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OLD AND NEW ON THE HERMITE-HADAMARD INEQUALITY*

Abstract

The goal of this paper is to describe the panorama of Mathematics grown up from the celebrated inequality of Hermite and Hadamard. Both old and new results are presented, complemented and discussed within this framework.

1 Introduction

Read the Masters! said an old adage. And indeed, their work continues to surprise us by the deep and constantly modern ideas it contains. Classical results like the Pythagorean Theorem, the Fundamental Theorem of Algebra, and Fermat's last Theorem constitute a great legacy and a source of inspiration for the next generations. Recently, O. B. Bekken [3] touched this point, recalling us the story of N. Abel and his mathematics. The aim of this paper is to illustrate the same idea, but in a less known case; namely, in what is now known as the Hermite-Hadamard inequality. The *Hermite-Hadamard inequality* gives us an estimate of the (integral) mean value of a continuous convex function. Precisely, if $f : [a, b] \to \mathbb{R}$ is such a function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}\,. \tag{HH}$$

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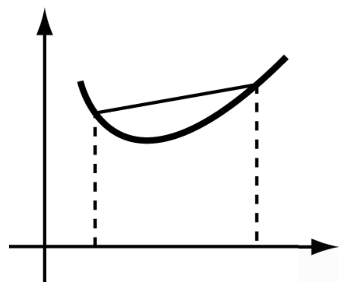

Figure 1: Convexity: the points of the graph are under the chord.

Moreover, equality holds in either side only for affine functions (i.e., for functions of the form $mx+n$). (HH) was first noticed by Ch. Hermite [14] in 1883 and rediscovered ten years later by J. Hadamard [11]. However the priority of Hermite was not recorded until recently and his note was not even mentioned in Hermite's Collected papers (published par E. Picard). See D. S. Mitrinović and I. B. Lacković [18] for the whole history.

The convexity of the function $f : [a,b] \to \mathbb{R}$ means that the points of Graph $f|[u,v]$ are under the chord (or on the chord) joining the end-points $(u, f(u))$ and $(v, f(v))$, for every $u, v \in [a,b]$. See Fig. 1. Then

$$f(x) \leq f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

from which, by integration, we infer the right hand side of (HH).

Assuming that f is also continuous, we actually get

$$\frac{1}{b-a}\int_a^b f(x)\,dx < \frac{f(a)+f(b)}{2} \tag{RHH}$$

except when f is *affine*; i.e., $f(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$.

The left part of (HH) is also easy to prove.

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{b-a}\left(\int_a^{(a+b)/2} f(x)\,dx + \int_{(a+b)/2}^b f(x)\,dx\right) \\ &= \frac{1}{2}\int_0^1 \left[f\left(\frac{a+b-t(b-a)}{2}\right) + f\left(\frac{a+b+t(b-a)}{2}\right)\right] dt \\ &\geq f\left(\frac{a+b}{2}\right). \end{aligned} \tag{LHH}$$

Quite interesting, each of the two sides of (HH) characterizes convex functions. More precisely, *if I is an interval and $f : I \to \mathbb{R}$ is a continuous function whose restriction to every compact subinterval $[a, b]$ satisfies* (LHH), *then f is convex.* The same works when (LHH) is replaced by (RHH).

We shall illustrate the power of (HH) with several examples from Calculus.

Examples. i) For $f(x) = 1/(1+x)$, $x \geq 0$, Hermite [14] noticed that $x - \frac{x^2}{2+x} < \log(1 + x) < x - \frac{x^2}{2(1+x)}$. In particular,

$$\frac{1}{n + 1/2} < \log(n + 1) - \log n < \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n + 1}\right) \tag{S}$$

for every $n \in \mathbb{N}^\star$, and this fact is instrumental in deriving Stirling's formula,

$$n! \sim \sqrt{2\pi} \cdot n^{n+1/2} e^{-n}.$$

See [2].

ii) For $f = \exp$, (HH) yields

$$e^{(a+b)/2} < \frac{e^b - e^a}{b - a} < \frac{e^a + e^b}{2} \quad \text{for } a \neq b \text{ in } \mathbb{R};$$

i.e.,

$$\sqrt{xy} < \frac{x - y}{\log x - \log y} < \frac{x + y}{2} \quad \text{for } x \neq y \text{ in } (0, \infty), \tag{GLA}$$

which represents the Geometric, Logarithmic and Arithmetic Mean Inequality. See [5].

iii) For $f(x) = \sin x$, $x \in [0, \pi]$, we obtain

$$\frac{\sin a + \sin b}{2} < \frac{\cos a - \cos b}{b - a} < \sin\left(\frac{a + b}{2}\right)$$

and this implies that $\tan x > x > \sin x$ (for $x \in [0, \pi/2]$).

At first glance, what can be added to the discussion above is that (HH) actually works for *all* convex functions $f : [a, b] \to \mathbb{R}$. In fact, every convex function on an interval $[a, b]$ comes from a continuous convex function whose endpoints are modified (by moving up). This fact is a consequence of the following elementary result. *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then either f is monotonic or there is a point $c \in (a, b)$ such that $f|[a, c]$ is decreasing and $f|[c, b]$ is increasing.* Consequently, for any such function f, the limits $f(a+)$ and $f(b-)$ exist in $\mathbb{R}$ and the function

$$\widetilde{f}(x) = \begin{cases} f(a+) & \text{if } x = a \\ f(x) & \text{if } x \in (a, b) \\ f(b-) & \text{if } x = b \end{cases}$$

is continuous and convex.

