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ON SOME POINTWISE DEFECTS OF PROPERTIES IN REAL ANALYSIS

Abstract

In this paper we continue the research in Real Analysis, in the spirit of the very recent book [1]. Firstly, as a refinement of the global defect of integrability introduced in [1], § 5.1, we consider here a pointwise defect of integrability and study its properties and connections with the pointwise defect of continuity already introduced in [1], § 5.1. Secondly, as refinements of the global defects of monotonicity and of convexity introduced in the same book [1], § 5.2, we consider and study pointwise variants.

1 Introduction

Let U be an abstract set and P a given property of some elements in U. Evidently P divides U into two disjoint sets:

$$U_P = \{x \in U; x \text{ has the property } P\}$$

and

$$U_{\overline{P}} = \{x \in U; x \text{ does not satisfy the property } P\}.$$

A tool of investigation of U_P and $U_{\overline{P}}$ might be the introduction (not necessarily in an unique way) of a quantity $E(x) \in \mathbb{R}$, defined for all $x \in U$, such that

$$x \in U_P \text{ if and only if } E(x) = 0.$$

In this way, for $x \in U_{\overline{P}}$ the quantity $|E(x)|$ can be considered to measure the "deviation" of x from the property P and can be called the defect of

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property P, at x. Of course the notion of defect is of interest only when it has appropriate analytic properties.

In the very recent book [1] we have studied this idea in Set Theory, Topology, Measure Theory, Real Analysis, Functional Analysis, Complex Analysis, Algebra, Geometry, Number Theory and Fuzzy Logic.

Concerning continuity, differentiability, integrability, monotonicity and convexity of real functions of one real variable, the following concepts were studied.

Definition 1.1. (i) (see, for example [1], p. 185, Definitions 5.1, 5.2) Let $f : E \to \mathbb{R}$ and $x_0 \in E \subset \mathbb{R}$. The defect of continuity of f at x_0 is the quantity

$$d_{cont}\left(f\right)\left(x_0\right) = \inf\left\{\delta\left[f\left(V \cap E\right)\right]; V \in \mathcal{V}\left(x_0\right)\right\},$$

where $\mathcal{V}\left(x_0\right)$ denotes the class of all neighborhoods of x_0 and

$$\delta\left[A\right] = \sup\left\{\left|a_1 - a_2\right|; a_1, a_2 \in A\right\}$$

denotes the diameter of the set $A \subset \mathbb{R}$.
Of course, $d_{cont}\left(f\right)\left(x_0\right)$ is a very old concept in analysis, usually called modulus of oscillation of f at x_0, which was introduced and studied by Bernhard Riemann and Paul Dubois-Reymond. Here we call it defect of continuity only for the homogeneity of language.

(ii) (see, for example [1], p. 189, Definition 5.4) Let $f : [a, b] \to \mathbb{R}$ and $x_0 \in [a, b]$. The defect of differentiability of f at x_0 is the quantity

$$d_{dif}\left(f\right)\left(x_0\right) = \inf\left\{\delta\left[F\left(V \cap [a, b] \setminus \{x_0\}\right)\right]; V \in \mathcal{V}\left(x_0\right)\right\},$$

where $F : [a, b] \setminus \{x_0\} \to \mathbb{R}$ is defined by $F\left(x\right) = \frac{f(x) - f(x_0)}{x - x_0}$.

(iii) (see, for example [1], p. 190, Definition 5.6) Let $f : [a, b] \to \mathbb{R}$ be bounded. The defect of Riemann integrability of f on the interval $[a, b]$ is the quantity

$$d_{int}\left(f\right)\left([a, b]\right) = \overline{\int_a^b} f\left(x\right)\, dx - \underline{\int_a^b} f\left(x\right)\, dx,$$

where $\overline{\int_a^b}$ and $\underline{\int_a^b}$ denote the upper and lower Darboux integrals, respectively. We note that $d_{int}\left(f\right)\left([a, b]\right)$ is only a simple rewording of a nineteenth century criterion of Darboux.

(iv) (see, for example [1], p. 194, Definition 5.7) Let $f : E \to \mathbb{R}$ and $E \subset \mathbb{R}$. The defect of monotonicity of f on E is the quantity

$$d_M\left(f\right)\left(E\right) = \sup\left\{\left|f\left(x_1\right) - f\left(x\right)\right| + \left|f\left(x_2\right) - f\left(x\right)\right| - \left|f\left(x_1\right) - f\left(x_2\right)\right|;\right.$$
$$\left. x_1, x, x_2 \in E, x_1 \leq x \leq x_2\right\}.$$

(v) (see, for example [1], p. 199) Let $f : [a, b] \to \mathbb{R}$. The defect of convexity of f on $[a, b]$ is the quantity

$$d_{conv}(f)([a,b]) = \sup\{f(\lambda x + (1-\lambda)y) - (\lambda f(x) + (1-\lambda)f(y)); \\ \lambda \in [0,1], x, y \in [a,b]\}.$$

Remark 1.1. It is easily seen that while the defects in Definition 1.1, (i) and (ii) are pointwise ones, those in Definition 1.1, (iii), (iv) and (v) are global ones.

The main aim of this paper is to refine the above global defects by defining and studying their pointwise variants.

Section 2 deals with the pointwise defect of integrability while in Section 3 we consider pointwise defects of monotonicity. Unlike the global defect of monotonicity in Definition 1.1, (iv), one can use the pointwise defects to characterize increasing and decreasing monotonicities. Section 4 deals with pointwise defects of convexity and Section 5 contains two simple applications to the best approximation problem. At the end some open questions are presented in Section 6.

2 Pointwise Defect of Integrability

A pointwise version of the concept in Definition 1.1, (iii), can be defined as follows.

Definition 2.1. Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. The (pointwise) defect of Riemann integrability of f at $x_0 \in (a, b)$, is the quantity

$$d_{int}(f)(x_0) = \limsup_{h \searrow 0} \left\{ \frac{1}{2h} \left(\overline{\int}_{x_0-h}^{x_0+h} f(x)\, dx - \underline{\int}_{x_0-h}^{x_0+h} f(x)\, dx \right) \right\}.$$

If $x_0 = a$, then

$$d_{int}(f)(x_0) = \limsup_{h \searrow 0} \left\{ \frac{1}{h} \left(\overline{\int}_{a}^{a+h} f(x)\, dx - \underline{\int}_{a}^{a+h} f(x)\, dx \right) \right\}$$

and if $x_0 = b$, then

$$d_{int}(f)(x_0) = \limsup_{h \searrow 0} \left\{ \frac{1}{h} \left(\overline{\int}_{b-h}^{b} f(x)\, dx - \underline{\int}_{b-h}^{b} f(x)\, dx \right) \right\}.$$

Remark 2.1. By the definition of $\limsup_{h\searrow 0}$, we can write (if e.g. $x_0 \in (a,b)$)

$$d_{int}(f)(x_0) = \inf_{\delta>0}\left\{\sup_{h\in(0,\delta)}\left\{\frac{1}{2h}\left(\overline{\int}_{x_0-h}^{x_0+h} f(x)\,dx - \underline{\int}_{x_0-h}^{x_0+h} f(x)\,dx\right)\right\}\right\}$$
$$= \lim_{\delta\searrow 0}\left\{\sup_{h\in(0,\delta)}\left\{\frac{1}{2h}\left(\overline{\int}_{x_0-h}^{x_0+h} f(x)\ dx - \underline{\int}_{x_0-h}^{x_0+h} f(x)\ dx\right)\right\}\right\}.$$

The following properties hold.

Theorem 2.1. *Let $f, g : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$ and $x_0 \in [a,b]$.*

(i) $0 \le d_{int}(f)(x_0) \le M - m$, *where* $M = \sup\{f(x); x \in [a,b]\}$ *and* $m = \inf\{f(x); x \in [a,b]\}$.

(ii) If f is locally Riemann integrable on x_0 (i.e., integrable on a subinterval containing x_0), then $d_{int}(f)(x_0) = 0$. If f is Riemann integrable on $[a,b]$, then $d_{int}(f)(x_0) = 0$, for all $x_0 \in [a,b]$.

