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Andrzej Nowik, Department of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland. email: nowik@manta.univ.gda.pl

Patrick Reardon, Department of Mathematics, SE Oklahoma State University, Durant, OK 74701, USA. email: preardon@sosu.edu

A DICHOTOMY THEOREM FOR THE ELLENTUCK TOPOLOGY

Abstract

Results are deduced from the following dichotomy which reduces descriptive properties concerning the Ellentuck topology to two well-known examples. **Theorem:** Every nonempty perfect set in the Ellentuck topology contains a closed copy of the Sorgenfrey line or a closed copy of the rational numbers. This leads to a Marczewski-Burstin representation for Marczewski sets in the Ellentuck topology.

1 Introduction

The aim of this paper is to prove the following dichotomy for the Ellentuck topology: every perfect set contains a closed copy of the Sorgenfrey line or a closed copy of the rationals. The Ellentuck topology is a special case (where $X = \omega$) of the exponential space associated with a topological space X. Introduced by Vietoris in 1922 [17], it has been studied by numerous authors (see for example [12], [16], [13], and [8]). The dichotomy leads to a Marczewski-Burstin representation for the class of Ellentuck Marczewski measurable sets in terms of closed copies of the Sorgenfrey line, each of which is a classical G_δ, completely Ramsey null, and of Lebesgue measure zero. Consequently, the classical Marczewski measurable sets form a proper subclass of the Ellentuck Marczewski measurable sets. It is well known that the corresponding result for the Baire property is false.

Relationships among the σ-algebras of Borel sets, C-sets, sets with the Baire property in both the wide and restricted senses, and sets with the Marczewski property are investigated under both the classical and Ellentuck

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topologies. J. B. Brown showed that under the continuum hypothesis there is a classical always first category set which is not Ramsey [3], and Darji in [5] improved this by constructing such a set using Martin's Axiom. In this note a weaker example is shown to exist in ZFC alone, namely a classical always first category set which is not an Ellentuck C-set. Some of the separations proved are known, and in these cases counterexamples with a simple descriptive structure relative to the Ellentuck topology are constructed. One of these answers a question due to Darji concerning the class of uniformly completely Ramsey sets [5].

Notation. The notation $X \subset Y$ means that X is a *proper* subset of Y. For $x, y \in [\omega]^{\leq\omega}$, let $[x, y] = \{z \in [\omega]^\omega : x \subseteq z \subseteq y\}$. The *classical topology* E on $[\omega]^\omega$ is generated by the sets $[x \cap n, x]$ where x is cofinite. The *Ellentuck topology* EL on $[\omega]^\omega$ is generated by sets of the form $[x \cap n, x]$ where $x \in [\omega]^\omega$. The *Sorgenfrey topology* S on $(0, 1]$ is generated by intervals of the form $(a, b]$. A set $M \subseteq [\omega]^\omega$ is *completely Ramsey* (CR) if for every $x \in [\omega]^\omega$ and $n < \omega$, there exists $z \in [x \cap n, x]$ and $m \geq n$ such that $[z \cap m, z] \subseteq M$ or $[z \cap m, z] \subseteq M^c$, and *completely Ramsey null* (CR_0) if the latter condition holds in every case. A set M has the *Baire property* (in the wide sense) if it is the symmetric difference of an open set and a first category set, the *Baire property in the restricted sense* if it has the Baire property relative to every perfect subspace, and is *always first category* if it is first category relative to every perfect subspace.

We distinguish the following classes in a topological space $\langle X, \mathcal{T}\rangle$: the Borel sets $B(\mathcal{T})$, the C-sets $C(\mathcal{T})$ (the smallest σ-algebra containing $B(\mathcal{T})$ and closed under the Souslin operation $\mathcal{A}$), sets with the Baire property $B_w(\mathcal{T})$, sets with the Baire property in the restricted sense $B_r(\mathcal{T})$, and the ideal $AFC(\mathcal{T})$ of always first category sets. If K is a subset of X, then $Cl_{\mathcal{T}}(K)$ denotes the closure of K relative to $\mathcal{T}$ and $\mathcal{T}|_K$ the relative subspace topology of K. If ϕ is a property such as "closed", "dense", etc. that has meaning in any topological space, then $\mathcal{T} - \phi$ denotes the subsets of X which satisfy ϕ relative to $\mathcal{T}$. Most of the topological terms in this paper refer to the Ellentuck topology, so the prefix $\mathcal{T}$ is often dropped whenever $\mathcal{T} = EL$.