If we look at the site http://rgmia.vu.au of *Research Group in Mathematical Inequalities and Applications* (Victoria University in Melbourne), we will find a large number of titles in this area, including books. However, in this paper we take a fairly different approach, which we hope will give the readers more insight into the world of the Hermite-Hadamard inequality and show the deep connection with Choquet's theory. This theory is about probability measures μ defined on the Borel subsets of some compact convex sets K and the Hermite-Hadamard inequality represents the main result in the case where $K = [a, b]$ and $\mu = dx/(b-a)$. The mid point $(a+b)/2$ is the barycenter of K with respect to the mass distribution associated with μ; so (HH) says that the value of f at the barycenter does not exceed the mean value of f over K, which in turn is less than or equal with the mean value of f over the set $\operatorname{Ext} K = \{a, b\}$, of all extreme points of K.

The starting point of Choquet's theory was the remark that every point of a compact convex set (of a locally convex Hausdorff space) is the barycenter of a Borel probability measure on K, which is concentrated on the closure of the extreme points of K. This is related to the celebrated Krein-Milman Theorem, that says that every compact convex set (as above) is the closed convex hull of its extreme points. The book of R. R. Phelps [27] gives a nice account on Choquet's theory but says nothing about the Hermite-Hadamard inequality. In fact, it was recognized only very recently (at *Inequalities* 2001, Timişoara, July 9–13) that the two theories have a strong intersection and could evolve into a more general theory, allowing the replacement of probability measures by some signed measures (see [22], [21]). Once again, this story proves that the development of mathematics is not linear and the apparition of new results changes the landscape of mathematics. It is exactly how sun, wind and water continuously reshape our planet!

This paper is organized as follows. First we investigate the Hermite-Hadamard inequality from a number of points of view: the use of subdifferential, the connection with the quadrature formulae, the precision in (LHH) and (RHH) and the possible improvements of (HH). Then we give a short account on classical Choquet's theory and pass to a generalization of it, in the same spirit we pass from the classical Jensen Inequality to the Jensen-Steffensen Inequality (as stated in [17] and [26]). This provides a framework to bring together a number of nice results and extend automatically many others.

The theory of convex function is at its core a theory about comparing arithmetic means (of certain random variables with their composition by a given function). One may develop parallel theories (like that of log-convexity), by considering pairs of means (on the domain and respectively on the codomain

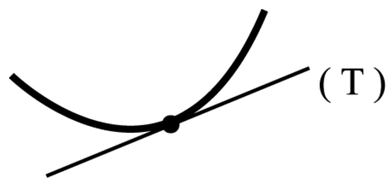

Figure 2: Convexity: the support line exists at each interior point.

of definition) and prove Hermite-Hadamard type inequalities in their setting. See [20], [23], [33]. But this is beyond our story.

2 An Alternative Approach to (LHH)

It is possible to prove (LHH) by a dual geometric argument, the existence of support lines at each interior point, a fact which leads to the important notion of *subdifferential* of a convex function.

The basic remark is that a real function f is convex on an interval I if and only if for each $c \in I$, the function

$$s_c : x \to \frac{f(x) - f(c)}{x - c} \tag{G}$$

is increasing on $I \setminus \{c\}$; for, notice that $u < v < w$ means that v is a convex combination of u and w. This fact (due to L. Galvani [10]) yields the existence of the one-sided derivatives at all interior points of I.

$$x, y \in \operatorname{int} I,\ x < y \text{ implies } f'(x-0) \le f'(x+0) \le f'(y-0) \le f'(y+0).$$

Moreover, the set of all points of non differentiability is at most countable. See [32] for details. The existence of finite one-sided derivatives at the interior points yields the continuity of f at these points. See Fig. 2.

The next result proves the existence of a support line at each interior point of the interval of definition.

Lemma 1. *A function $f : (a, b) \to \mathbb{R}$ is convex if and only if for each point $c \in (a, b)$ there exists a number λ such that*

$$f(x) \ge f(c) + \lambda(x - c) \text{ for every } x \in [a, b]. \tag{SL}$$

Moreover, λ can be chosen arbitrarily in the interval $[f'(c-0), f'(c+0)]$.

PROOF. See [32]. □

Coming back to the case of a continuous convex function $f : [a,b] \to \mathbb{R}$, by integrating the inequality (SL) we obtain the following generalization of (LHH).

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq f(c) + \lambda \frac{b+a-2c}{2},$$

for any $c \in (a,b)$ (if $c = (a+b)/2$ we obtain (LHH)). Equality occurs for linear functions (i.e., for $f = f(c) + \lambda(x-c)$ on (a,b)).

3 On the Precision in the Hermite-Hadamard Inequality

The Hermite-Hadamard inequality can be easily derived by restricting ourselves to the case of convex functions which are piecewise C^2. In fact, the continuous convex functions on intervals can be approximated (uniformly) by piecewise linear functions. Due to the additivity of the integral, we can restrict ourselves to the C^2-pieces. In the light of this discussion the following result constitutes an extension of the Hermite-Hadamard inequality.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function such that there exist real constants m and M so that $m \leq f'' \leq M$. Then*

$$m\frac{(b-a)^2}{24} \leq \frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \leq M\frac{(b-a)^2}{24}$$

and

$$m\frac{(b-a)^2}{12} \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \leq M\frac{(b-a)^2}{12}.$$

PROOF. In fact, the functions $f - mx^2/2$ and $Mx^2/2 - f$ are convex and thus we can apply the Hermite-Hadamard inequality to them and the proof follows. □

The discussion above indicates that it is meaningful to generalize (HH) also by working with other special classes of functions than those in Theorem 1. In particular, it is more convenient to indicate estimates in the framework of Lipschitz constants. If $f : [a,b] \to \mathbb{R}$ is a Lipschitz function with

$$||f||_{Lip} := \sup\left\{\left|\frac{f(x)-f(y)}{x-y}\right|;\ x \neq y\right\} = M < \infty,$$

then two basic inequalities estimating the integral mean, which are related to (HH) are known:

- the *inequality of Ostrowski,*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M(b-a); \tag{O}$$

- the *inequality of Iyengar*,

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{M(b-a)}{4} - \frac{1}{4M(b-a)}\left(f(b)-f(a)\right)^2 . \tag{I}$$

The proofs of both inequalities are quite straightforward. For example, (O) can be derived as follows.

$$\begin{aligned} \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| &= \left| \frac{1}{b-a}\int_a^b \left(f(x)-f(t)\right)\,dt \right| \\ &\leq \frac{1}{b-a}\int_a^b |f(x)-f(t)|\,dt \\ &\leq \frac{M}{b-a}\int_a^b |x-t|\,dt \\ &= \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M(b-a) \end{aligned}$$

for every $x \in [a,b]$. As for (I), see Iyengar's paper [15], which provides a short argument. An alternative approach is indicated in Section 7 below.