(iii) $d_{int}(f+g)(x_0) \le d_{int}(f)(x_0) + d_{int}(g)(x_0)$.

(iv) $d_{int}(\lambda f)(x_0) = |\lambda|\, d_{int}(f)(x_0), \forall \lambda \in \mathbb{R}$.

PROOF. (i) It is immediate.

(ii) It is also immediate because the Riemann integrability implies the equality between the lower and upper Darboux integrals.

(iii) The properties of subadditivity of upper Darboux integral, superadditivity of lower Darboux integral and subadditivity of upper limit prove (iii).

(iv) The properties

$$\underline{\int}_c^d \lambda f(x)\ dx = \lambda \underline{\int}_c^d f(x)\ dx, \overline{\int}_c^d \lambda f(x)\ dx = \lambda \overline{\int}_c^d f(x)\ dx, \forall \lambda > 0,$$
$$\underline{\int}_c^d \lambda f(x)\ dx = \lambda \overline{\int}_c^d f(x)\ dx, \overline{\int}_c^d \lambda f(x)\ dx = \lambda \underline{\int}_c^d f(x)\ dx, \forall \lambda < 0,$$

hold for every $c, d \in \mathbb{R}, c < d$, and the positive homogeneity of upper limits imply the equality. □

Example 2.1. *For the Dirichlet function $f : [0,1] \to \mathbb{R}$, defined by $f(x) = 0$ if x is a rational number and $f(x) = 1$, otherwise, it easily follows that $d_{int}(f)(x_0) = 1$, for all $x_0 \in [0,1]$.*

In what follows, consider the well-known Baire functions

$$M(x) = \lim_{\delta \searrow 0} M_\delta(x) \text{ and } m(x) = \lim_{\delta \searrow 0} m_\delta(x),$$

where

$$M_\delta(x) = \sup\{f(t); t \in [a,b] \cap (x-\delta, x+\delta)\},$$
$$m_\delta(x) = \inf\{f(t); t \in [a,b] \cap (x-\delta, x+\delta)\}.$$

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be bounded on $[a,b]$. Then, for all $x_0 \in (a,b)$, we have*

$$d_{int}(f)(x_0) = \lim_{\delta \searrow 0} \left\{ \sup_{h \in (0,\delta)} \left\{ \frac{1}{2h} (\mathcal{L}) \int_{x_0-h}^{x_0+h} d_{cont}(f)(x)\, dx \right\} \right\}, \quad (1)$$

where $(\mathcal{L}) \int$ denotes the Lebesgue integral (if $x_0 = a$ and $x_0 = b$ in (1) appear $(\mathcal{L}) \int_a^{a+h}$ and $(\mathcal{L}) \int_{b-h}^{b}$, respectively).

Proof. By e.g. [5], p. 175–176, it follows that the Baire functions $M(x), m(x)$ are Lebesgue measurable and that

$$\overline{\int}_{x_0-h}^{x_0+h} f(x)\, dx = (\mathcal{L}) \int_{x_0-h}^{x_0+h} M(x)\, dx,$$
$$\underline{\int}_{x_0-h}^{x_0+h} f(x)\, dx = (\mathcal{L}) \int_{x_0-h}^{x_0+h} m(x)\, dx.$$

Consequently, by the above Remark 2.1 we get

$$d_{int}(f)(x_0) = \lim_{\delta \searrow 0} \left\{ \sup_{h \in (0,\delta)} \left\{ \frac{1}{2h} (\mathcal{L}) \int_{x_0-h}^{x_0+h} (M(x) - m(x))\, dx \right\} \right\}.$$

But by e.g. [7], p. 165 we have

$$\begin{aligned} d_{cont}(f)(x_0) &= \lim_{\delta \searrow 0} \{\sup(f([a,b] \cap (x_0-\delta, x_0+\delta))) \\ &\quad - \inf(f([a,b] \cap (x_0-\delta, x_0+\delta)))\} \\ &= \lim_{\delta \searrow 0} \{M_\delta(x_0) - m_\delta(x_0)\} \\ &= \lim_{\delta \searrow 0} M_\delta(x_0) - \lim_{\delta \searrow 0} m_\delta(x_0) \\ &= M(x_0) - m(x_0), \end{aligned}$$

for all $x_0 \in [a,b]$, which immediately proves the theorem. □

Corollary 2.3. *Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if $d_{int}(f)(x_0) = 0$, for all $x_0 \in [a, b]$.*

PROOF. If f is Riemann integrable on $[a, b]$, then by Theorem 2.1, (ii) we get $d_{int}(f)(x_0) = 0$, for all $x_0 \in [a, b]$.

Now, suppose that $d_{int}(f)(x_0) = 0$, for all $x_0 \in [a, b]$. By (1) we get

$$d_{int}(f)(x_0) \geq \lim_{\delta \searrow 0} \frac{1}{2\delta} (\mathcal{L}) \int_{x_0-\delta}^{x_0+\delta} d_{cont}(f)(x)\, dx,$$

which implies $\lim_{\delta \searrow 0} \frac{1}{2\delta} (\mathcal{L}) \int_{x_0-\delta}^{x_0+\delta} d_{cont}(f)(x)\, dx = 0$, for all $x_0 \in [a, b]$. Since the integrand $d_{cont}(f)(x) = M(x) - m(x)$ is nonnegative, we can deduce that $\forall x_0 \in [a, b]$, $\forall \varepsilon > 0$, $\exists \delta_{x_0,\varepsilon}$ such that for all $a \leq x_0 \leq b$ and $0 < t < \delta_{x_0,\varepsilon}$, if $[x_0, x_0 + t] \subset [a, b]$, then

$$(\mathcal{L}) \int_{x_0}^{x_0+t} (M(x) - m(x))\, dx < \varepsilon t$$

while if $[x_0 - t, x_0] \subset [a, b]$, then

$$(\mathcal{L}) \int_{x_0-t}^{x_0} (M(x) - m(x))\, dx < \varepsilon t.$$

For every $\varepsilon > 0$, define F_ε to be the collection of all intervals $[x_0, x_0+t] \subset [a, b]$ and $[x_0 - t, x_0] \subset [a, b]$ for $0 < t < \delta_{x_0,\varepsilon}$. Applying Cousin's Lemma (see e.g. [2], p. 9) there exists a partition of $[a, b]$,

$$[a, b] = [a_0, a_1] \cup [a_1, a_2] \cup \cdots \cup [a_{n-2}, a_{n-1}] \cup [a_{n-1}, a_n],$$

$a_0 = a$, $a_n = b$, where each $[a_i, a_{i+1}] \in F_\varepsilon$, $i = \overline{0, n-1}$. Adding term by term the inequalities for the subintervals of the partition, we obtain

$$(\mathcal{L}) \int_a^b (M(x) - m(x))\, dx < \varepsilon(b - a), \forall \varepsilon > 0.$$

We get $(\mathcal{L}) \int_a^b (M(x) - m(x))\, dx = 0$ and because $0 \leq M(x) - m(x)$, it follows $M(x) - m(x) = 0$, a.e. $x \in [a, b]$. Consequently by e.g. [5], p. 172, Theorem 1, it follows that f is almost everywhere continuous on $[a, b]$ and therefore it is Riemann integrable on $[a, b]$. □

Remark 2.2. By Theorem 2.2 and Corollary 2.3, it follows that formula (1) can be considered in fact a generalization of the well-known result which states that a bounded function f is Riemann integrable on $[a, b]$, if and only if it is almost everywhere continuous on $[a, b]$. Indeed, this immediately follows from [1], p. 186, Theorem 5.1, (i), which states that f is continuous on x_0 if and only if $d_{cont}(f)(x_0) = 0$.

3 Pointwise Defects of Monotonicity

The pointwise variant of above Definition 1.1, *(iv)*, is the following.

