}\right);$$

i.e.,

$$\frac{h(t)}{h'(t)}-t=\frac{\varphi'(y)}{\varphi'(ty)}\left(\frac{h\left(\frac{\varphi(ty)}{\varphi(y)}\right)}{h'\left(\frac{\varphi(ty)}{\varphi(y)}\right)}-\frac{\varphi(ty)}{\varphi(y)}\right),\ t\in(\alpha,\beta);\ y>0. \tag{6}$$

Setting $H(t) := \frac{h(t)}{h'(t)} - t,\ t \in (\alpha, \beta)$, we get

$$H(t) = \frac{\varphi'(y)}{\varphi'(ty)} H\left(\frac{\varphi(ty)}{\varphi(y)}\right),\ t \in (\alpha, \beta);\ y > 0, \tag{7}$$

and, of course, H is differentiable in (α, β). Suppose that there is a $t_0 \in (\alpha, \beta)$, $t_0 \neq 1$, such that $H(t_0) = 0$. Then we would have $H\left(\frac{\varphi(t_0 y)}{\varphi(y)}\right) = 0$ for all $y > 0$. Hence either $H(t) = 0$ in a neighborhood of t_0 or $\frac{\varphi(t_0 y)}{\varphi(y)} = t_0$ for all $y > 0$. The first case cannot occur because, by the definition of H, we would have $h(t) = ct$ in a neighborhood of t_0, and, consequently, by (2), $M(x, y) = yh\left(\frac{x}{y}\right) = kx$ for some $k > 0$ and for all $x, y > 0$ such that $\frac{x}{y}$ belongs to the neighborhood of t_0. Since M is a strict mean, we have $k < 1$. Hence, by (1), $\varphi(kx) = \varphi(M(x, y)) = M(\varphi(x), \varphi(y)) = k\varphi(x)$; that is, $\frac{\varphi(kx)}{kx} = \frac{\varphi(x)}{x}$ for all $x > 0$. Thus φ coincides with a linear function at the points x and kx. By Lemma 1, the function φ must be linear, which is the desired contradiction. In the second case we would have $\frac{\varphi(t_0 y)}{t_0 y} = \frac{\varphi(ty)}{y}$ for all $y > 0$, and again, φ would be a linear function. Thus we have shown that $H(t) \neq 0$ for all $t \in (\alpha, \beta),\ t \neq 1$.

Setting $y = 1$ here we get $\varphi'(t) = \varphi'(1)\frac{H(\varphi(t))}{H(t)},\ t \in (\alpha, \beta),\ t \neq 1$. Whence, the differentiability of H implies that φ is twice differentiable in $(\alpha, \beta) \setminus \{1\}$. Taking (7) into account, we infer that φ is twice differentiable in $(0, \infty)$. Differentiating both sides of (7) with respect to $t \in (\alpha, \beta)$ we obtain

$$H'(t) = -\frac{\varphi'(y)\varphi''(ty)y}{[\varphi'(ty)]^2} H\left(\frac{\varphi(ty)}{\varphi(y)}\right) + \frac{\varphi'(y)y}{\varphi(y)} H'\left(\frac{\varphi(ty)}{\varphi(y)}\right)$$

for all $t \in (\alpha, \beta);\ y > 0$. Taking $t := 1$ here and replacing y by x, we get

$$H(1)x\frac{\varphi''(x)}{\varphi'(x)} - H'(1)x\frac{\varphi'(x)}{\varphi(x)} + H'(1) = 0,\ x > 0. \tag{8}$$

Note that $H(1) \neq 0$ as, in the opposite case, we would get

$$H'(1)x\frac{\varphi'(x)}{\varphi(x)} - H'(1) = 0,\ x > 0.$$

Since $h(1) = 1$ and, by assumption, $h'(1) \neq 1$, we have

$$H'(1) = \frac{h\,(t)}{h'\,(t)} - t = \frac{1}{h'\,(1)} - 1 \neq 0.$$

Hence $x\frac{\varphi'(x)}{\varphi(x)} - 1 = 0,\ x > 0$, and, consequently, there would exist a $c > 0$ such that $\varphi(x) = cx,\ x > 0$, which is a contradiction.