4 Some Straightforward Improvements on the Hermite-Hadamard Inequality

Suppose that $f : [a,b] \to \mathbb{R}$ is a convex function. By applying the Hermite-Hadamard inequality on each of the intervals $[a,(a+b)/2]$ and $[(a+b)/2,b]$ we get

$$f\left(\frac{3a+b}{4}\right) \leq \frac{2}{b-a}\int_a^{(a+b)/2} f(x)\,dx \leq \frac{1}{2}\left(f(a) + f\left(\frac{a+b}{2}\right)\right)$$

and

$$f\left(\frac{a+3b}{4}\right) \leq \frac{2}{b-a}\int_{(a+b)/2}^b f(x)\,dx \leq \frac{1}{2}\left(f\left(\frac{a+b}{2}\right) + f(b)\right);$$

so summing up (side by side) we obtain the following refinement of (HH).

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \leq & \frac{1}{2}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right) \\ \leq & \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{2}\left[f\Big(\frac{a+b}{2}\Big)+\frac{f(a)+f(b)}{2}\right] \\ \leq & \frac{1}{2}\left(f(a)+f(b)\right). \end{aligned}$$

By continuing the division process, the integral mean of f can be approximated as close as we want by the mean values of f at the dyadic points of $[a, b]$.

Ioan Raşa [31] made the following remark in connection with the above refinement on (HH). *If* $f : [a, b] \to \mathbb{R}$ *is a convex function, then*

$$\frac{1}{2}\left(f\left(\frac{a+b}{2}-c\right)+f\left(\frac{a+b}{2}+c\right)\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx$$

for every $c \in [0, (b-a)/4]$, *and* $c = (b-a)/4$ *is maximal with this property.* For a proof, apply the remark after Lemma 1 to $f|[a, (a+b)/2]$ and $f|[(a+b)/2, b]$.

5 Various Quadrature Formulae as Sources of Hermite-Hadamard Type Inequalities

By applying twice the integration by parts formula to a function $f \in C^2([a, b], \mathbb{R})$ we get the following quadrature formula.

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2}\left[f(a)+f(b)\right] - \frac{1}{b-a}\int_a^b f''(x)\frac{(b-x)(x-a)}{2}\,dx.$$

In particular, if $f'' \geq 0$, then this formula implies (RHH).

In the case of C^4-functions,

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx = & \frac{1}{8}\left[f(a)+3f\left(\frac{3a+b}{4}\right)+3f\left(\frac{a+3b}{4}\right)+f(b)\right] \\ & -\frac{1}{b-a}\int_a^b f^{(iv)}(x)\varphi(x)\,dx, \end{aligned}$$

where φ is a piecewise polynomial nonnegative function such that

$$\frac{1}{b-a}\int_a^b \varphi(x)\,dx = \frac{(b-a)^4}{6480}.$$

This yields the following improvement on (RHH) for functions $f \in C^4([a,b])$ with $f^{(iv)} \geq 0$.

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{8}\left[f(a) + 3f\left(\frac{3a+b}{4}\right) + 3f\left(\frac{a+3b}{4}\right) + f(b)\right] \leq \frac{f(a)+f(b)}{2}. \tag{RHH*}$$

Examples. *i*) An application of (RHH*) (for $f(x) = e^x|[\ln a, \ln b]$) was given by F. Burk [5], who recovered the following inequality of Tung Po-Lin [37].

$$\frac{b-a}{\ln b - \ln a} < \left(\frac{a^{1/3} + b^{1/3}}{2}\right)^3. \tag{LP$_3$}$$

ii) In the case of the function $f(x) = \ln x$, $x \in [a,b]$, we are led to the following upper estimate for the *identric mean*,

$$\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \leq a^{1/8}\left(\frac{3a+b}{4}\right)^{3/8}\left(\frac{a+3b}{4}\right)^{3/8} b^{1/8},$$

and, according to the AG-inequality, this is a refinement of the last inequality in the following chain of inequalities,

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} < \frac{a+b}{2} \tag{GLIA}$$

valid for $a, b > 0$, $a \neq b$. See [1] and compare also the second inequality with (LP$_3$).

The above examples are actually related to higher-order convexity, a theory initiated by T. Popoviciu [28], [30], in 1934. See also [26], [32]. A function $f : [a,b] \to \mathbb{R}$ is said to be *n-convex* ($n \in \mathbb{N}$) if for all choices of $n+1$ distinct points $x_0 < \cdots < x_n$ in $[a,b]$, the nth order divided difference of f satisfies

$$f[x_0, \ldots, x_n] \geq 0.$$

The divided differences are given inductively by

$$\begin{aligned}
f\,[x_0, x_1] &= \frac{f(x_0) - f(x_1)}{x_0 - x_1} \\
f\,[x_0, x_1, x_2] &= \frac{f\,[x_0, x_1] - f\,[x_1, x_2]}{x_0 - x_2} \\
&\ldots\ldots \\
f[x_0, \ldots, x_n] &= \frac{f\,[x_0, \ldots\ldots, x_{n-1}] - f\,[x_1, \ldots, x_n]}{x_0 - x_n}.
\end{aligned}$$

Thus the 1-convex functions are the nondecreasing functions, while the 2-convex functions are precisely the classical convex functions. As noticed by Popoviciu, if f is n times differentiable, with $f^{(n)} \geq 0$, then f is n-convex.