Definition 3.1. Let $f:(a,b)\rightarrow\mathbb{R}$ be a non-constant function and $x_0\in(a,b)$. The defect of monotonicity of f on x_0 is the quantity

$$d_M\left(f\right)\left(x_0\right)=\limsup_{\varepsilon_i\searrow 0,i\in\{1,2\}}E_{f,x_0}\left(\varepsilon_1,\varepsilon_2\right),$$

where $E_{f,x_0}\left(\varepsilon_1,\varepsilon_2\right)$ is the fraction

$$\frac{\left|f\left(x_0-\varepsilon_1\right)-f\left(x_0\right)\right|+\left|f\left(x_0+\varepsilon_2\right)-f\left(x_0\right)\right|-\left|f\left(x_0-\varepsilon_1\right)-f\left(x_0+\varepsilon_2\right)\right|}{\left|f\left(x_0-\varepsilon_1\right)-f\left(x_0+\varepsilon_2\right)\right|}.$$

If f is constant, then by definition we take $d_M\left(f\right)\left(x_0\right)=0,\forall x_0\in(a,b)$.

Remark 3.1. If f is a non-constant function, then obviously we can write

$$d_M\left(f\right)\left(x_0\right)=\lim_{\delta_i\searrow 0}\left\{\sup_{\varepsilon_i\in(0,\delta_i),i\in\{1,2\}}E_{f,x_0}\left(\varepsilon_1,\varepsilon_2\right)\right\}.$$

The following properties can easily be proved.

Theorem 3.1. *Let $f:(a,b)\rightarrow\mathbb{R}$ and $x_0\in(a,b)$.*

(i) $d_M\left(f\right)\left(x_0\right)\geq 0$.

(ii) *If f is monotonous in a neighborhood of x_0, then $d_M\left(f\right)\left(x_0\right)=0$.*

(iii) $d_M\left(\lambda f\right)\left(x_0\right)=d_M\left(f\right)\left(x_0\right),\forall\lambda\in\mathbb{R}\setminus\{0\}$.

(iv) *If $a=-b,b>0$ and $f\left(-x\right)=f\left(x\right),\forall x\in(a,b)$ or $f\left(-x\right)=-f\left(x\right)$, $\forall x\in(a,b)$, then $d_M\left(f\right)\left(-x_0\right)=d_M\left(f\right)\left(x_0\right)$.*

(v) $d_M\left(1-f\right)\left(x_0\right)=d_M\left(f\right)\left(x_0\right)$.

Lemma 3.2. *Let $f:(a,b)\rightarrow\mathbb{R}$ be locally continuous at $x_0\in(a,b)$ (i.e., $\exists\,\varepsilon_1,\varepsilon_2>0,I=(x_0-\varepsilon_1,x_0+\varepsilon_2)\subset(a,b)$ such that f is continuous on I). If x_0 is a locally strict extremum point of f, then $d_M\left(f\right)\left(x_0\right)=+\infty$.*

Proof. From the continuity of f on $I=(x_0-\varepsilon_1,x_0+\varepsilon_2)$, there exist $\varepsilon_1^{(n)}\searrow 0,\varepsilon_2^{(n)}\searrow 0$, such that $f\left(x_0-\varepsilon_1^{(n)}\right)=f\left(x_0-\varepsilon_2^{(n)}\right)\neq f\left(x_0\right),\forall n\in\mathbb{N}$. It follows that for all $\delta_1,\delta_2>0$, sufficiently small, we have

$$\sup_{\varepsilon_i\in(0,\delta_i),i\in\{1,2\}}\left\{\frac{\left|f\left(x_0-\varepsilon_1\right)-f\left(x_0\right)\right|+\left|f\left(x_0+\varepsilon_2\right)-f\left(x_0\right)\right|}{\left|f\left(x_0-\varepsilon_1\right)-f\left(x_0+\varepsilon_2\right)\right|}\right.$$

$$\left. - \frac{|f(x_0-\varepsilon_1)-f(x_0+\varepsilon_2)|}{|f(x_0-\varepsilon_1)-f(x_0+\varepsilon_2)|} \right\} = +\infty,$$

which implies $d_M(f)(x_0) = +\infty$. □

Theorem 3.3. *Let $f:(a,b)\to\mathbb{R}$ be continuous on (a,b), such that f is not constant on some subintervals of (a,b). Then f is monotonic on (a,b) if and only if $d_M(f)(x_0)=0$, for all $x_0\in(a,b)$.*

PROOF. If f is monotonic on (a,b), then it is immediate that $d_M(f)(x_0) = 0, \forall x_0 \in (a,b)$. Conversely, suppose that $d_M(f)(x_0)=0, \forall x_0\in(a,b)$, but f would be not monotonic on (a,b). Then there exist $x_1,x_2,x_3, a<x_1<x_2<x_3<b$, satisfying:

(*i*) $f(x_2)<f(x_1), f(x_2)<f(x_3)$

or

(*ii*) $f(x_2)>f(x_1), f(x_2)>f(x_3)$.

Case (*i*). Suppose, for example, $f(x_1)\le f(x_3)$ (the subcase $f(x_1)>f(x_3)$ is similar). It follows that f has in (x_1,x_3) a (locally) strict minimum point x^*, which by Lemma 3.2 implies $d_M(f)(x^*)=+\infty$, a contradiction.

Case (*ii*). Similarly, it follows that f has in (x_1,x_3) a strict maximum point x^*; i.e., we again get the contradiction $d_M(f)(x^*)=+\infty$. □

Remark 3.2. The condition that f cannot be constant on some subintervals of (a,b) is necessary. Indeed, if we define $f:(0,1)\to\mathbb{R}$ as the continuous polygonal line passing through the points $(0,1), \left(\frac{1}{3},\frac{1}{2}\right), \left(\frac{2}{3},\frac{1}{2}\right)$ and $(1,1)$, a simple calculation show us that $d_M(f)(x_0)=0, \forall x_0\in(0,1)$, while f is not monotonic on $(0,1)$.

Example 3.1. In [3], p. 66, the following example of nowhere monotone function on $(0,1)$ is given. Let $f(x)=x$ if x is rational and $f(x)=1-x$ if x is irrational. Let $x_0\in\left(0,\frac{1}{2}\right)\cap\mathbb{Q}$ and $\left(\varepsilon_1^{(n)}\right)_{n\in\mathbb{N}}, \left(\varepsilon_2^{(n)}\right)_{n\in\mathbb{N}}$ be two sequences such that $x_0-\varepsilon_1^{(n)}\in(0,1)\cap\mathbb{R}\backslash\mathbb{Q}, x_0+\varepsilon_2^{(n)}\in(0,1)\cap\mathbb{R}\backslash\mathbb{Q}, \forall n\in\mathbb{N}, \varepsilon_i^{(n)}\searrow 0, n\to\infty, i\in\{1,2\}$ and $\varepsilon_2^{(n)}<1-2x_0, \forall n\in\mathbb{N}$. We get that $E_{f,x_0}\left(\varepsilon_1^{(n)},\varepsilon_2^{(n)}\right)$ is the expression

$$\frac{\left|1-x_0+\varepsilon_1^{(n)}-x_0\right|+\left|1-x_0-\varepsilon_2^{(n)}-x_0\right|-\left|1-x_0+\varepsilon_1^{(n)}-1+x_0+\varepsilon_2^{(n)}\right|}{\left|1-x_0+\varepsilon_1^{(n)}-1+x_0+\varepsilon_2^{(n)}\right|}$$

$$=\frac{1-2x_0+\varepsilon_1^{(n)}+1-2x_0-\varepsilon_2^{(n)}-\varepsilon_1^{(n)}-\varepsilon_2^{(n)}}{\varepsilon_1^{(n)}+\varepsilon_2^{(n)}}=\frac{2(1-2x_0)-2\varepsilon_2^{(n)}}{\varepsilon_1^{(n)}+\varepsilon_2^{(n)}}.$$

Passing to limit with $n \to \infty$ and taking into account Definition 3.1 we obtain $d_M(f)(x_0) = +\infty$.