Putting $p := 1 - \frac{H'(1)}{H(1)}$, we can write equation (8) in the following equivalent form

$$\frac{\varphi''(x)}{\varphi'(x)} - (1-p)\frac{\varphi'(x)}{\varphi(x)} + \frac{1-p}{x} = 0,\ x > 0.$$

For $p = 1$ the only functions satisfying this differential equations are linear. Solving this differential equation for $p \neq 1$ we obtain

1. if $0 \neq p \neq 1$, then, for some $a, b \in \mathbb{R}$, $a > 0$, $b > 0$,

$$\varphi(x) = (ax^p + b)^{1/p},\ x > 0; \tag{9}$$

2. if $p = 0$, then, for some $a, b \in \mathbb{R}$, $0 \neq a \neq 1$, $b \neq 0$,

$$\varphi(x) = bx^a,\ x > 0, \tag{10}$$

(we have excluded here the constant and linear functions).

Now we shall find the form of the mean M in each of these two cases. In the first case, when $0 \neq p \neq 1$, from (3) we have

$$\left(a\left[yh\left(\frac{x}{y}\right)\right]^p + b\right)^{1/p} = (ay^p + b)^{1/p}\, h\left(\frac{(ax^p+b)^{1/p}}{(ay^p+b)^{1/p}}\right),\ x, y > 0.$$

Replacing $a^{1/p}x$ and $a^{1/p}y$, here respectively by x and y we obtain

$$\left(\left[yh\left(\frac{x}{y}\right)\right]^p + b\right)^{1/p} = (y^p + b)^{1/p}\, h\left(\left(\frac{x^p+b}{y^p+b}\right)^{1/p}\right),\ x, y > 0.$$

Multiplying both sides by an arbitrary $c > 0$ (cf. Remark 2, part 4) we get, for all $x, y > 0$,

$$\left(\left[cyh\left(\frac{cx}{cy}\right)\right]^p + c^p b\right)^{1/p} = ((cy)^p + c^p b)^{1/p}\, h\left(\left(\frac{(cx)^p + c^p b}{(cy)^p + c^p b}\right)^{1/p}\right).$$

Replacing cx, cy, $c^p b$, here respectively, by x, y and r, we obtain

$$\left[yh\left(\frac{x}{y}\right)\right]^p + r = (y^p + r)\left[h\left(\left(\frac{x^p+r}{y^p+r}\right)^{1/p}\right)\right]^p \text{ for all } r, x, y > 0.$$

Hence, for all $r, x, y > 0$,

$$[M(x,y)]^p = \left[yh\left(\frac{x}{y}\right)\right]^p = (y^p + r)\left[h\left(\left(\frac{x^p+r}{y^p+r}\right)^{1/p}\right)\right]^p - r.$$

Taking into account that the right hand side does not depend on $r > 0$, and the relation $h(1) = 1$, we obtain, for all $x, y > 0$,

$$\begin{aligned}
[M(x,y)]^p &= \lim_{r\to\infty}\left\{(y^p+r)\left[h\left(\left(\frac{x^p+r}{y^p+r}\right)^{1/p}\right)\right]^p - r\right\} \\
&= y^p \lim_{r\to\infty} h\left(\left(\frac{x^p+r}{y^p+r}\right)^{1/p}\right)^p + \lim_{r\to\infty}\frac{\left[h\left(\left(\frac{x^p+r}{y^p+r}\right)^{1/p}\right)\right]^p - 1}{\frac{1}{r}} \\
&= h(1)y^p + \lim_{r\to\infty}\frac{\left(\frac{x^p+r}{y^p+r}\right)^{1/p} - 1}{\frac{1}{r}}\,\frac{\left[h\left(\left(\frac{x^p+r}{y^p+r}\right)^{1/p}\right)\right]^p - \left[h\left(1^{1/p}\right)\right]^p}{\left(\frac{x^p+r}{y^p+r}\right)^{1/p} - 1} \\
&= y^p - h'(1)(y^p - x^p).
\end{aligned}$$

Consequently, $M(x,y) = (wx^p + (1-w)y^p)^{1/p}$, $x, y > 0$, where $w := h'(1)$. Since $w \in (0,1)$, M is a weighted power mean.