Based on Gauss type quadrature formulae, M. Bessenyei and Z. Páles [4] were able to extend the Hermite-Hadamard inequality to the case of n -convex functions. Other directions of generalization were proposed by J. Pečarić and his collaborators. See [6] and [25].

The idea of integrating by parts (used at the beginning of this section) can be adapted to the several variables setting, via the Green formula. This leads us to Hermite-Hadamard type formulae for subharmonic functions. Let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. Then the Dirichlet problem

$$\begin{cases} \Delta\varphi = 1 & \text{on } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{DP}$$

has a unique solution, which is < 0 on Ω (according to the maximum principle). By Green's formula, for every $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ we have

$$\int_\Omega \begin{vmatrix} u & \varphi \\ \Delta u & \Delta\varphi \end{vmatrix} dV = \int_{\partial\Omega} \begin{vmatrix} u & \varphi \\ \nabla u & \nabla\varphi \end{vmatrix} \cdot n \, dS;$$

i.e., in view of (DP),

$$\begin{aligned} \int_\Omega u \, dV &= \int_\Omega u\Delta\varphi \, dV \\ &= \int_\Omega \varphi\Delta u \, dV + \int_{\partial\Omega} u \left(\nabla\varphi \cdot n\right) \, dS - \int_{\partial\Omega} \varphi \left(\nabla u \cdot n\right) \, dS \\ &= \int_\Omega \varphi\Delta u \, dV + \int_{\partial\Omega} u \left(\nabla\varphi \cdot n\right) \, dS \end{aligned}$$

for every $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. We are then led to the following Hermite-Hadamard type inequality.

Theorem 2. *If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is subharmonic (i.e., $\Delta u \geq 0$ on Ω) and φ satisfies (DP), then $\int_\Omega u \, dV < \int_{\partial\Omega} u \left(\nabla\varphi \cdot n\right) \, dS$, except for harmonic functions (when equality occurs).*

The equality case needs the remark that $\int_\Omega \varphi\Delta u \, dV = 0$ yields $\varphi\Delta u = 0$ on Ω, and thus $\Delta u = 0$ on Ω. Note that $\varphi\Delta u$ is continuous and nonpositive since $\varphi < 0$ on Ω.

By using the same technique we also see that the following multidimensional version of Theorem 1 holds.

Theorem 3. *Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be such that $m \leq \Delta u \leq M$ on Ω and let φ satisfy (DP). Then*

$$m \int_\Omega \varphi \, dV \leq \int_\Omega u \, dV - \int_{\partial\Omega} u \, (\nabla\varphi \cdot n) \, dS \leq M \int_\Omega \varphi \, dV.$$

In the case of balls in $\mathbb{R}^3$, the conclusion of Theorem 3 is

$$u(a) \leq \frac{1}{Vol\ \overline{B}_R(a)} \iiint_{\overline{B}_R(a)} u(x) \, dV < \frac{1}{Area\ S_R(a)} \iint_{S_R(a)} u(x) \, dS \qquad \text{(VS)}$$

for every $u \in C^2(B_R(a)) \cap C^1(\overline{B}_R(a))$ with $\Delta u \geq 0$, which is not harmonic. In fact, in this case $\varphi(x) = \frac{1}{6}\left(|x|^2 - R^2\right)$ satisfies (DP) and

$$\nabla\varphi \cdot n = x/3 \cdot x/|x| = R/3.$$

The inequality (VS) works for all convex functions on balls $\overline{B}_R(a)$; see S. S. Dragomir [7] for details. Let us mention that in the several variables case the notions of convexity and subharmonicity are different.

Finally, Theorem 3 can be extended easily to the case of all uniformly elliptic operators of the form

$$Pu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(p_{ij}\frac{\partial u}{\partial x_i}\right) + q(x)u$$

where $q \in C(\overline{\Omega})$, $q \geq 0$, $p_{ij} \in C^1(\overline{\Omega})$, $p_{ij} = p_{ji}$ $(i, j = 1, \ldots, n)$ and that there exists a constant $\gamma > 0$ such that $\sum_{i,j=1}^{n} p_{ij}\xi_i\xi_j \geq \gamma \sum_{i=1}^{n} |\xi_i|^2$ for every $x \in \overline{\Omega}$ and every $\xi_1, \ldots, \xi_n \in \mathbb{R}$.

6 Choquet's Theory

At the heart of Choquet's theory is the notion of barycenter, which can be defined as follows. Let K be a compact convex subset of a locally convex Hausdorff space E and suppose there is given a Borel probability measure μ on K (which can be thought of as a mass distribution on K). The *μ-barycenter* of K is defined as the unique point x_μ of K such that

$$x'(x_\mu) = \int_K x'(x) \, d\mu(x) \qquad \text{(B)}$$

for every continuous linear functional x' on E. See [27], Proposition 1.1, for details. When E is the Euclidean n-dimensional space, then norm and the weak convergence are the same and thus $x_\mu = \int_K x \, d\mu(x)$.

An immediate consequence of (B) is the validity of the (LHH) inequality in great generality.

Lemma 2. *Let K and μ be defined as above. Then, for every continuous convex function $f : K \to \mathbb{R}$, $f(x_\mu) \leq \int_K f(x)\, d\mu(x)$.*

For details, see the remark before Lemma 4.1 in [27].