Let $x_0 \in \left(\frac{1}{2}, 1\right) \cap \mathbb{Q}$ and $\left(\varepsilon_1^{(n)}\right)_{n\in\mathbb{N}}, \left(\varepsilon_2^{(n)}\right)_{n\in\mathbb{N}}$ two sequences as above, but $\varepsilon_1^{(n)} < 2x_0 - 1, \forall n \in \mathbb{N}$. We get

$$E_{f,x_0}\left(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}\right) =$$

$$\frac{\left|1 - x_0 + \varepsilon_1^{(n)} - x_0\right| + \left|1 - x_0 - \varepsilon_2^{(n)} - x_0\right| - \left|1 - x_0 + \varepsilon_1^{(n)} - 1 + x_0 + \varepsilon_2^{(n)}\right|}{\left|1 - x_0 + \varepsilon_1^{(n)} - 1 + x_0 + \varepsilon_2^{(n)}\right|}$$

$$= \frac{2x_0 - 1 - \varepsilon_1^{(n)} + 2x_0 - 1 + \varepsilon_2^{(n)} - \varepsilon_1^{(n)} - \varepsilon_2^{(n)}}{\varepsilon_1^{(n)} + \varepsilon_2^{(n)}} = \frac{2(2x_0 - 1) - 2\varepsilon_1^{(n)}}{\varepsilon_1^{(n)} + \varepsilon_2^{(n)}}.$$

As above we obtain $d_M(f)(x_0) = +\infty$.

If $x_0 = \frac{1}{2}$ and $\left(\varepsilon_1^{(n)}\right)_{n\in\mathbb{N}}, \left(\varepsilon_2^{(n)}\right)_{n\in\mathbb{N}}$ are two sequences such that $x_0 - \varepsilon_1^{(n)} \in (0,1) \cap \mathbb{Q}, x_0 + \varepsilon_2^{(n)} \in (0,1) \cap \mathbb{R}\backslash\mathbb{Q}, \forall n \in \mathbb{N}, \varepsilon_i^{(n)} \searrow 0, n \to \infty, i \in \{1, 2\}$, we get

$$E_{f,x_0}\left(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}\right) = \frac{\varepsilon_1^{(n)} + \varepsilon_2^{(n)} - \left|\varepsilon_1^{(n)} - \varepsilon_2^{(n)}\right|}{\left|\varepsilon_1^{(n)} - \varepsilon_2^{(n)}\right|}.$$

Therefore $d_M(f)(x_0) = +\infty$ (for example, if $\varepsilon_1^{(n)} = \frac{1}{n}, \varepsilon_2^{(n)} = \frac{1}{\sqrt{n^2+1}}$, then $\lim_{n\to\infty} E_{f,x_0}\left(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}\right) = \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+1}-n} = +\infty$).

Similarly, if $x_0 \in (0,1) \cap \mathbb{R}\backslash\mathbb{Q}$, then $d_M(f)(x_0) = +\infty$.

More refined pointwise defects of monotonicities than those in Definition 3.1, can be introduced in such a way that can characterize the sense of monotonicity. We begin with this definition.

Definition 3.2. ([7], p. 119) Let $f : (a,b) \to \mathbb{R}$ and $x_0 \in (a,b)$. We say that f is (pointwise) increasing at x_0 if $\exists\, \delta > 0$ (sufficiently small) such that $\frac{f(x)-f(x_0)}{x-x_0} \geq 0, \forall x \neq x_0, |x - x_0| < \delta$. Analogously, f is called decreasing on x_0 if $\exists\, \delta > 0$ such that $\frac{f(x)-f(x_0)}{x-x_0} \leq 0, \forall x \neq x_0, |x - x_0| < \delta$.

Theorem 3.4. *([7], p. 120) $f : (a,b) \to \mathbb{R}$ is increasing (decreasing) on (a,b) if and only if f is increasing (decreasing) at each $x_0 \in (a,b)$ (in the sense of Definition 3.2).*

The pointwise deviations from the monotonicities in Definition 3.2 can be measured by the following.

Definition 3.3. Let $f:(a,b)\to\mathbb{R}$ and $x_0\in(a,b)$. The defect of increasing monotonicity of f on x_0 is defined by

$$d_{IM}(f)(x_0)=\max\left\{0,-\overline{D}(f)(x_0)\right\},$$

where $\overline{D}(f)(x_0)=\limsup_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$.

Analogously, the defect of decreasing monotonicity of f on x_0 is defined by

$$d_{DM}(f)(x_0)=\max\left\{0,\underline{D}(f)(x_0)\right\},$$

where $\underline{D}(f)(x_0)=\liminf_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$.

Theorem 3.5. *Let $f:(a,b)\to\mathbb{R}$.*

(i) *f continuous on (a,b) is increasing on (a,b) if and only if $d_{IM}(f)(x_0)=0$, for all $x_0\in(a,b)$. f continuous on (a,b) is decreasing on (a,b) if and only if $d_{DM}(f)(x_0)=0$, for all $x_0\in(a,b)$.*

(ii) *If $f\in C^1(a,b)$, then $d_{IM}(f)(x_0)=\max\{0,-f'(x_0)\}$ and $d_{DM}(f)(x_0)=\max\{0,f'(x_0)\}$, for all $x_0\in(a,b)$.*

(iii) *If f is increasing on (a,b), then $d_{DM}(f)(x_0)=\underline{D}(f)(x_0)$, for all $x_0\in(a,b)$. If f is decreasing on (a,b), then $d_{IM}(f)(x_0)=-\overline{D}(f)(x_0)$, for all $x_0\in(a,b)$.*

(iv) *If $g\in C^1(a,b), f\in C^1(g(a,b))$, then*

$$\begin{aligned}d_{IM}(f\circ g)(x_0)&=|f'(g(x_0))|\,d_{IM}(g)(x_0)+g'(x_0)\,d_{IM}(f)(g(x_0))\\&=|f'(g(x_0))|\,d_{DM}(g)(x_0)-g'(x_0)\,d_{DM}(f)(g(x_0))\end{aligned}$$

and

$$\begin{aligned}d_{DM}(f\circ g)(x_0)&=|f'(g(x_0))|\,d_{IM}(g)(x_0)+g'(x_0)\,d_{DM}(f)(g(x_0))\\&=|f'(g(x_0))|\,d_{DM}(g)(x_0)-g'(x_0)\,d_{IM}(f)(g(x_0)),\end{aligned}$$

for all $x_0\in(a,b)$.

(v) *If $f\in C^1(a,b), f$ is invertible and $f'(x)\neq 0,\forall x\in(a,b)$, then*

$$d_{IM}\left(f^{-1}\right)(y_0)=-\frac{d_{IM}(f)(x_0)}{f'(x_0)\,|f'(x_0)|}=\frac{f'(x_0)-d_{DM}(f)(x_0)}{f'(x_0)\,|f'(x_0)|}$$

and

$$d_{DM}\left(f^{-1}\right)(y_0) = \frac{d_{DM}(f)(x_0)}{f'(x_0)\,|f'(x_0)|} = \frac{f'(x_0) + d_{IM}(f)(x_0)}{f'(x_0)\,|f'(x_0)|},$$

for every $y_0 = f(x_0)$.

(vi) *If* $f \in C^1(A, B)$, *then for all* $a, b \in (A, B)$, $a < b$, *there exist* $\alpha, \beta \in (a, b)$ *such that*

$$-d_{IM}(f)(\alpha) \leq \frac{f(b) - f(a)}{b - a} \leq d_{DM}(f)(\beta).$$

Proof. (i) If f is increasing on (a, b), then by Theorem 3.4 it follows that for every $x_0 \in (a, b)$, $\frac{f(x)-f(x_0)}{x-x_0} \geq 0$, for all $x \neq x_0, x$ sufficiently close to x_0. This implies $\overline{D}(f)(x_0) \geq 0$ and therefore $d_{IM}(f)(x_0) = \max\left\{0, -\overline{D}(f)(x_0)\right\} = 0$, for all $x_0 \in (a, b)$. Conversely, by $d_{IM}(f)(x_0) = 0$ we get $-\overline{D}(f)(x_0) \leq 0$; i.e., $\overline{D}(f)(x_0) \geq 0, \forall x_0 \in (a, b)$, which by a well-known result (see e.g. [4], p. 222) implies that f is increasing on (a, b). The proof of the second statement is similar.

(ii) It is immediate.

(iii) $\frac{f(x)-f(x_0)}{x-x_0} \geq 0, \forall x, x_0 \in (a, b), x \neq x_0$ implies $\underline{D}(f)(x_0) \geq 0, \forall x_0 \in (a, b)$. Therefore $d_{DM}(f)(x_0) = \underline{D}(f)(x_0), \forall x_0 \in (a, b)$. If f is decreasing, then the proof is similar.