Galvin and Prikry showed that classical Borel sets are completely Ramsey [7]. This was extended by Silver [15] and Mathias [10] to classical analytic sets. Elementary topological proofs of the Silver/Mathias result were given by Louveau in [9] and Ellentuck in [6], whose proof relied on the following.

Proposition 1.1. *For every $M \subseteq [\omega]^\omega$, M is completely Ramsey if and only if $M \in B_w(EL)$; M is completely Ramsey null if and only if M is first category if and only if M is nowhere dense.*

2 Closed Sorgenfrey Subspaces

If $x \not\subseteq y$ and $y \not\subseteq x$, then x is *incomparable* to y, denoted by $x \perp y$. Otherwise x is *comparable* to y, denoted by $x \sim y$. A subset of $[\omega]^\omega$ is an *incomparable set* if any two of its members are incomparable. A *chain of sets* is a sequence $\langle U_i : i \in I \subseteq \omega\rangle$, finite or infinite, such that if $i < j, x \in U_i$ and $y \in U_j$, then $x \subseteq y$. The term *linear* means linear with respect to inclusion.

Let $\mathcal{S}$ denote the Sorgenfrey line $\langle (0,1], S\rangle$. Van Douwen in [16] and Popov in [13] showed $[\omega]^\omega$ contains a closed linear copy of $\mathcal{S}$. A set $M \subseteq [\omega]^\omega$ has *Lebesgue measure zero* if its image under the characteristic function has measure zero with respect to the completion of the product measure on 2^ω.

Lemma 2.1. *Suppose $P \subseteq [\omega]^\omega$ is a closed linear copy of $\mathcal{S}$. Then $\mathcal{C}l_E(P)$ is a classical nowhere dense perfect set, and P differs from $\mathcal{C}l_E(P)$ by a countable set, is completely Ramsey null, and has Lebesgue measure zero.*

PROOF. Assume $P \subseteq [\omega]^\omega$ is a closed linear copy of $\mathcal{S}$. P has no isolated points so it has no $E-$isolated points either. Hence $\mathcal{C}l_E(P)$ is $E-$perfect. Suppose $x \in \mathcal{C}l_E(P) \setminus P$. Set $x_L = \bigcup\{z \in P : z \subset x\}$. P is closed under arbitrary unions so $x_L \in P$. Hence $x_L \subset x$. Assume $y \in P$ and $y \not\subseteq x$. Let p denote the least element of $y \setminus x$ and $q > p$ an arbitrary element of x. Choose $u \in [x \cap (q+1), (\omega \setminus (q+1)) \cup x] \cap P$. As $y \in P$ and $p \notin u$, we have $u \subseteq y$. This implies $q \in y$ and thus $x \subseteq y$. Therefore for each $x \in \mathcal{C}l_E(P) \setminus P$,

$$P = \{z \in P : z \subseteq x_L\} \cup \{z' \in P : x \subset z'\}.$$

It follows that $\mathcal{C}l_E(P)$ is linear. Neither open nor $E-$open sets are linear; so $\mathcal{C}l_E(P)$ is nowhere dense. Thus CR_0 by Proposition 1.1, and is classically nowhere dense as well. Associate each $x \in \mathcal{C}l_E(P)\setminus P$ with an element of $x \setminus x_L$. It follows that $\mathcal{C}l_E(P)\setminus P$ is countable. $\langle P, EL|_P\rangle$ is separable and uncountably dense so there is a countable sequence of coinfinite sets $\langle x_i \in P : i < \omega\rangle$ whose union is $\bigcup P$. Therefore $P \subseteq \bigcup_{i<\omega}[\emptyset, x_i]$. For each $i < \omega$, $[\emptyset, x_i]$ has Lebesgue measure zero and the same is true of P. $\square$

We remark that the analysis above indicates that P resembles a Cantor-like subset of $\mathcal{S}$, as the intervals $\{z : x_L \subset z \subseteq x\}$ contain no points of P.