The extension of the right hand inequality in (HH) is a bit more subtle and constitutes the core of Choquet's theory. A key role is played by the notion of majorization. G. H. Hardy, J. E. Littlewood and G. Pólya [12], [13], were the first to recognize the importance of this notion, but they only considered the particular case when K consists of finitely many points. Choquet's theory needs the notion of majorization at the level of Borel probability measures (on arbitrary compact convex spaces K). Given two Borel probability measures μ and λ on K, we say that μ is *majorized* by λ (denoted $\mu \prec \lambda$) if

$$\int_K f(x)\, d\mu(x) \leq \int_K f(x)\, d\lambda(x)$$

for every continuous convex function $f : K \to \mathbb{R}$. As noticed in [27], $\prec$ is a partial ordering on the set of all Borel probability measures on K. Given a Borel probability measure μ on K there always exists a Borel probability measure $\lambda \succ \mu$, which is maximal with this property; use Zorn's Lemma. The delicate point is to show that the maximal measure λ is *concentrated* on the set $\operatorname{Ext} K$, of all extreme points of K, in the sense that $\lambda(B) = 0$ for every Baire subset $B \subset K$ such that $B \cap \operatorname{Ext} K = \emptyset$. In the particular case when K is also metrizable, things are much better.

Theorem 4. (*G. Choquet*; *see [27]*). *Suppose that K is a metrizable compact convex set* (*in a locally convex Hausdorff space*). *Then the set* $\operatorname{Ext} K$ *of all extreme points of K is a G_δ-subset of K and for every Borel probability measure μ on K there exists a Borel probability measure λ on K supported on* $\operatorname{Ext} K$ (*i.e.,* $\lambda(K \setminus \operatorname{Ext} K) = 0$) *such that*

$$f(x_\mu) \leq \int_K f(x)\, d\mu(x) \leq \int_{\operatorname{Ext} K} f(x)\, d\lambda(x) \tag{Ch}$$

for every continuous convex function $f : K \to \mathbb{R}$.

Choquet's theory has deep applications to several areas of mathematics such as function algebras, invariant measures and potential theory. The book of R. R. Phelps [27] contains a good account of this matter. We shall add here a few words concerning the connection of Theorem 4 with some old and new inequalities.

When K is the interval $[a, b]$ endowed with the normalized Lebesgue measure $dx/(b - a)$, then x_μ is exactly the midpoint $(a + b)/2$ and $\operatorname{Ext} K = \{a, b\}$.

Any probability measure λ concentrated on $\operatorname{Ext} K$ is necessarily a convex combination of Dirac measures; i.e., $\lambda = \alpha\delta_a + (1-\alpha)\delta_b$ for some $\alpha \in [0,1]$. Checking the right side inequality in (Ch) for $f = x - a$ and $f = b - x$ we get $1 - \alpha \geq 1/2$ and $\alpha \geq 1/2$; i.e., $\alpha = 1/2$. Consequently, in this case (Ch) coincides with (HH) and we conclude that Theorem 4 provides a complete generalization of the Hermite-Hadamard inequality.

Many interesting inequalities relating weighted means represent averages over the $(n-1)$-dimensional simplex

$$\Delta_n = \{\mathbf{u} = (u_1, \ldots, u_n);\ u_1, \ldots, u_n \geq 0,\ u_1 + \cdots + u_n = 1\}.$$

Clearly, Δ_n is compact and convex and its extreme points are the "corners" $(1,0,\ldots,0), \ldots, (0,0,\ldots,1)$. An easy consequence of Theorem 4 is the following refinement of the classical Jensen inequality.

Theorem 5. *For every continuous convex function $f : [a,b] \to \mathbb{R}$, every n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$ of elements of $[a,b]$ and every Borel probability measure μ on Δ_n,*

$$f\left(\sum_{k=1}^{n} w_k x_k\right) \leq \int_{\Delta_n} f(\mathbf{x} \cdot \mathbf{u})\, d\mu \leq \sum_{k=1}^{n} w_k f(x_k). \tag{J}$$

Here $(w_1, \ldots, w_n)$ denotes the barycenter of Δ_n with respect to μ. The above inequality is reversed if f is concave on $[a,b]$.

The *weighted identric mean* $I(\mathbf{x}, \mu)$ is defined by the formula

$$I(\mathbf{x}, \mu) = \exp \int_{\Delta_n} \ln(\mathbf{x} \cdot \mathbf{u})\ d\mu(\mathbf{u})$$

and the *weighted logarithmic mean* $L(\mathbf{x}, \mu)$ is defined by the formula

$$L(\mathbf{x}, \mu) = \left(\int_{\Delta_n} \frac{1}{\mathbf{x} \cdot \mathbf{u}}\ d\mu(\mathbf{u})\right)^{-1}.$$

By (J), we infer easily that $L(\mathbf{x}, \mu) \leq I(\mathbf{x}, \mu)$ and both lie between the weighted arithmetic mean $A(\mathbf{x}, \mu) = \sum_{k=1}^{n} w_k x_k$ and the weighted geometric mean $G(\mathbf{x}, \mu) = \prod_{k=1}^{n} x_k^{w_k}$; i.e.,

$$G(\mathbf{x}, \mu) \leq L(\mathbf{x}, \mu) \leq I(\mathbf{x}, \mu) \leq A(\mathbf{x}, \mu),$$

a fact which extends (GLIA).

Further applications of (J) to Ky Fan type inequalities were recently obtained by E. Neuman and J. Sándor [19].

As we shall show in the next section, all results above extend mutatis mutandis to a more general context, where the place of probability measures is taken by real Borel measures that satisfy some "end positivity" conditions.

7 Choquet's Theory for Signed Measures

Steffensen's extension of the Jensen inequality was the first result showing the possibility of proving convexity inequalities involving signed linear combinations.