(iv) Because the above property (ii) implies $d_{IM}(h)(x_0) = \frac{|h'(x_0)|-h'(x_0)}{2}$ and $d_{DM}(h)(x_0) = \frac{|h'(x_0)|+h'(x_0)}{2}$, for every function $h \in C^1(a, b)$ and $x_0 \in (a, b)$, we have

$$\begin{aligned}
d_{IM}(f \circ g)(x_0) &= \frac{|f'(g(x_0))|\,|g'(x_0)| - f'(g(x_0))\,g'(x_0)}{2} \\
&= \frac{|f'(g(x_0))|\left(|g'(x_0)| - g'(x_0)\right) + g'(x_0)\left(|f'(g(x_0))| - f'(g(x_0))\right)}{2} \\
&= |f'(g(x_0))|\,d_{IM}(g)(x_0) + g'(x_0)\,d_{IM}(f)(g(x_0)).
\end{aligned}$$

or

$$\begin{aligned}
d_{IM}(f \circ g)(x_0) &= \frac{|f'(g(x_0))|\,|g'(x_0)| - f'(g(x_0))\,g'(x_0)}{2} \\
&= \frac{|f'(g(x_0))|\left(|g'(x_0)| + g'(x_0)\right) - g'(x_0)\left(|f'(g(x_0))| + f'(g(x_0))\right)}{2} \\
&= |f'(g(x_0))|\,d_{DM}(g)(x_0) - g'(x_0)\,d_{DM}(f)(g(x_0)).
\end{aligned}$$

The proof of the second part is analogous.

(v) Replacing g with f^{-1} in the first property (iv) we obtain

$$\begin{aligned}0 =&d_{IM}\left(1_{(a,b)}\right)(y_0) = \left|f'\left(f^{-1}(y_0)\right)\right| d_{IM}\left(f^{-1}\right)(y_0)\\ &+\left(f^{-1}\right)'(y_0)\, d_{IM}(f)\left(f^{-1}(y_0)\right)\end{aligned}$$

and

$$\begin{aligned}0 =&d_{IM}\left(1_{(a,b)}\right)(y_0) = \left|f'\left(f^{-1}(y_0)\right)\right| d_{DM}\left(f^{-1}\right)(y_0)\\ &-\left(f^{-1}\right)'(y_0)\, d_{DM}(f)\left(f^{-1}(y_0)\right),\end{aligned}$$

where $1_{(a,b)}(x) = x$, for all $x \in (a,b)$. Denoting $y_0 = f(x_0)$ the first equality implies $d_{IM}\left(f^{-1}\right)(y_0) = -\dfrac{d_{IM}(f)(x_0)}{f'(x_0)\,|f'(x_0)|}$ and the second equality implies $d_{DM}\left(f^{-1}\right)(y_0) = \dfrac{d_{DM}(f)(x_0)}{f'(x_0)\,|f'(x_0)|}$. The proof of the others equalities is similar starting from the property $d_{DM}\left(1_{(a,b)}\right)(y_0) = 1, \forall y_0 \in (a,b)$.

(vi) Because $\frac{|f'(x)|+f'(x)}{2} \geq f'(x), \forall x \in (a,b)$, we get $\int_a^b d_{DM}(f)(x)\,dx \geq f(b) - f(a)$. On the other hand, there exists $\beta \in (a,b)$ such that $\int_a^b d_{DM}(f)(x)\,dx = (b-a)\, d_{DM}(f)(\beta)$. These imply the desired inequality. The proof of the other inequality is similar. □

Example 3.2. If f is not continuous on (a,b), then Theorem 3.5, (i), fails to be valid. Indeed, let $f : (0,1) \to \mathbb{R}$ be defined by $f(x) = 0$ if x is rational and $f(x) = 1$ if x is irrational. If $x_0 \in \mathbb{Q} \cap (0,1)$, then

$$\begin{aligned}&\sup\left\{\frac{f(x)-f(x_0)}{x-x_0}; x \in (x_0-\delta, x_0+\delta) \cap (0,1), x \neq x_0\right\}\\ \geq &\sup\left\{\frac{f(x)}{x-x_0}; x \in (\mathbb{R}\setminus\mathbb{Q}) \cap (x_0-\delta, x_0+\delta) \cap (0,1)\right\} = +\infty, \forall \delta > 0.\end{aligned}$$

Therefore

$$\overline{D}(f)(x_0) = \limsup_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = +\infty.$$

We get $d_{IM}(f)(x_0) = 0$.

If $x_0 \in (\mathbb{R}\setminus\mathbb{Q}) \cap (0,1)$, then

$$\begin{aligned}&\sup\left\{\frac{f(x)-f(x_0)}{x-x_0}; x \in (x_0-\delta, x_0+\delta) \cap (0,1), x \neq x_0\right\}\\ \geq &\sup\left\{\frac{f(x)-1}{x-x_0}; x \in \mathbb{Q} \cap (x_0-\delta, x_0+\delta) \cap (0,1)\right\} = +\infty, \forall \delta > 0.\end{aligned}$$

Therefore

$$\overline{D}(f)(x_0) = \limsup_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = +\infty.$$

We get $d_{IM}(f)(x_0) = 0$.

As above we obtain $\underline{D}(f)(x_0) = -\infty$ and $d_{DM}(f)(x_0) = 0$.

Remark 3.3. The properties in Theorem 3.5, $(iv), (v)$, can be considered generalizations of the well-known results which state that the composition of two increasing (decreasing) functions is also increasing, the composition of an increasing function with a decreasing function is a decreasing function, the inverse of an increasing function is increasing and the inverse of a decreasing function is decreasing.

4 Pointwise Defect of Convexity

A pointwise analogue of the global defect of convexity in Definition 1.1, (v), might be the following.

Definition 4.1. Let $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$. The pointwise defect of convexity of f at x_0 is the quantity

$$d_{conv}(f)(x_0) =$$

$$\limsup_{x_1, x_2 \to x_0} \left\{ \sup_{\lambda \in [0,1]} \frac{f(\lambda x_1 + (1-\lambda) x_2) - \lambda f(x_1) - (1-\lambda) f(x_2)}{(x_1 - x_2)^2} \right\}.$$

Analogously, the pointwise defect of concavity of f on x_0 is the quantity

$$d_{conc}(f)(x_0) =$$

$$\limsup_{x_1, x_2 \to x_0} \left\{ \sup_{\lambda \in [0,1]} \frac{\lambda f(x_1) + (1-\lambda) f(x_2) - f(\lambda x_1 + (1-\lambda) x_2)}{(x_1 - x_2)^2} \right\}.$$

We present properties of these defects.

Theorem 4.1. *Let $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$.*

(i) $d_{conv}(f)(x_0) \geq 0$ *and* $d_{conc}(f)(x_0) \geq 0$.

(ii) If f is convex on (a, b), then $d_{conv}(f)(x_0) = 0$. If f is concave on (a, b), then $d_{conc}(f)(x_0) = 0$.

(iii) *If f is strongly concave on (a,b); i.e., there exists $M>0$ such that*

$$M\lambda(1-\lambda)(x_1-x_2)^2 \le f(\lambda x_1+(1-\lambda)x_2)-\lambda f(x_1)-(1-\lambda)f(x_2), \tag{2}$$

for all $\lambda\in[0,1], x_1,x_2\in[a,b]$, then $d_{conv}(f)(x_0)\ge \frac{\overline{M}}{4}$, where $\overline{M}=\sup\{M; M \text{ verifies } (2)\}$.

(iv) *If f is locally convex on x_0 (i.e., convex in a neighborhood of x_0), then $d_{conv}(f)(x_0)=0$.*

(v) *If $[x_1,\lambda x_1+(1-\lambda)x_2,x_2;f]$ denotes the divided difference, then*

$$d_{conv}(f)(x_0)=\limsup_{x_1,x_2\to x_0}\left\{\sup_{\lambda\in[0,1]} -\lambda(1-\lambda)[x_1,\lambda x_1+(1-\lambda)x_2,x_2;f]\right\}$$

$$d_{conc}(f)(x_0)=\limsup_{x_1,x_2\to x_0}\left\{\sup_{\lambda\in[0,1]} \lambda(1-\lambda)[x_1,\lambda x_1+(1-\lambda)x_2,x_2;f]\right\}.$$

(vi) $d_{conv}(Ax+B)(x_0)=d_{conc}(Ax+B)(x_0)=0, \forall A,B\in\mathbb{R}, \forall x_0\in\mathbb{R}$.