Lemma 2.2. *Suppose X has no isolated points. If there exists $y \in X$ which is not a limit point of any incomparable set in X, then there is a point $z \in X$ and a set $U \subseteq [z, y]$ such that U is open relative to X and y is a limit point of U.*

PROOF. Assume X has no isolated points and $y \in X$ is not the limit point of any incomparable set in X. Suppose for all $k < \omega$ and every finite $\{z_i : i < k\} \subseteq X \cap [\emptyset, y] \setminus \{y\}$,

$$([y \cap k, y] \cap X) \not\subseteq \bigcup_{i<k} ([\emptyset, z_i] \cup [z_i, y]) .$$

For each k, choose $z_k \in [y \cap k, y] \cap X \setminus \left(\bigcup_{i<k}[\emptyset, z_i] \cup [z_i, y]\right)$. But then $\{z_i : i < \omega\}$ is an incomparable subset of X with y as a limit point. This is a contradiction. Therefore there exist $k < \omega$ and $\{z_i : i < k\} \subseteq X \cap [\emptyset, y] \setminus \{y\}$ such that $([y \cap k, y] \cap X) \subseteq \bigcup_{i<k}([\emptyset, z_i] \cup [z_i, y])$. Choose $N \geq k$ such that $z_i \cap N$ is a proper subset of $y \cap N$ for all $i < k$. It follows that $([y \cap N, y] \cap X) \subseteq \bigcup_{i<k}[z_i, y]$. This partitions $[y \cap N, y] \cap X$ into finitely-many pieces, one of which must contain a set open relative to X with y as a limit point. □

Lemma 2.3. *Suppose X has no isolated points. If there exists $y \in X$ which is not a limit point of any incomparable set in X, then there are points $z_0, z_1, \ldots$ in X and natural numbers $n_0, n_1, \ldots$ such that $\langle([z_i \cap n_i, z_i] \cap X) : i < \omega\rangle$ is a chain of nonempty sets with y as a limit point.*

PROOF. Suppose X has no isolated points and there exists $y \in X$ which is not a limit point of an incomparable set in X. Apply Lemma 2.2 to obtain $z_0 \in X$ and a nonempty set $U_0 \subseteq [z_0, y]$ which is open relative to X and such that y is a limit point of U_0. Choose $n_0 = 0$ and observe that $[z_0 \cap n_0, z_0] \cap X \subseteq [\emptyset, y] \cap X$. For the next step, Lemma 2.2 applied to U_0 yields $z_1 \in U_0$ and a set $U_1 \subseteq [z_1, y]$ which is open relative to U_0 and hence to X, and such that y is a limit point of U_1. Choose $n_1 < \omega$ such that $[z_1 \cap n_1, z_1] \cap X \subseteq U_0$. Observe that $\langle [z_0 \cap n_0, z_0] \cap X, [z_1 \cap n_1, z_1] \cap X\rangle$ is a chain of nonempty sets.

In general, suppose $k < \omega$ is arbitrary and sequences $\langle z_i \in X : i < k\rangle$ and $\langle n_i < \omega : i < k\rangle$ and a set $U_{k-1} \subseteq [z_{k-1}, y]$ have been defined such that $\langle [z_i \cap n_i, z_i] \cap X : i < k\rangle$ is a chain of sets and U_{k-1} is open relative to X and has y as a limit point. Apply Lemma 2.2 to U_{k-1} to get $z_k \in U_{k-1}$ and a set $U_k \subseteq [z_k, y]$ which is open relative to U_{k-1} and hence to X, and such that y is a limit point of U_k. Finally, choose $n_k < \omega$ such that $[z_k \cap n_k, z_k] \cap X \subseteq U_{k-1}$. It can be easily verified that $\langle [z_i \cap n_i, z_i] \cap X : i < k+1\rangle$ is a chain of sets and the induction is complete. It follows that $\langle [z_i \cap n_i, z_i] \cap X : i < \omega\rangle$ is a chain of nonempty sets with y as a limit point. □