Now consider the second case when $p = 0$. From (3) we have

$$b\left[yh\left(\frac{x}{y}\right)\right]^a = by^a h\left(\frac{bx^a}{by^a}\right), \quad x, y > 0.$$

Putting $t := \frac{x}{y}$ for $x, y > 0$, we obtain the functional equation

$$[h(t)]^a = h(t^a), \quad t > 0.$$

Define $F : \mathbb{R} \to \mathbb{R}$ by $F := \log \circ h \circ \exp$. Then $F(0) = 0$, F is differentiable at 0, $F(0) = h'(1)$, and F satisfies the functional equation $F(au) = aF(u)$, $u \in \mathbb{R}$. Since this equation is equivalent to $a^{-1}F(u) = F(a^{-1}u)$, $(u \in \mathbb{R})$, we can assume, without loss of generality, that $|a| < 1$. Hence, by induction, $F(a^n u) = a^n F(u)$ for all $u \in \mathbb{R}$ and $n \in \mathbb{N}$. Thus $F(u) = \frac{F(a^n u)}{a^n u} u$, $u \in \mathbb{R}$, $n \in \mathbb{N}$. Letting $n \to \infty$ we get $F(u) = F'(0)u$, $u \in \mathbb{R}$, and, consequently, $h(t) = t^w$, $t > 0$. Of course we have $0 < w < 1$. Thus in this case $M(x,y) = x^w y^{1-w}$, $x, y > 0$, where $w := h'(1)$ which proves that M is a weighted geometric mean. □

Remark 4. Note that in the case $p \neq 0$ every function φ of the form (9) with positive a and b is M-affine, and in the case $p = 0$, every function of the form (10) with positive a and b is M-affine.

Remark 5. Let $M:(0,\infty)^2\to(0,\infty)$ be a homogeneous mean and let $h,h^{\star}:(0,\infty)\to(0,\infty)$ be defined by $h(x):=M(x,1),\ h^{\star}(x):=M(1,x),\ x>0$. Then $h^{\star}(x)=xh\left(\frac{1}{x}\right),\ x>0$. If moreover h is differentiable at the point 1 and $h'(1)=0$, then $\left(h^{\star}\right)'(1)=1$ and vice versa.

To show that the assumption $0\neq h'(1)\neq 1$ is essential consider the following.

Remark 6. *Let $M:(0,\infty)^2\to(0,\infty)$ be a homogeneous mean. Suppose that $h:(0,\infty)\to(0,\infty)$ defined by $h(x):=M(x,1),\ x>0$, is twice differentiable (in a neighborhood of 1) and $h'(1)=0,\quad h''(1)\neq 0$. If $\varphi:(0,\infty)\to(0,\infty)$ is a twice differentiable M-affine function, then either φ is linear or constant. The same remains true if twice differentiability is replaced by nth differentiability and $h'(1)=h''(1)=...=h^{(n-1)}(1)=0,\ h^{(n)}(1)\neq 0$.*

Proof. Differentiating twice both sides of (3) with respect to x we obtain

$$\begin{aligned}&\varphi''\left(yh\left(\frac{x}{y}\right)\right)\left[h'\left(\frac{x}{y}\right)\right]^2+\frac{2}{y}\varphi'\left(yh\left(\frac{x}{y}\right)\right)h''\left(\frac{x}{y}\right)\\&=h''\left(\frac{\varphi(x)}{\varphi(y)}\right)\frac{[\varphi'(x)]^2}{\varphi(y)}+h'\left(\frac{\varphi(x)}{\varphi(y)}\right)\varphi''(x).\end{aligned}$$

Taking here $y:=x$ and making use of the assumptions $h'(1)=0,\ h''(1)\neq 0$, we get $h''(1)\,\varphi'(x)\left(\frac{[\varphi'(x)]}{\varphi(x)}-\frac{1}{x}\right)=0$. If φ is not constant, then $\frac{[\varphi'(x)]}{\varphi(x)}=\frac{1}{x}$, and, consequently, φ is linear. The same argument works in the case $n\geq 3$ as after n times differentiation of both sides of (3) and the substitution $y:=x$ only two summands do not disappear and we again get the above differential equation. □

As a consequence of Theorem 1 we obtain the following.