Theorem 6. (*The Jensen-Steffensen Inequality [17], [26]*). *Let $x_1 \le x_2 \le \cdots \le x_n$ be points in $[a, b]$ and let $p_1, p_2, \ldots, p_n$ be real numbers such that the sums $P_k = \sum_{j=1}^k p_j$ satisfy $0 \le P_k \le P_n$ and $P_n > 0$. Then for every real valued convex function f defined on $[a, b]$ we have the inequality*

$$f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k\, x_k\right) \le \frac{1}{P_n}\sum_{k=1}^{n} p_k\, f(x_k).$$

More recently, A. M. Fink [8] proved an extension of (HH) which also escapes Choquet's theory.

Theorem 7. *If $f : [a, b] \to \mathbb{R}$ is a continuous convex function, then*

$$f(x_\mu) \le \frac{1}{\mu([a,b])}\int_a^b f(x)\, d\mu(x) \le \frac{b - x_\mu}{b-a}\cdot f(a) + \frac{x_\mu - a}{b-a}\cdot f(b) \qquad \text{(FHH)}$$

for every real Borel measure μ on $[a, b]$ which is " end positive" in the sense that

$$\mu([a,b]) > 0, \ \int_a^t (t-x)\, d\mu(x) \ge 0, \ \text{and} \ \int_t^b (x-t)\, d\mu(x) \ge 0, \qquad \text{(EP)}$$

for every $t \in [a, b]$. (As above, $x_\mu = \int_a^b x\, d\mu(x)/\mu([a, b])$ represents the barycenter of μ.)

Clearly, all Borel probability measures are "end positive". An example in the category of signed measures is the restriction of $\left(\prod_{k=1}^n (x_k^2 + a_k)\right) dx_1 \ldots dx_n$ to $[-1, 1]^n$, for every $a_1, \ldots, a_n > -1/3$; use mathematical induction.

It was precisely Theorem 7 which led the first author to the problem whether Choquet's theory admits a generalization outside the framework of probability measures. The answer was affirmative and it was presented to the Conference *Inequalities 2001*, held in Timişoara. See [21].

The class of measures which makes it possible to bring together Choquet's theory and a number of results such as Theorems 6 and 7 above has its roots in an old paper of T. Popoviciu [29] (dedicated to the n-convex functions).

Definition 1. *A Popoviciu measure is any real Borel measure μ on K such that*
$$\mu(K) > 0 \text{ and } \int_K f^+(x)\, d\mu(x) \geq 0 \tag{PM}$$
for every continuous convex function f on K.

When K is an interval $[a,b]$ and μ is a real Borel measure on $[a,b]$, with $\mu([a,b]) > 0$, the condition (PM) coincides with the condition of end positivity (EP) mentioned above, a fact which was known to T. Popoviciu. In fact, (PM) yields
$$\mu(K) > 0 \text{ and } \int_K (x'(x) + t)^+\, d\mu(x) \geq 0 \text{ for every } x' \in E' \text{ and every } t \in \mathbb{R}$$
and the dual of $\mathbb{R}$ consists only of homoteties $x' : x \to sx$. T. Popoviciu's argument for the other implication, (EP) $\Rightarrow$(PM), was as follows. If $f \geq 0$ is a piecewise linear, continuous and convex function, then f can be represented as a finite combination, with non-negative coefficients, of functions of the form 1, $(x-t)^+$ and $(t-x)^+$, so that $\int_K f(x)\, d\mu(x) \geq 0$. In the general case, we have to approximate f^+ by piecewise linear continuous and convex functions. It is worth noticing that T. Popoviciu [29] was interested in a slightly different problem; precisely, when a real Radon measure on an interval $[a,b]$ is non-negative for all n-convex functions on that interval. However, he did not mention any possible connection with the Hermite-Hadamard inequality.

The notion of barycenter can be introduced in the same way and an inspection of the argument of Lemma 2 shows that it remains valid in the general context of Popoviciu measures:

Proposition 1. (*The generalized Jensen-Steffensen inequality*). *Suppose that μ is a Popoviciu measure on a compact convex set K. Then*
$$f(x_\mu) \leq \frac{1}{\mu(K)} \int_K f(x)\, d\mu(x) \tag{JS}$$
for every continuous convex function $f : K \to \mathbb{R}$.

Clearly, if μ is a real Borel measure on K, with $\mu(K) > 0$, and (JS) holds for some point $x_\mu \in K$ and every continuous convex function $f : K \to \mathbb{R}$, then μ is a Popoviciu measure.

The next step is to enlarge the concept of majorization.

Definition 2. *Given two Popoviciu measures μ and λ on the compact convex set K, we say that μ is* majorized *by λ (denoted $\mu \prec \lambda$) if*
$$\frac{1}{\mu(K)} \int_K f\, d\mu \leq \frac{1}{\lambda(K)} \int_K f\, d\lambda$$

for every continuous convex function $f : K \to \mathbb{R}$.

Using techniques already developed in [27], one can prove that every Popoviciu measure is majorized by a Borel probability measure. Combining this fact with Choquet's Theorem (see Theorem 4 above), we arrive at the following result.

Theorem 8. (*The generalization of Choquet's Theorem.* See [21]). *Let μ be a Popoviciu measure on a metrizable compact convex subset K of a locally convex Hausdorff space E. Then there exists a probability Borel measure λ on K such that the following two conditions are satisfied*:

i) $\mu \prec \lambda$ *and* λ *and* μ *have the same barycenter;*

ii) λ *is supported by* $\operatorname{Ext} K$ *and*

$$f(x_\mu) \leq \frac{1}{\mu(K)} \int_K f(x)\, d\mu(x) \leq \int_{\operatorname{Ext} K} f(x)\, d\lambda(x) \qquad (\mathrm{Ch}^*)$$

for every continuous convex function $f : K \to \mathbb{R}$.