Lemma 2.4. *Let $\mathbb{Q}_I$ denote the rationals in $(0, 1]$. The following are equivalent for all closed $M \subseteq [\omega]^\omega$.*

1. *M contains a closed linear copy of $\mathcal{S}$.*

2. *M contains an uncountable linear set.*

3. *M contains a nonempty set X without isolated points and no point of X is a limit point of an incomparable subset of X.*

4. *There is an increasing $\phi\colon \langle \mathbb{Q}_I, \leq\rangle \to \langle M, \subseteq\rangle$.*

PROOF. Suppose M is closed. (1) $\Rightarrow$ (2) is trivial. To see (2) $\Rightarrow$ (3), suppose $X \subseteq M$ is a linear uncountable set. Then $X \setminus \{x \in X : (\exists n < \omega)([x \cap n, x] \cap X \text{ is countable})\}$ is nonempty, linear, and has no isolated points.

(3) $\Rightarrow$ (4): Suppose $X \subseteq M$ is nonempty, has no isolated points, and no point of X is a limit point of an incomparable set in X. We will construct a dense set in X which is linear and indexed by $s \in \omega^{<\omega}$. Choose $z_\emptyset \in X$ to begin the construction. Apply Lemma 2.3 to $Y = [\emptyset, z_\emptyset] \cap X$ to get points $z_0, z_1, \ldots$ in Y and natural numbers $n_0, n_1, \ldots$ such that $z_\emptyset$ is a limit point of $\{z_i : i < \omega\}$ and $\langle [z_i \cap n_i, z_i] \cap Y : i < \omega\rangle$ is a chain of sets. In general, assume $t \in \omega^{<\omega}$ is arbitrary and that $z_{t^\frown i}$ and $n_{t^\frown i}$ have been defined for all $i < \omega$ such that z_t is a limit point of $\{z_{t^\frown i} : i < \omega\}$ and $\langle [z_{t^\frown i} \cap n_{t^\frown i}, z_{t^\frown i}] \cap X : i < \omega\rangle$ is a chain of subsets of $[z_t \cap n_t, z_t]$. Let $k < \omega$ be arbitrary and set $s = t^\frown k$. Apply Lemma 2.3 to $Y = [z_s \cap n_s, z_s] \cap X$ to get points $z_{s^\frown 0}, z_{s^\frown 1}, \ldots$ in Y and natural numbers $n_{s^\frown 0}, n_{s^\frown 1}, \ldots$ such that z_s is a limit point of $\{z_{s^\frown i} : i < \omega\}$ and $\langle [z_{s^\frown i} \cap n_{s^\frown i}, z_{s^\frown i}] \cap Y : i < \omega\rangle$ is a chain of sets. This completes the induction. Set $D = \{z_s : s \in 2^{<\omega}\}$ and observe that $\langle D, \subseteq\rangle \setminus \{z_\emptyset\}$ is countable, linear, dense and unbounded. Let $\phi\colon \langle \mathbb{Q}_I, \leq\rangle \to \langle D, \subseteq\rangle$ be an order isomorphism.

(4) $\Rightarrow$ (1): Assume $\phi\colon \langle \mathbb{Q}_I, \leq\rangle \to \langle M, \subseteq\rangle$ is increasing. Then ϕ generates an order isomorphism $\Psi\colon \langle (0,1], \leq\rangle \to \langle M, \subseteq\rangle$ given by $\Psi(x) = \bigcup_{y \in Q_I}\{\phi(y) : y < x\}$. Set $X = \text{range}(\Psi)$ and observe that X is linear. Clearly the Sorgenfrey topology on $(0,1]$ and the order topology on X generated by intervals of the form $(x,y]$ are homeomorphic. It only remains to show that the order topology on X is indeed the subspace topology of X.