Corollary 1. *Let $M:(0,\infty)^2\to(0,\infty)$ be a strict, symmetric, and homogeneous mean. Suppose that the function $h:(0,\infty)\to(0,\infty)$ defined by $h(x):=M(x,1),\ x>0$, is twice differentiable. If there exists an M-affine function, continuous at a point which is neither constant nor linear, then there is a $p\in\mathbb{R}$ such that*

$$M(x,y)=\begin{cases}\left(\frac{x^p+y^p}{2}\right)^{1/p} & \textit{for } p\neq 0\\ \sqrt{xy} & \textit{for } p=0.\end{cases}$$

4 A Generalization Involving M-Convex Functions

Theorem 2. *Let $M : (0,\infty)^2 \to (0,\infty)$ be a strict continuous mean. Suppose that:*

1. *there are $a, b > 0$, $a < 1 < b$, $\frac{\log b}{\log a} \notin \mathbb{Q}$, such that the linear functions $(0,\infty) \ni x \to ax$, $(0,\infty) \ni x \to bx$ are both M-convex (or both M-concave),*
2. *the function $h(x) := M(x,1)$, $x > 0$, is twice differentiable, and $0 \neq h'(1) \neq 1$.*

If there exists an M-affine function, continuous at least at one point, which is neither constant nor linear, then there is a $p \in \mathbb{R}$ such that

$$M(x,y) = \begin{cases} (wx^p + (1-w)y^p)^{1/p} & \text{for } p \neq 0. \\ x^w y^{1-w} & \text{for } p = 0 \end{cases}, \ x, y > 0,$$

where $w := h'(1)$.

PROOF. The assumed convexity of the functions $(0,\infty) \ni x \to ax$ and $(0,\infty) \ni x \to bx$ implies that

$$aM(x,y) \leq M(ax, ay), \ bM(x,y) \leq M(bx, by), \ x, y > 0.$$

Hence, by induction, for all $n, m \in \mathbb{N}$ and $x, y > 0$,

$$a^m M(x,y) \leq M(a^m x, a^m y); \ b^n M(x,y) \leq M(b^n x, b^n y),$$

whence

$$a^m b^n M(x,y) \leq M(a^m b^n x, a^m b^n y); \ m, n, \in \mathbb{N}, x, y > 0.$$

The assumptions on a and b imply that the set $\{a^m b^n : m, n, \in \mathbb{N}\}$ is dense in $(0,\infty)$. The continuity of M implies that $tM(x,y) \leq M(tx, ty)$; $t, x, y > 0$, which, obviously yields the homogeneity of M. Now our theorem follows from Theorem 1. □

5 Non-Homogeneous Means - A Characterization of Weighted Quasi-Arithmetic Means

By Remark 3, if $g : J \to J$ is M-affine, then, for every $n \in \mathbb{N}$, its nth iterate g^n is M-affine If, moreover, g is invertible, then the inverse g^{-1} is M-affine on $g(J)$, and the family of iterates $\{g^k : k \in \mathbb{Z}\}$ is a group consisting of M-affine functions.

We begin with recalling the following.

Definition 2. Let $J \subset \mathbb{R}$ be an interval. A one-parameter family $\{g^u : u \in \mathbb{R}\}$ of continuous functions $g^u : J \to J$ such that $g^u \circ g^v = g^{u+v}$, $u, v \in \mathbb{R}$; $g^0 = id|_J$ is said to be an iteration group (cf. M. Kuczma [8], p.197-198). If for every $x \in J$ the function $(-\infty, \infty) \ni u \to g^u(x)$ is continuous or measurable, the iteration group is called, respectively, continuous or measurable.

Remark 7. Suppose that $\{g^u : u \in \mathbb{R}\}$ is an iteration group in an interval J. Then the function $F : J \times \mathbb{R} \to J$, $F(x, u) := g^u(x)$, satisfies the translation equation $F(F(x, u), v) = F(x, u + v)$, $x \in J$, $u, v \in \mathbb{R}$. If J is open and $\{g^t : t \in \mathbb{R}\}$ is a continuous iteration group, then (J. Aczél, [2], p. 248), there is a surjective homeomorphic function $\gamma : J \to \mathbb{R}$, determined uniquely up to an additive constant (cf. [2], p. 246), such that $F(x, u) = \gamma^{-1}(\gamma(x) + u)$, $x \in J$, $u \in \mathbb{R}$ and, consequently, $g^u(x) = \gamma^{-1}(\gamma(x) + u)$, $x \in J$, $u \in \mathbb{R}$. Setting $\alpha := \exp \circ \gamma$ we can write this iteration group in the form $g^u(x) = \alpha^{-1}(e^u \alpha(x))$, $x \in J; u \in \mathbb{R}$, where $\alpha : J \to (0, \infty)$ is a surjective homeomorphism determined uniquely up to a multiplicative positive constant. The function α is referred to as a *generator* of the iteration group. Note that the family $\{f^t : t > 0\}$ defined by $f^t := g^{\log t}$, $t > 0$, is a "multiplicative" iteration group; that is, $f^s \circ f^t = f^{st}$, $s, t > 0$, and