(vii) $d_{conv}(-f)(x_0)=d_{conc}(f)(x_0)$.

(viii) *If $a=-b, b>0$ and $f(-x)=f(x), \forall x\in(a,b)$, then $d_{conv}(f)(-x_0) = d_{conv}(f)(x_0)$ and $d_{conc}(f)(-x_0)=d_{conc}(f)(x_0)$.*

(ix) *If $a=-b, b>0$ and $f(-x)=-f(x), \forall x\in(a,b)$, then $d_{conv}(f)(-x_0) = d_{conc}(f)(x_0)$ and $d_{conc}(f)(-x_0)=d_{conv}(f)(x_0)$.*

PROOF. (i) They are immediate because for $x_1\ne x_2$ and $\lambda=0$ or $\lambda=1$, we get $f(\lambda x_1+(1-\lambda)x_2)-\lambda f(x_1)-(1-\lambda)f(x_2)=0$.

(ii) $f(\lambda x_1+(1-\lambda)x_2)-\lambda f(x_1)-(1-\lambda)f(x_2)\le 0, \forall\lambda\in[0,1], \forall x_1,x_2\in[a,b]$ immediately implies $d_{conv}(f)(x_0)=0, \forall x_0\in(a,b)$. Similarly if f is concave.

$(iii),(iv)$ Are immediate.

(v) Simple calculations show that for $x_1\ne x_2$,

$$\frac{\lambda f(x_1)+(1-\lambda)f(x_2)-f(\lambda x_1+(1-\lambda)x_2)}{(x_1-x_2)^2}=$$

$$\lambda(1-\lambda)[x_1,\lambda x_1+(1-\lambda)x_2,x_2;f].$$

$(vi),(vii)$ Are obvious.

$(viii)$

$$d_{conv}(f)(-x_0)=$$

$$\begin{aligned}
&\limsup_{x_1,x_2\to -x_0}\left\{\sup_{\lambda\in[0,1]}\frac{f\left(\lambda x_1+(1-\lambda)x_2\right)-\lambda f\left(x_1\right)-(1-\lambda)f\left(x_2\right)}{\left(x_1-x_2\right)^2}\right\}\\
&=\limsup_{y_1,y_2\to x_0}\left\{\sup_{\lambda\in[0,1]}\frac{f\left(-\lambda y_1-(1-\lambda)y_2\right)-\lambda f\left(-y_1\right)-(1-\lambda)f\left(-y_2\right)}{\left(-y_1+y_2\right)^2}\right\}\\
&=\limsup_{y_1,y_2\to x_0}\left\{\sup_{\lambda\in[0,1]}\frac{f\left(\lambda y_1+(1-\lambda)y_2\right)-\lambda f\left(y_1\right)-(1-\lambda)f\left(y_2\right)}{\left(-y_1+y_2\right)^2}\right\}\\
&=d_{conv}\left(f\right)\left(x_0\right).
\end{aligned}$$

The proof of the second equality is similar.

(ix) It is similar to $(viii)$. □

Example 4.1. *The function $f\left(x\right)=-x^2$ is strongly concave on $\left(a,b\right), a,b\in\mathbb{R}$, with $M\in\left(0,1\right]$. We obtain $d_{conv}\left(f\right)\left(x_0\right)=\frac{1}{4}, \forall x_0\in\left(a,b\right)$ that is the equality in Theorem 4.1, property (iii).*

Example 4.2. *For $f:\left[-1,1\right]\to\mathbb{R}, f\left(x\right)=\left|x\right|$, we easily get $d_{conv}\left(f\right)\left(x_0\right)=0, \forall x_0\in\left(-1,1\right), d_{conc}\left(f\right)\left(x_0\right)=0, \forall x_0\in\left(-1,0\right)\cup\left(0,1\right)$ and $d_{conc}\left(f\right)\left(0\right)=+\infty$.*

Corollary 4.2. *If $f\in C^2\left(a,b\right)$, then for all $x_0\in\left(a,b\right)$ we have*

$$\begin{aligned}
d_{conv}\left(f\right)\left(x_0\right)&=\max\left\{0,-\frac{f''\left(x_0\right)}{8}\right\},\\
d_{conc}\left(f\right)\left(x_0\right)&=\max\left\{0,\frac{f''\left(x_0\right)}{8}\right\}.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&\sup_{\lambda\in[0,1]}\left\{-\lambda\left(1-\lambda\right)\left[x_1,\lambda x_1+\left(1-\lambda\right)x_2,x_2;f\right]\right\}\\
&=\max\left\{0,\sup_{\lambda\in(0,1)}\lambda\left(1-\lambda\right)\left(-\left[x_1,\lambda x_1+\left(1-\lambda\right)x_2,x_2;f\right]\right)\right\}.
\end{aligned}$$

Without loss of generality, we can suppose $x_1<x_2$. By the mean value theorem, there exists $\xi_\lambda\in\left(x_1,x_2\right)$ with $-\left[x_1,\lambda x_1+\left(1-\lambda\right)x_2,x_2;f\right]=-\frac{f''\left(\xi_\lambda\right)}{2}$, $\forall\lambda\in\left(0,1\right)$, which implies

$$d_{conv}\left(f\right)\left(x_0\right)=\max\left\{0,\limsup_{x_1,x_2\to x_0}\left\{\sup_{\lambda\in(0,1)}\frac{\lambda\left(1-\lambda\right)}{2}\left(-f''\left(\xi_\lambda\right)\right)\right\}\right\}.$$

But

$$\inf_{x\in(x_1,x_2)}\left(-f''(x)\right)\leq -f''(\xi_\lambda)\leq \sup_{x\in(x_1,x_2)}\left(-f''(x)\right), \tag{3}$$

which implies

$$\inf_{x\in(x_1,x_2)}\left(-f''(x)\right)\sup_{\lambda\in(0,1)}\frac{\lambda(1-\lambda)}{2}\leq \sup_{\lambda\in(0,1)}\left\{\frac{\lambda(1-\lambda)}{2}\left(-f''(\xi_\lambda)\right)\right\}.$$

From $\sup_{\lambda\in(0,1)}\frac{\lambda(1-\lambda)}{2}=\frac{1}{8}$, by passing above to $\limsup_{x_1,x_2\to x_0}$ and taking into account the continuity of f'' on (a,b), we get

$$-\frac{f''(x_0)}{8}\leq \limsup_{x_1,x_2\to x_0}\left\{\sup_{\lambda\in(0,1)}\frac{\lambda(1-\lambda)}{2}\left(-f''(\xi_\lambda)\right)\right\}. \tag{4}$$

Concerning $f''(x_0)$ we have three possibilities: (i) $f''(x_0)<0$; (ii) $f''(x_0)>0$; (iii) $f''(x_0)=0$.