Suppose $x \in [\omega]^\omega$, $n < \omega$, and $y \in [x \cap n, x] \cap X$ are arbitrary. Choose $z \in [y \cap n, y] \cap X \setminus \{y\}$. Then $(z, y] \cap X \subseteq [x \cap n, x] \cap X$, whence the order topology on X refines the subspace topology. For the other direction, suppose $a, b \in X$, $a \subset b$, and $y \in (a,b] \cap X$ are arbitrary. $[y \cap (k+1), y] \cap X \subseteq (a,b] \cap X$ for any $k \in y - a$. Thus the subspace topology refines the order topology. $\square$

3 Marczewski Sets

A set M in a topological space $\langle X, \mathcal{T}\rangle$ has the *Marczewski property* if every $\mathcal{T}-$perfect set has a $\mathcal{T}-$perfect subset Q such that $Q \subseteq M$ or $Q \subseteq M^c$, and has the *Marczewski null property* if the latter condition holds in every case. Define $(s)(\mathcal{T})$ and $(s)_0(\mathcal{T})$ to be, respectively, the sets in X with the Marczewski property and the Marczewski null property.

It is not hard to see that $B(E) \subseteq B(EL)$ and $C(E) \subseteq C(EL)$. The analogous result for the pair $B_w(E)$ and $B_w(EL)$ is false, and it is consistent with ZFC that it is not true for the pair $B_r(E)$ and $B_r(EL)$ either (see [3] or [5]). In this section it is shown that, unlike B_w and B_r, $(s)(E)$ is a proper subset of $(s)(EL)$.

Lemma 3.1. *Suppose $P \subseteq [\omega]^\omega$ is perfect and for every disjoint family $\{[x_i \cap n, x_i] \cap P : \ i < k\}$, there is an incomparable set $\{z_0, \ldots, z_{k-1}\}$ with $z_i \in [x_i \cap n, x_i] \cap P \setminus \{x_i\}$ for all $i < k$. Then P contains a closed copy of the rationals.*

PROOF. Suppose P satisfies the hypotheses. Choose $x_0, x_1 \in P$ such that $x_0 \in [\emptyset, x_1] \cap P \setminus \{x_1\}$, and $n_1 > 0$ such that $x_0 \cap n_1$ is a proper subset of $x_1 \cap n_1$. Apply the hypothesis to get $x_{00} \in [x_0 \cap n_1, x_0] \cap P \setminus \{x_0\}$ and $x_{10} \in [x_1 \cap n_1, x_1] \cap P \setminus \{x_1\}$ such that $x_{00} \perp x_{10}$. Set $x_{01} = x_0$ and $x_{11} = x_1$ and choose $n_2 > n_1$ such that $x_{s0} \cap n_2$ is a proper subset of $x_{s1} \cap n_2$ for $s \in \{0, 1\}$.

In general, suppose $n_k < \omega$ has been defined. Further suppose $\{x_s : \ s \in 2^{\leq k}\} \subseteq P$ is such that for all $t \in 2^{<k}$, $x_{t1} = x_t$ and $x_{t0} \cap n_k$ is a proper subset of $x_{t1} \cap n_k$. Apply the hypothesis to obtain an incomparable set $\{x_{s0} : \ s \in 2^k\}$ such that $x_{s0} \in [x_s \cap n_k, x_s] \cap P \setminus \{x_s\}$ for all $s \in 2^k$, and define $x_{s1} = x_s$. Finally, choose $n_{k+1} > n_k$ so that $x_{s0} \cap n_{k+1}$ is a proper subset of $x_{s1} \cap n_{k+1}$. This completes the construction. Set $D = \{x_s : s \in 2^{<\omega}\}$.

Clearly D is countable and has no isolated points; it only remains to show it is closed. Set $B = \bigcap_{n<\omega} \bigcup_{s \in 2^n} [x_s \cap n_{|s|}, x_s]$. B is closed and contains D. Let $y \in B \setminus D$. There is a sequence $c_0 \subset c_1 \subset \ldots$ of binary strings such that $y \in \bigcap_{k<\omega} [x_{c_k} \cap n_{|c_k|}, x_{c_k}]$. As $y \notin D$ assume that each c_k ends in "0". Suppose x_s is an arbitrary element of D. Choose distinct strings c_h and $s' \supseteq s$ such that $|c_h| = |s'| \geq |s| + 2$ and s' ends in "0". If $x_s \subseteq y$, then $x_{s'} \subset x_s \subseteq y \subset x_{c_h}$. But $x_{s'} \perp x_{c_h}$ by construction of D, a contradiction. Thus $x_s \notin [\emptyset, y]$ and it follows that D is closed. $\square$