$$f^t(x) = \alpha^{-1}(t\alpha(x)),\ t > 0, x \in J. \tag{11}$$

In the sequel it is convenient to write the iteration groups in their multiplicative forms.

Let us mention that M. C. Zdun [14] proved that every measurable iteration group is continuous.

A motivation for the present section is the following obvious comment.

Remark 8. The family $\{f^t : t > 0\}$ of linear functions $f^t : (0, \infty) \to (0, \infty)$, $f^t(x) := tx$, $x > 0$ is a continuous (multiplicative) iteration group. Moreover, a mean $M : (0, \infty)^2 \to (0, \infty)$ is homogeneous if, and only if, every function of this family is M-affine.

Now we prove this assertion.

Theorem 3. *Let $J \subset \mathbb{R}$ be an open interval and $M : J^2 \to J$ a strict mean. Suppose that there exists a continuous iteration group $\{f^t : t > 0\}$ of the form (11) which consists of M-affine functions. Furthermore, suppose that $h : (0, \infty) \to (0, \infty)$ defined by $h(u) := \alpha(M(\alpha^{-1}(u), 1)$, $u > 0$ is twice differentiable, and $0 \neq h'(1) \neq 1$. If there exists an M-affine function, continuous at a point, that is neither constant nor an element of the iteration group $\{f^t : t > 0\}$, then*

$$M(x, y) = \beta^{-1}\left(w\beta(x) + (1 - w)\beta(y)\right),\ x, y \in J$$

for some continuous and strictly monotonic function $\beta : J \to (0,\infty)$ and $w = h'(1)$; that is, M is a weighted quasi-arithmetic mean.

PROOF. By assumption each function of the iteration group $\{f^t : t > 0\}$ is M-affine; i.e., $f^t(M(x,y)) = M(f^t(x), f^t(y))$, $t > 0, x, y \in J$. There exists (cf. Remark 7) a surjective homeomorphism $\alpha : J \to (0,\infty)$ such that $f^t(x) = \alpha^{-1}(t\alpha(x))$, $t > 0, x \in J$. Hence

$$\alpha^{-1}(t\alpha(M(x,y))) = M\big(\alpha^{-1}\big(t\alpha(x)\big), \alpha^{-1}\big(t\alpha(y)\big)\big),\ t > 0, x, y \in J.$$

Take arbitrary $u, v > 0$. There are $x, y \in J$ such that $x = \alpha^{-1}(u)$ and $y = \alpha^{-1}(v)$. Setting these numbers into the above formula, we obtain

$$\alpha\big(M\big(\alpha^{-1}(tu), \alpha^{-1}(tv)\big)\big) = t\alpha\big(M\big(\alpha^{-1}(u), M\big(\alpha^{-1}(v)\big)\big),\ t, u, v > 0,$$

which shows that the function $K : (0,\infty)^2 \to (0,\infty)$ defined by $K(u,v) := \alpha(M(\alpha^{-1}(u), \alpha^{-1}(v)))$, $u, v > 0$, is homogeneous. It is also obvious that K is a strict mean. By Theorem 1, K is a weighted power mean with a power $p \in \mathbb{R}$ and the weight $w = h'(1)$. Whence

$$M(x,y) = \begin{cases} \alpha^{-1}\left[\left(w[\alpha(x)]^p + (1-w)[\alpha(y)]^p\right)^{1/p}\right] & \text{for } p \neq 0 \\ \alpha^{-1}\left[\alpha(x)^w \alpha(y)^{1-w}\right] & \text{for } p = 0 \end{cases},\quad x, y \in J.$$

To complete the proof it is enough to take $\beta(x) := \alpha(x)^p$, $x \in J$, in the case $p \neq 0$, and $\beta := \ln \circ \alpha$ in the case $p = 0$. □

Remark 9. If M is a weighted quasi-arithmetic mean with generator β, then the family $\left\{\beta^{-1} \circ t \circ \beta : t > 0\right\}$ is an iteration group and every function of this family is M-affine.