Case (i). There exists a neighborhood V_0 of x_0 such that $f''(x)<0, \forall x\in V_0$, which implies $\sup_{x\in(x_1,x_2)}\left(-f''(x)\right)>0$, for all $x_1,x_2\in V_0$. By (3) we obtain

$$\sup_{\lambda\in(0,1)}\left\{\frac{\lambda(1-\lambda)}{2}\left(-f''(\xi_\lambda)\right)\right\}\leq \sup_{x\in(x_1,x_2)}\left(-f''(x)\right)\sup_{\lambda\in(0,1)}\left\{\frac{\lambda(1-\lambda)}{2}\right\},$$

for all $x_1,x_2\in V_0, x_1<x_2$, wherefrom passing to $\limsup_{x_1,x_2\to x_0}$, we get

$$\limsup_{x_1,x_2\to x_0}\left\{\sup_{\lambda\in(0,1)}\frac{\lambda(1-\lambda)}{2}\left(-f''(\xi_\lambda)\right)\right\}\leq -\frac{f''(x_0)}{8}.$$

Combined with (4) it follows that

$$d_{conv}(f)(x_0)=-\frac{f''(x_0)}{8}=\max\left\{0,-\frac{f''(x_0)}{8}\right\}.$$

Case (ii). There exists a neighborhood V_0 of x_0 such that $f''(x)>0, \forall x\in V_0$, which by (3) implies $-f''(\xi_\lambda)<0$, for all $x_1,x_2\in V_0$, and therefore

$$\limsup_{x_1,x_2\to x_0}\left\{\sup_{\lambda\in(0,1)}\frac{\lambda(1-\lambda)}{2}\left(-f''(\xi_\lambda)\right)\right\}\leq 0.$$

As a consequence,

$$d_{conv}(f)(x_0)=0=\max\left\{0,-\frac{f''(x_0)}{8}\right\}.$$

Case (iii). By hypothesis we have $f''(x_0) = 0$. By (4) it follows

$$0 \leq \limsup_{x_1, x_2 \to x_0} \left\{ \sup_{\lambda \in (0,1)} \frac{\lambda(1-\lambda)}{2} \left(-f''(\xi_\lambda)\right) \right\}.$$

Suppose that

$$0 < l = \limsup_{x_1, x_2 \to x_0} \left\{ \sup_{\lambda \in (0,1)} \frac{\lambda(1-\lambda)}{2} \left(-f''(\xi_\lambda)\right) \right\}.$$

Then, for $0 < l_1 < l$, there exists a neighborhood V_0 of x_0 such that we have

$$0 < l_1 < \sup_{\lambda \in (0,1)} \left\{ \lambda(1-\lambda)\left(-\left[x_1, \lambda x_1 + (1-\lambda) x_2, x_2; f\right]\right) \right\},$$

for all $x_1, x_2 \in V_0$. It follows that there exists $\lambda_0 \in (0,1)$ (depending on x_1, x_2 too) such that

$$0 < l_1 < \lambda_0 (1-\lambda_0)\left(-\left[x_1, \lambda_0 x_1 + (1-\lambda_0) x_2, x_2; f\right]\right);$$

i.e.,

$$0 < \frac{l_1}{\lambda_0(1-\lambda_0)} < -\left[x_1, \lambda_0 x_1 + (1-\lambda_0) x_2, x_2; f\right], \forall x_1, x_2 \in V_0.$$

But $\dfrac{l_1}{\lambda_0(1-\lambda_0)} \geq 4l_1 > 0$. Passing here to limit with $x_1, x_2 \to x_0$, it follows

$$0 < 4l_1 \leq \frac{l_1}{\lambda_0(1-\lambda_0)} \leq -\frac{f''(x_0)}{2};$$

that is, $f''(x_0) < 0$, a contradiction. As a conclusion,

$$\limsup_{x_1, x_2 \to x_0} \left\{ \sup_{\lambda \in (0,1)} \frac{\lambda(1-\lambda)}{2} \left(-f''(\xi_\lambda)\right) \right\} = 0,$$

which implies

$$d_{conv}(f)(x_0) = \max\{0, 0\} = 0 = \max\left\{0, -\frac{f''(x_0)}{8}\right\}.$$

The proof of the second formula in statement is similar, which proves the theorem. □

As an immediate consequence we obtain the next assertion.

Corollary 4.3. *Let* $f \in C^2(a,b)$.

(i) f *is convex on* (a,b) *if and only if* $d_{conv}(f)(x_0) = 0$, *for all* $x_0 \in (a,b)$.

(ii) f *is concave on* (a,b) *if and only if* $d_{conc}(f)(x_0) = 0$, *for all* $x_0 \in (a,b)$.

(iii) $d_{IM}(f')(x_0) = 8d_{conv}(f)(x_0)$ *and* $d_{DM}(f')(x_0) = 8d_{conc}(f)(x_0)$, *for all* $x_0 \in (a,b)$.

The results proved in Corollary 4.2 are also used in the proof of following result.

Corollary 4.4. (i) *If* $g \in C^2(a,b), f \in C^2(g(a,b))$, *then*

$$d_{conv}(f \circ g)(x_0) \leq (g'(x_0))^2 d_{conv}(f)(g(x_0)) + |f'(g(x_0))| d_{conv}(g)(x_0) + \frac{1}{8} g''(x_0) d_{IM}(f)(g(x_0))$$

and

$$d_{conc}(f \circ g)(x_0) \leq (g'(x_0))^2 d_{conc}(f)(g(x_0)) + |f'(g(x_0))| d_{conc}(g)(x_0) - \frac{1}{8} g''(x_0) d_{IM}(f)(g(x_0)),$$

for every $x_0 \in (a,b)$.

(ii) *If* $f \in C^2(a,b)$, f *is invertible and* $f'(x) \neq 0, \forall x \in (a,b)$, *then*

$$d_{conv}\left(f^{-1}\right)(y_0) \geq \frac{f''(x_0) d_{IM}(f)(x_0) - 8d_{conv}(f)(x_0)}{8|f'(x_0)|^3}$$

and

$$d_{conc}\left(f^{-1}\right)(y_0) \geq -\frac{f''(x_0) d_{IM}(f)(x_0) + 8d_{conc}(f)(x_0)}{8|f'(x_0)|^3},$$

for every $y_0 = f(x_0)$, $x_0 \in (a,b)$.

PROOF. (i) Because the property in Corollary 4.2 implies $d_{conv}(h)(x) = \frac{|h''(x)| - h''(x)}{16}, \forall x \in (a,b)$, for every function $h \in C^2(a,b)$, we get

$$d_{conv}(f \circ g)(x_0) = \frac{\left| f''(g(x_0))(g'(x_0))^2 + f'(g(x_0)) g''(x_0) \right|}{16}$$

$$-\frac{\left(f''\left(g\left(x_0\right)\right)\left(g'\left(x_0\right)\right)^2+f'\left(g\left(x_0\right)\right)g''\left(x_0\right)\right)}{16}$$

$$\leq \frac{\left|f''\left(g\left(x_0\right)\right)\right|\left(g'\left(x_0\right)\right)^2+\left|f'\left(g\left(x_0\right)\right)\right|\left|g''\left(x_0\right)\right|}{16}$$

$$-\frac{f''\left(g\left(x_0\right)\right)\left(g'\left(x_0\right)\right)^2-f'\left(g\left(x_0\right)\right)g''\left(x_0\right)}{16}$$

$$=\left(g'\left(x_0\right)\right)^2\frac{\left|f''\left(g\left(x_0\right)\right)\right|-f''\left(g\left(x_0\right)\right)}{16}+\left|f'\left(g\left(x_0\right)\right)\right|\frac{\left|g''\left(x_0\right)\right|-g''\left(x_0\right)}{16}$$

$$+\frac{1}{8}g''\left(x_0\right)\frac{\left|f'\left(g\left(x_0\right)\right)\right|-f'\left(g\left(x_0\right)\right)}{2}$$

$$=\left(g'\left(x_0\right)\right)^2 d_{conv}\left(f\right)\left(g\left(x_0\right)\right)+\left|f'\left(g\left(x_0\right)\right)\right|d_{conv}\left(g\right)\left(x_0\right)$$

$$+\frac{1}{8}g''\left(x_0\right)d_{IM}\left(f\right)\left(g\left(x_0\right)\right),\forall x_0\in(a,b).$$

The proof of the second inequality is similar.

(ii) Because $d_{conv}\left(1_{(a,b)}\right)(x)=0,\forall x\in(a,b)$, where $1_{(a,b)}$ is the identical function on (a,b), taking $g=f^{-1}$ in (i) we obtain

$$\left(\left(f^{-1}\left(y_0\right)\right)'\right)^2 d_{conv}\left(f\right)\left(f^{-1}\left(y_0\right)\right)+\left|f'\left(f^{-1}\left(y_0\right)\right)\right|d_{conv}\left(f^{-1}\right)\left(y_0\right)$$

$$+\frac{1}{8}\left(f^{-1}\right)''\left(y_0\right)d_{IM}\left(f\right)\left(f^{-1}\left(y_0\right)\right)\geq 0,\forall y_0\in f((a,b));$$

that is,

$$d_{conv}\left(f^{-1}\right)\left(y_0\right)\geq\frac{f''\left(x_0\right)d_{IM}\left(f\right)\left(x_0\right)-8d_{conv}\left(f\right)\left(x_0\right)}{8\left|f'\left(x_0\right)\right|^3}$$

for every $y_0=f\left(x_0\right),x_0\in(a,b)$. The proof of the second inequality is similar. □

Remark 4.1. For example, the first formula in Corollary 4.3 (iii) above can be viewed as a generalization of the following well-known result in Real Analysis. f is convex on (a,b) if and only if f' is increasing on (a,b).