Lemma 3.2. *If $P \subseteq [\omega]^\omega$ is perfect and does not contain a closed copy of the rationals, then there exist disjoint relative open subsets X and Y of P such that $a \in X$ and $b \in Y$ implies $a \subseteq b$.*

PROOF. Suppose P is perfect and does not contain a closed copy of the rationals. By Lemma 3.1 there is a disjoint family $\{[x_i \cap n, x_i] \cap P : \ i < k\}$ of relative open subsets of P such that if $w_i \in [x_i \cap n, x_i] \cap P \setminus \{x_i\}$ for all

$i < k$, then there exist indices $i \neq j < k$ such that $w_i \sim w_j$. Without loss of generality, assume that $x_0 < \ldots < x_{k-1}$ and that $n < \omega$ is large enough so $(x_q \cap n) \not\subseteq (x_p \cap n)$ for all $p < q < k$. If the conclusion to the theorem is false, then

$$\begin{gathered}\text{for any two indices } p \neq q < k, \text{ and}\\ \text{for any relative open sets } X \subseteq [x_p \cap n, x_p] \cap P \text{ and } Y \subseteq [x_q \cap n, x_q] \cap P,\\ \text{there exist points } a \in X \text{ and } b \in Y \text{ such that } a \perp b.\end{gathered} \tag{*}$$

We use this to construct an incomparable set $\{w_0, \ldots w_k\}$. The construction is by induction on $1 \leq \alpha < k$. If $\alpha = 1$, use $(*)$ to get $z_0 \in [x_0 \cap n, x_0] \cap P \setminus \{x_0\}$ and $z_1 \in [x_1 \cap n, x_1] \cap P \setminus \{x_1\}$ so that $z_0 \perp z_1$. Choose $n_1 \geq n$ such that $(z_0 \cap n_1) \perp (z_1 \cap n_1)$. In general, assume that $\alpha < k$ is arbitrary, and $n_\alpha \geq n$ and $(z_0, \ldots, z_\alpha) \in \prod_{\beta \leq \alpha} [x_\beta \cap n_\alpha, x_\beta] \cap P \setminus \{x_\beta\}$ have been defined such that $(z_\beta \cap n_\alpha) \perp (z_\gamma \cap n_\alpha)$ for all $\beta \neq \gamma \leq \alpha$. Use $(*)$ repeatedly as follows to complete this stage of the induction. Choose incomparable elements $v_0 \in [z_0 \cap n_\alpha, z_0] \cap P$ and $v^0_{\alpha+1} \in [x_{\alpha+1} \cap n_\alpha, x_{\alpha+1}] \cap P \setminus \{x_{\alpha+1}\}$; also choose $m_0 \geq n_\alpha$ such that $(v_0 \cap m_0) \perp (v^0_{\alpha+1} \cap m_0)$. For the next step, again use $(*)$ to choose incomparable elements $v_1 \in [z_1 \cap m_0, z_1] \cap P$ and $v^1_{\alpha+1} \in [v^0_{\alpha+1} \cap m_0, v^0_{\alpha+1}] \cap P$; and choose $m_1 \geq m_0$ such that $(v_1 \cap m_1) \perp (v^1_{\alpha+1} \cap m_1)$. Continue inductively for all $\beta \leq \alpha$. Finally, set $n_{\alpha+1} = m_\alpha$ and $v_{\alpha+1} = v^\alpha_{\alpha+1}$. This yields an incomparable set $\{v_i : i \leq \alpha + 1\}$ and completes the inductive step. As $\alpha < k$ is arbitrary there is an incomparable set $\{w_i : i < k\}$ such that $w_i \in [x_i \cap n, x_i] \cap P \setminus \{x_i\}$ for all $i < k$. But this contradicts our assumption that any such set must contain two comparable elements and the lemma follows. □