The last theorem makes critical the question when a Borel measure is also a Popoviciu measure. The answer is satisfactory in the one real variable case. In the discrete case, suppose that there are given real points $x_1 \leq \cdots \leq x_n$ and real weights $p_1, \ldots, p_n$. According to (EP), the discrete measure $\mu = \sum_{k=1}^n p_k \delta_{x_k}$ is a Popoviciu measure if and only if

$$\sum_{k=1}^n p_k > 0,\ \sum_{k=1}^m p_k(x_m - x_k) \geq 0 \text{ and } \sum_{k=m}^n p_k(x_k - x_m) \geq 0 \qquad (\mathrm{dEP})$$

for every $m \in \{1, \ldots, n\}$. A special case when (dEP) holds is the following, used by Steffensen in his extension of Jensen's inequality.

$$\sum_{k=1}^n p_k > 0 \text{ and } 0 \leq \sum_{k=1}^m p_k \leq \sum_{k=1}^n p_k \text{ for every } m \in \{1, \ldots, n\}. \qquad (\mathrm{dSt})$$

In fact, (dSt) $\Rightarrow$ (dEP) by Abel's summation formula (the discrete analogue of integration by parts).

A consequence of (dSt) and Theorem 8 is the following inequality of G. Szegö [36]. *If* $a_1 \geq a_2 \geq \cdots \geq a_{2m-1} \geq 0$ *and* f *is a convex function in* $[0, a_1]$, *then*

$$\sum_{k=1}^{2m-1} (-1)^{k-1} f(a_k) \geq f\left(\sum_{k=1}^{2m-1} (-1)^{k-1} a_k \right).$$

This corresponds to the measure $\mu = \sum_{k=1}^{2m-1}(-1)^{k-1}\delta_{a_k}$, whose barycenter is $x_\mu = \sum_{k=1}^{2m-1}(-1)^{k-1}a_k$.

In the case of absolutely continuous measures, $d\mu = p(x)\,dx$, the condition (EP) reads as

$$\int_a^b p(x)\,dx > 0,\ \int_a^t (t-x)p(x)\,dx \geq 0 \text{ and } \int_t^b (x-t)p(x)\,dx \geq 0 \qquad \text{(cEP)}$$

for every $t \in [a,b]$. In practice, the following continuous analogue of (dSt) suffices.

$$\int_a^b p(x)\,dx > 0 \text{ and } 0 \leq \int_a^t p(x)\,dx \leq \int_a^b p(x)\,dx \text{ for every } t \in [a,b]. \qquad \text{(cSt)}$$

In Section 5 we mentioned the notion of a n-convex function. The following problem arises naturally.

Problem 1. *Is it possible to extend Choquet's theory to the case of n-convex functions?*

A recent paper by M. Bessenyei and Z. Páles [4] provided a good starting point; namely, they proved the analogue of the Hermite-Hadamard inequality for higher-order convexity. However, as far as we know, even the analogue of Fink's aforementioned result to this context is unknown.

Books like [17] and [26] contain many valuable results related to Problem 1. We shall recall here an inequality of J. F. Steffensen [34], that sheds some light on the case of 0-convex functions. In the following form, it appeared in [38]. See also [26].

Theorem 9. *Let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $\lambda = \int_a^b g(t)\,dt \in [0, b-a]$. Then the following two conditions are equivalent:*

i) $0 \leq \int_a^x g(t)\,dt \leq x-a$ *and* $0 \leq \int_x^b g(t)\,dt \leq b-x$, *for every* $x \in [a,b]$;

ii) $\int_a^{a+\lambda} h(t)\,dt \leq \int_a^b h(t)g(t)\,dt \leq \int_{b-\lambda}^b h(t)\,dt$, *for every nondecreasing function* $h : [a,b] \to \mathbb{R}$.

PROOF. $i) \Rightarrow ii)$ In fact,

$$\begin{aligned}\int_a^b h(t)g(t)\,dt - \int_a^{a+\lambda} h(t)\,dt &= \int_a^{a+\lambda} h(t)\left(g(t)-1\right)\,dt + \int_{a+\lambda}^b h(t)g(t)\,dt \\ &= \int_a^{a+\lambda} h(t)\,d\left(\int_a^t g(s)\,ds - t + a\right) \\ &\quad - \int_{a+\lambda}^b h(t)\,d\left(\int_t^b g(s)\,ds\right) \\ &= \int_a^{a+\lambda}\left(\int_a^t g(s)\,ds - t + a\right)\,dh(t) \\ &\quad + \int_{a+\lambda}^b \left(\int_t^b g(s)\,ds\right)\,dh(t)\end{aligned}$$

which gives us the left hand inequality of ii). The other one can be obtained in a similar manner.

$ii) \Rightarrow i)$ Consider the functions $h = -\chi_{[a,x]}$ and $h = \chi_{[x,b]}$. $\square$

If $f : [a,b] \to \mathbb{R}$ is a Lipschitz function with $||f||_{\mathrm{Lip}} = M$, then f is differentiable a.e. and $0 \leq g = (f' + M)/(2M) \leq 1$ satisfies condition $i)$ of Theorem 9. For this choice of g and $h = x - a$, the condition $ii)$ yields Iyengar's inequality (I).

Theorem 9 outlines the importance of comparing Popoviciu measures by a weaker relation of majorization. Namely, for μ and λ two Popoviciu measures, $\mu \prec_w \lambda \Leftrightarrow \mu(h) \leq \lambda(h)$ for every continuous nondecreasing convex function h, where $\mu \prec_w \lambda$ means that $\mu([a,x]) \leq \lambda([a,x])$ and $\mu([x,b]) \leq \lambda([x,b])$ for every $x \in [a,b]$.

The role of Borel probability measures in the most different areas of mathematics is now well understood. It is natural to address the problem how important are the Popoviciu measures outside convexity, particularly, the following basic question.