The above considerations and the concept of convex (concave) function of order $n\in\{-1,0,1,2,\dots\}$ on (a,b) in [6], allow us to introduce the following.

Definition 4.2. Let $f:(a,b)\to\mathbb{R}$ and $x_0\in(a,b)$. The pointwise defect of convexity of order n of f at x_0 is the quantity

$$d_{conv}^{(n)}(f)(x_0)=\max\left\{0,-\limsup_{\substack{x_i\to x_0, i\in\{1,\ldots,n+2\}\\ x_i\neq x_j, i\neq j}}[x_1,\ldots,x_{n+2};f]\right\}.$$

Analogously, the pointwise defect of concavity of order n of f at x_0 is the quantity

$$d_{conc}^{(n)}(f)(x_0)=\max\left\{0,\limsup_{\substack{x_i\to x_0, i\in\{1,\ldots,n+2\}\\ x_i\neq x_j, i\neq j}}[x_1,\ldots,x_{n+2};f]\right\}.$$

Here $[x_1,\ldots,x_{n+2};f]$ denotes the divided difference.

Remark 4.2. If $f\in C^{n+1}(a,b)$, then by the mean value for $[x_1,\ldots,x_{n+2};f]$ in [6], we immediately get

$$d_{conv}^{(n)}(f)(x_0)=\max\left\{0,-\frac{f^{(n+1)}(x_0)}{(n+1)!}\right\},$$
$$d_{conc}^{(n)}(f)(x_0)=\max\left\{0,\frac{f^{(n+1)}(x_0)}{(n+1)!}\right\},$$

for every $x_0\in(a,b)$, such that for $n=0$ and $n=1$ we essentially recapture the pointwise defects in Definitions 3.3 and 4.1.

5 Applications

In what follows we present some simple applications. Let

$$IM(x_0)=\{g:(a,b)\to\mathbb{R}; g \text{ is differentiable and increasing on } x_0\},$$

where $x_0\in(a,b)$ and the pointwise increasing monotonicity is defined as in Definition 3.2, let

$$IM(a,b)=\{g:(a,b)\to\mathbb{R}; g \text{ is differentiable and increasing on } (a,b)\},$$

for f differentiable on $x_0\in(a,b)$, let

$$E_{IM}(f)(x_0)=\inf\{d_{IM}(f-g)(x_0); g\in IM(x_0)\},$$

for f differentiable on (a,b), let

$$\|f\|_{IM} = \sup\{d_{IM}(f)(x); x \in (a,b)\} \text{ and let}$$

$$E_{IM}(f)(a,b) = \inf\{\|f-g\|_{IM}; g \text{ is differentiable and increasing on } (a,b)\}.$$

Remark 5.1. $\|\cdot\|_{IM}$ is a special kind of norm, because $\|f\|_{IM} = 0$ if and only if f is monotonically increasing on (a,b), $\|\lambda f\|_{IM} = \lambda \|f\|_{IM}$ only for $\lambda \geq 0$, $\|f+g\|_{IM} \leq \|f\|_{IM} + \|g\|_{IM}$.

Theorem 5.1. *Let* $f:(a,b) \to \mathbb{R}$.

(i) If f is differentiable on $x_0 \in (a,b)$, then $E_{IM}(f)(x_0) \geq d_{IM}(f)(x_0)$.

(ii) If f is differentiable on (a,b), then $E_{IM}(f)(a,b) \geq \|f\|_{IM}$.

Proof. (i) For any $g \in IM(x)$ we have

$$\begin{aligned} d_{IM}(f)(x_0) &= \frac{|f'(x_0)| - f'(x_0)}{2} \\ &= \frac{|f'(x_0)| - |g'(x_0)|}{2} + \frac{|g'(x_0)| - g'(x_0)}{2} + \frac{g'(x_0) - f'(x_0)}{2} \\ &= \frac{|f'(x_0)| - |g'(x_0)|}{2} + \frac{g'(x_0) - f'(x_0)}{2} \\ &\leq \frac{|f'(x_0) - g'(x_0)|}{2} - \frac{f'(x_0) - g'(x_0)}{2} = d_{IM}(f-g)(x_0). \end{aligned}$$

Passing to infimum with $g \in IM(x_0)$ we get (i).

(ii) Passing to supremum with $x \in (a,b)$ in (i) we get

$$\begin{aligned} \|f\|_{IM} &\leq \sup_{x\in(a,b)} \{\inf\{d_{IM}(f-g)(x); g \in IM(x)\}\} \\ &\leq \inf_{g \in IM(a,b)} \{\sup\{d_{IM}(f-g)(x); x \in (a,b)\}\} = E_{IM}(f)(a,b), \end{aligned}$$

which proves the theorem. □

Let

$$CONV(x_0) = \left\{g:(a,b) \to \mathbb{R}; g \in C^2(a,b) \text{ and } g \text{ is convex on } x_0\right\},$$

where $x_0 \in X$ and the pointwise convexity in x_0 is as in Theorem 4.1, (iv), for $g \in C^2(a,b)$,

$$\begin{aligned} E_{CONV}(f)(x_0) &= \inf\{d_{conv}(f-g)(x_0); g \in CONV(x_0)\}, \\ \|f\|_{CONV} &= \sup\{d_{conv}(f)(x); x \in (a,b)\}, \end{aligned}$$

($\|\cdot\|_{CONV}$ is a special kind of norm, because $\|f\|_{CONV} = 0$ if and only if f is convex on (a,b), $\|\lambda f\|_{CONV} = \lambda \|f\|_{CONV}$ only for $\lambda \geq 0$, $\|f+g\|_{CONV} \leq \|f\|_{CONV} + \|g\|_{CONV}$),

$$E_{CONV}(f)(a,b) = \inf\left\{\|f-g\|_{CONV}\,; g \in C^2(a,b), g \text{ is convex on } (a,b)\right\}.$$

As was done above, we can prove the following.

Theorem 5.2. *Let $f \in C^2(a,b)$. We have:*

(i) $E_{CONV}(f)(x) \geq d_{conv}(f)(x), \forall x \in (a,b)$.

(ii) $E_{CONV}(f)(a,b) \geq \|f\|_{CONV}$.

PROOF. From Corollary 4.2, we have

$$d_{conv}(f)(x_0) = \max\left\{0, -\frac{f''(x_0)}{8}\right\} = \frac{|f''(x_0)| - f''(x_0)}{16}.$$

Reasoning as in the proof of the theorem above, we get the desired conclusion. □

6 Open Problems

Concerning the above results, the study of the following questions would be of interest.

Question 1. What connections exist between $d_{IM}(f')(x_0)$ and $d_{conv}(f)(x_0)$ in Corollary 4.3, (iii) when f is only in $C^1(a,b)$ or only differentiable on (a,b) (and it is not in $C^2(a,b)$)? We conjecture something of the form

$$M_1 d_{IM}(f')(x_0) \leq d_{conv}(f)(x_0) \leq M_2 d_{IM}(f')(x_0),$$

where M_1, M_2 are independent of $x_0 \in (a,b)$.

Question 2. Do Theorems 5.1 and 5.2 remain valid in the case when the functions f and g in the definitions of $E_{IM}(f)(x_0)$, $E_{IM}(f)(a,b)$, $E_{CONV}(f)(x_0)$, $E_{CONV}(f)(a,b)$ are supposed to be non- smooth; i.e., are only continuous?

Question 3. Are Theorem 3.5, (vi) valid in the case when f is only continuous, and Corollary 4.4, in the case when f and g are only of C^1-class or only differentiable?

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