Theorem 3.3 (Dichotomy). *Every perfect set contains a closed copy of the rationals or a closed linear copy of the Sorgenfrey line.*

Proof. Assume P is perfect and does not contain a closed copy of the rationals. Construct a Cantor scheme as follows. Define $X_\emptyset = P$ and inductively, for $n < \omega$ and $t \in 2^n$, use Lemma 3.2 to obtain basic relative open sets $X_{t0}, X_{t1} \subseteq X_t$ such that $a \in X_{t0}$ and $b \in X_{t1}$ implies $a \subseteq b$. Basic open subsets of perfect sets are again perfect, hence X_{t0} and X_{t1} are perfect and the induction can continue. This completes the construction.

Let $s \in 2^{<\omega}$ denote a string that doesn't end in "1". For each $n < \omega$, write $X_{s1^n0} = [a^s_n \cap m^s_n, a^s_n] \cap P$, and assume without loss of generality that $a^s_n \in P$. By construction, $a^s_0 \subseteq a^s_1 \subseteq \ldots$, and it follows that $(\bigcup_{n<\omega} a^s_n)$ is a limit point of $\{a^s_n : n < \omega\}$. For each $K < \omega$, the tail $\{a^s_n : K < n\} \subseteq X_{s1^K}$, whence $(\bigcup_{n<\omega} a^s_n) \in \bigcap_{n<\omega} X_{s1^n}$. Set $c_s = (\bigcup_{n<\omega} a^s_n)$. Suppose $s \neq t \in 2^{<\omega}$ are strings that don't end in "1", let $\ell = \max(|s|, |t|)$, and set $p = s1^{\ell-|s|} \cap t1^{\ell-|t|}$.

Assume without loss of generality that $p0 \subseteq s1^{\ell-|s|}$ and $p1 \subseteq t1^{\ell-|t|}$. Then $c_s \in X_{p0}$, $c_t \in X_{p1}$, and thus $c_s \subseteq c_t$ by construction. As s and t were arbitrary, $\{c_s : s \in 2^{<\omega}\}$ is linear and clearly has no isolated points. Therefore P contains a closed linear copy of $\mathcal{S}$ by Lemma 2.4. $\square$

A class of sets $\mathcal{C}$ is said to have a *Marczewski-Burstin representation* in terms of a class Γ if membership in $\mathcal{C}$ can be characterized as follows: M belongs to $\mathcal{C}$ if and only if for every $P \in \Gamma$, there is some $Q \in \Gamma$ such that $Q \subseteq P$ and either $Q \subseteq M$ or $Q \subseteq M^c$. (See [4] for more on Marczewski-Burstin representations.) The *hereditary class* associated with $\mathcal{C}$, denoted by $\mathrm{H}(\mathcal{C})$, is the collection of all $M \in \mathcal{C}$ such that every subset of M is also in $\mathcal{C}$. Let $\mathcal{G}$ denote the collection of closed linear copies of $\mathcal{S}$ in $[\omega]^\omega$ and Π^0_1 the collection of classical G_δ sets. $\mathcal{G}$ is a proper subclass of $\Pi^0_1 \cap \mathcal{L}_0 \cap CR_0$ by Lemma 2.1 and in addition satisfies the important closure property that any perfect subset of a member of $\mathcal{G}$ is also in $\mathcal{G}$.

Theorem 3.4 (MB representation)**.** *For all $M \in [\omega]^\omega$,*

1. *$M \in (s)(EL)$ if and only if for every $P \in \mathcal{G}$, there exists $Q \in \mathcal{G}$, such that $Q \subseteq P$ and either $Q \subseteq M$ or $Q \subseteq M^c$.*
2. *$M \in H((s)(EL))$ if and only if for every $P \in \mathcal{G}$, there exists $Q \in \mathcal{G}$ such that $Q \subseteq P$ and $Q \subseteq M^c$.*