Problem 2. *What's the physical signification of Popoviciu measures?*

8 The Case of Quasi Popoviciu Measures

Some classical convexity inequalities escapes even the above extension of Choquet's theory because the measure of the whole space is 0. However, we can still handle such cases by considering small perturbations $\mu_\varepsilon = \mu + \varepsilon\delta_z$ (for suitable $\varepsilon > 0$ and some z in K). They are Popoviciu measures, which makes it possible to apply Theorem 8. Then

$$f(x_\mu) \cdot (\mu(K) + \varepsilon) \leq \int_K f(x)\, d\mu(x) + \varepsilon f(z)$$

for every continuous convex function f on K. Letting $\varepsilon \to 0$, we arrive at the following result.

Proposition 2. *Let μ be a real Borel measure on a compact convex set K such that $\mu(K) = 0$ and $\int_K f^+\, d\mu(x) \geq 0$ for every continuous convex function f on K. Then $\int_K f(x)\, d\mu(x) \geq 0$ for every continuous convex function f on K.*

Due to Popoviciu's characterization of convex functions (mentioned in Section 6), better results can be proved on intervals.

Proposition 3. *Let μ be a real Borel measure defined on $[a,b]$ such that $\mu([a,b]) = 0$, $\int_a^t (t-x)\, d\mu(x) \geq 0$ and $\int_t^b (x-t)\, d\mu(x) \geq 0$ for every $t \in \mathbb{R}$. Then $\int_a^b f(x)\, d\mu(x) \geq 0$ for every convex function f on $[a,b]$.*

As an immediate consequence we obtain L. Fuchs' extension of the majorization principle.

Theorem 10. (*L. Fuchs [9]; see also [17], pp. 165-166*). *Let $f : [a,b] \to \mathbb{R}$ be a convex function. Then for every $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in [a,b]$ and every $p_1, \ldots, p_n \in \mathbb{R}$ such that:*

i) $x_1 > \cdots > x_n, \quad y_1 > \cdots > y_n$

ii) $\sum_{k=1}^{r} p_k x_k \leq \sum_{k=1}^{r} p_k y_k$ *for every* $r = 1, \ldots, n-1$

iii) $\sum_{k=1}^{n} p_k x_k = \sum_{k=1}^{n} p_k y_k$

we have the inequality $\sum_{k=1}^{n} p_k f(x_k) \leq \sum_{k=1}^{n} p_k f(y_k)$.

We pass now to the case of absolutely continuous measures.

Proposition 4. *Let $p(x)$ be a continuous or a monotonic density on an interval $[a,b]$, such that*

$$\int_a^b p(x)\, dx = 0,\ \int_a^t p(x)\, dx \geq 0 \ \text{and} \ \int_t^b p(x)\, dx \geq 0 \qquad \text{(qcSt)}$$

for every $t \in [a,b]$. Then $\int_a^b f(x)p(x)\, dx \geq 0$ for every convex function f on $[a,b]$.

When the graph of $p(x)$ is symmetric with respect to the line $x = (a+b)/2$, it suffices to ask (qcSt) only for t in the appropriate half interval. This remark allows us to retrieve the following result due to L. Lupaş [16]. *Suppose that* $g : [-a, a] \to \mathbb{R}$ *is an even function, nondecreasing on* $[0, a]$, *and* $f : [-a, a] \to \mathbb{R}$ *is a convex function. Then*

$$\frac{1}{2a}\int_{-a}^{a} f(x)g(x)\,dx \geq \left(\frac{1}{2a}\int_{-a}^{a} f(x)\,dx\right)\left(\frac{1}{2a}\int_{-a}^{a} g(x)\,dx\right).$$

In fact, $p(x) = g(x) - \frac{1}{2a}\int_{-a}^{a} g(x)\,dx$ is an even weight such that

$$\int_{-a}^{-t} p(x)\,dx = \int_{t}^{a} p(x)\,dx \geq 0$$

for every $t \in [0, a]$.

9 The Story Continues

While the left part of the Hermite-Hadamard inequality imposes a restriction on Popoviciu measures as a general framework for the whole inequality, the right part works outside of this restriction. More precisely, on each interval $[a, b]$ there are Borel measures μ which are not Popoviciu, yet

$$\frac{1}{\mu([a,b])}\int_{a}^{b} f(x)\,d\mu(x) \leq \frac{b - x_\mu}{b-a}\cdot f(a) + \frac{x_\mu - a}{b-a}\cdot f(b)$$

works for every convex function on [a,b]. Here $x_\mu = \int_a^b x\,d\mu(x)/\mu([a,b])$ is given by the same formula as in the case of Borel probability measures. In fact, as noticed by A. M. Fink [8],

$$\frac{3}{2}\int_{-1}^{1} f(x)(x^2 - x)dx \leq 1\cdot f(-1) + 0\cdot f(1) = f(-1)$$

for every convex function $f : [-1, 1] \to \mathbb{R}$. The proof of A. M. Fink appeals to the Green's function associated to $L(y) = y''$ (with homogeneous boundary conditions).

Of course, the phenomenon outlined above occurs in all dimensions. Recall the case of $d\mu = \prod_{k=1}^{n}(x_k^2 - x_k)dx_1 \ldots dx_n$ on $[-1, 1]^n$. This leads us naturally to a series of open problem. The most important one seems to be the question of characterizing the Borel measures μ (defined on a metrizable compact convex set K) such that $\mu(K) > 0$ and $\frac{1}{\mu(K)}\int_K f(x)\,d\mu(x) \leq \int_{\mathrm{Ext}\,K} f(x)\,d\lambda(x)$

for some Borel probability measure λ on $\operatorname{Ext} K$ and all continuous convex functions f on K.

A second big problem concerns the extension of the Hardy-Littlewood-Pólya theorem of majorization to this context. The case of Borel probability measures (and usual convexity) is covered in R. R. Phelps [27] via dilations of measures and orderings equivalent to majorization. However, almost nothing is known at the level of Popoviciu measures.

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