The following counterpart of Theorem 2 for non-homogeneous means is a characterization of the weighted quasi-arithmetic means.

Theorem 4. *Let $J \subset \mathbb{R}$ be an open interval and $M : J^2 \to J$ a strict continuous mean. Suppose that there is a homeomorphism $\alpha : J \to (0,\infty)$ such that*

1. *for some $a, b > 0$, $a < 1 < b$, the number $\frac{\log b}{\log a}$ is irrational and the functions $\alpha^{-1} \circ (a\alpha)$ and $\alpha^{-1} \circ (b\alpha)$ are both M-convex (or both M-concave);*
2. *the function $h : (0,\infty) \to (0,\infty)$ defined by $h(x) := \alpha(M(\alpha^{-1}(x), 1))$, $x > 0$, is twice differentiable and $0 \neq h'(1) \neq 1$.*

If there exists an M-affine function, continuous at a point which is neither constant nor of the form $\alpha^{-1} \circ (t\alpha)$ for a $t > 0$, then

$$M(x, y) = \beta^{-1} \left(w\beta(x) + (1 - w)\beta(y) \right), \qquad x, y \in J,$$

for some continuous and strictly monotonic function $\beta : J \to (0, \infty)$ and $w = h'(1)$; that is, M is a weighted quasi-arithmetic mean.

PROOF. By the M-convexity of the functions $\alpha^{-1} \circ (a\alpha)$ and $\alpha^{-1} \circ (b\alpha)$ we have

$$\alpha^{-1}(a\alpha(M(x, y))) \leq M(\alpha^{-1}(a(\alpha^{-1}(x)), \alpha^{-1}(a(\alpha^{-1}(y)))$$

and

$$\alpha^{-1}(b\alpha(M(x, y))) \leq M(\alpha^{-1}(b(\alpha^{-1}(x)), \alpha^{-1}(b(\alpha^{-1}(y)))$$

for all $x, y > 0$. Hence, taking into account that $\alpha^{-1} \circ (a\alpha)$ and $\alpha^{-1} \circ (b\alpha)$ are increasing, by induction, we obtain, for all $m \in \mathbb{N}$ and $x, y > 0$,

$$\alpha^{-1}(a^m\alpha(M(x, y))) \leq M(\alpha^{-1}(a^m(\alpha^{-1}(x)), \alpha^{-1}(a^m(\alpha^{-1}(y))),$$

and for all $n \in \mathbb{N}$ and $x, y > 0$,

$$\alpha^{-1}(b^n\alpha(M(x, y))) \leq M(\alpha^{-1}(b^n(\alpha^{-1}(x)), \alpha^{-1}(b^n(\alpha^{-1}(y))).$$

From these two inequalities we get, for all $m, n \in \mathbb{N}$ and $x, y > 0$,

$$\alpha^{-1}(a^m b^n\alpha(M(x, y))) \leq M(\alpha^{-1}(a^m b^n(\alpha^{-1}(x)), \alpha^{-1}(a^m b^n(\alpha^{-1}(y))).$$

Now the density of the set $\{a^m b^n : m, n, \in \mathbb{N}\}$ in $(0, \infty)$ and the continuity of M imply that, for all $t, x, y > 0$,

$$\alpha^{-1}(t\alpha(M(x, y))) \leq M(\alpha^{-1}(t(\alpha^{-1}(x)), \alpha^{-1}(t(\alpha^{-1}(y)));$$

that is, for every $t > 0$ the function $\alpha^{-1} \circ (t\alpha)$ is M-convex. Since, for every $t > 0$, the function $\alpha^{-1} \circ (t\alpha)$ is increasing, its inverse, $\alpha^{-1} \circ (t^{-1}\alpha)$ is M-concave (cf. Remark 3). It follows that $\alpha^{-1} \circ (t\alpha)$ is M-affine for every $t > 0$. Since the family $\{f^t : t > 0\}$ with $f^t := \alpha^{-1} \circ (t\alpha)$ is an iteration group, our result follows from Theorem 3. □

6 Some Conclusions for M-Convex and "M-Affinely Convex" Functions

Let us introduce the following notion.

Definition 3. Let $J \subset \mathbb{R}$ and $I \subset J$ be intervals and $M : J^2 \to J$ a mean. A function $f : I \to J$ is said to be M-affinely convex if for every $x_0 \in I$ there is an M-affine function $\varphi : J \to J$ such that $f(x_0) = \varphi(x_0)$ and $\varphi(x) \leq f(x)$ for all $x \in I$.

For a function $f : I \to J$ denote by $E(f)$ the epigraph of f; i.e., the set $E(f) := \{(x, y) \in I \times \mathbb{R} : f(x) \leq y\}$.

Remark 10. A function $f : I \to J$ is M-affinely convex if, and only if, there is a family Φ of M-affine functions $\varphi : I \to J$ such that $E(f) = \bigcap \{E(\varphi) : \varphi \in \Phi\}$.

Theorem 5. *Suppose that $M : J^2 \to J$ is a mean in an interval J which is increasing with respect to each variable. Then every M-affinely convex function is M-convex.*

PROOF. Let $I \subset J$ be an interval and suppose that $f : I \to J$ is M-affinely convex. Take $x, y \in I$. By Definition 3 there is an M-affine function $\varphi : J \to J$ such that $f(M(x, y)) = \varphi(M(x, y))$ and $\varphi(u) \leq f(u)$ for all $u \in I$. Hence, by the M-affinity of φ and the increasing monotonicity of M, we have $f(M(x, y)) = \varphi(M(x, y)) = M(\varphi(x), \varphi(y)) \leq M(f(x), f(y))$. □

Remark 11. Given a continuous and strictly monotonic function $\beta : J \to \mathbb{R}$ and $w \in (0, 1)$, denote by $M_\beta : J^2 \to J$ the quasi-arithmetic mean

$$M_\beta(x, y) = \beta^{-1} (w\beta(x) + (1 - w)\beta(y)), \ x, y \in J.$$

Suppose that a function $f : I \to J$ is measurable (or the closure of the graph of f does not have interior points). Then, obviously,

1. if β is increasing, then f is M_β-convex iff the function $\beta \circ f \circ \beta^{-1}$ is convex,
2. if β is decreasing, then f is M_β-convex iff the function $\beta \circ f \circ \beta^{-1}$ is concave.

Now it is easy to see that

- f is M_β-convex iff it is M_β-affinely convex.

We obtain the following an immediate consequence of Theorem 1.

Proposition 1. *Let $M : (0, \infty)^2 \to (0, \infty)$ be a strict homogeneous non power mean. If $h := M(\cdot, 1)$ is twice continuously differentiable and $0 \neq h'(1) \neq 1$, then the following conditions are equivalent:*

1. *a function* $f : (0,\infty) \to (0,\infty)$ *is* M*-affinely convex.*
2. f *is either constant or linear or* $f(x) = \max(a, cx),\ x \in (0,\infty)$, *for some* $a, c > 0$.

Example 1. The logarithmic mean $L : (0,\infty)^2 \to (0,\infty)$,

$$L(x,y) := \begin{cases} \frac{x-y}{\log x - \log y} & \text{for } x \neq y \\ x & \text{for } x = y \end{cases}$$

is homogeneous and non-power. By Theorem 1 (cf. also [11]), every continuous at a point L-affine function is either constant or linear. Since the function $\exp|_{(0,\infty)}$ is L-convex (cf. 10]), taking into account the above Proposition, we infer that the notions of L-convexity and L-affine convexity are not equivalent.

7 Open Problems and Final Remarks

In Theorems 1-4 we assume twice differentiability of the mean. It is an open question wether these results remain true under weaker regularity conditions. Let us mention that in a recent paper [3], J. Aczél, R. Duncan Luce motivated by some problems in utility theory and psychophysics, considered the functional equation $H(K(s,t)) = L(H(s), H(t)),\ s \geq t \geq 1$, where K and L are homogeneous functions, which is more general than (1). Assuming that H is twice differentiable and strictly increasing, and the functions K and L are twice differentiable, the authors determine the forms of H and K. According to a personal communication, this functional equation will be also considered in [4].

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