PROOF. The "only if" part of (1) follows easily since $\mathcal{G}$ is closed under perfect subsets. For the other direction, let $M \subseteq [\omega]^\omega$ and assume that for $P \in \mathcal{G}$, there exists $Q \in \mathcal{G}$ such that $Q \subseteq P$ and either $Q \subseteq M$ or $Q \subseteq M^c$. Let R be a perfect set. By the Dichotomy Theorem, it contains a closed copy of the rationals or a closed linear copy of $\mathcal{S}$. Suppose $P \subseteq R$ is a closed copy of the rationals. Every dense subset of P contains a closed copy of the rationals (see [14]). At least one of $M \cap P$ or $P \setminus M$ is dense in a relative clopen subset of P, hence contains a closed copy of the rationals. On the other hand, suppose $P \subseteq R$ is a closed linear copy of $\mathcal{S}$. By assumption there is a closed linear copy of $\mathcal{S}$, call it Q, so that $Q \subseteq P$ and either $Q \subseteq M$ or $Q \subseteq M^c$. Therefore $M \in (s)(EL)$.

For the characterization of H((s)(EL)), note that for $Q \in \mathcal{G}$, a standard Bernstein construction relative to the perfect subsets of Q yields a subset of Q which is not in $(s)(EL)$. The rest now follows by (1). $\square$

A collection $R \subseteq [\omega]^\omega$ is said to be an *almost disjoint family* if $|x \cap y| < \omega$ for any two $x, y \in R$.

Theorem 3.5. *$(s)(E) \subset (s)(EL)$ and $H((s)(EL)) \not\subseteq (s)(E)$.*

Proof. Suppose $M \subseteq [\omega]^\omega$ is in $(s)(E)$ and $P \in \mathcal{G}$. By Lemma 2.1, P contains an $E-$perfect subset Q. Let $R \subseteq Q$ be $E-$perfect such that $R \subseteq M$ or $R \subseteq M^c$. Then R is closed and inherits linearity from P. As every collection of disjoint open subsets of the Sorgenfrey line is countable, $R \setminus \{x \in R : (\exists n < \omega)([x \cap n, x] \cap R \text{ is countable})\}$ is a perfect subset of P, hence belongs to $\mathcal{G}$. It follows by Theorem 3.4 that $M \in (s)(EL)$.

Let $R \subseteq [\omega]^\omega$ be an almost disjoint family which is $E-$perfect. Then $|R \cap P| \leq 1$ for every $P \in \mathcal{G}$. Partition R into sets B and $R \setminus B$, both of which meet every $E-$perfect subset of R. Then $B \in H((s)(EL))$ by Theorem 3.4, but $B \notin (s)(E)$. □

Corollary 3.6. *$H((s)(EL)) \not\subseteq CR$.*

Proof. Brendle in [2] gave a ZFC example of a set in $(s)_0(E) \setminus CR$. □

Because of Corollary 3.6 the next result is somewhat surprising, although it should be noted that $(s)_0(EL)$ is not a σ-ideal.

Proposition 3.7. *$(s)_0(EL) \subseteq CR_0$.*

Proof. Assume M is not CR_0. Proposition 1.1 implies M is dense in some basic open set. Any set which satisfies the latter condition contains a closed copy of the rationals [14]. It follows that M is not $(s)_0(EL)$. □

Example 3.8. *There is a subset of $[\omega]^\omega$ which is CR_0, $\mathcal{L}_0$, and $E-$nowhere dense but not $(s)(EL)$.*

Proof. Suppose P is a closed linear copy of $\mathcal{S}$. Via a standard Bernstein construction relative to P, obtain sets B and B^c that intersect every perfect subset of P. It follows that B is not in $(s)(EL)$. By Lemma 2.1, P is CR_0, $\mathcal{L}_0$ and $E-$nowhere dense, therefore so is B. □

A set $\mathcal{F} \subseteq [\omega]^\omega$ is a *filter* if it is nonempty and closed under supersets and finite intersections. Every filter is closed and CR_0. A filter $\mathcal{F}$ is an *ultrafilter* if for every $x \subseteq \omega$, exactly one of x or $\omega \setminus x$ belongs to $\mathcal{F}$, and *principal* if there is some $g \in [\omega]^\omega$ such that $\mathcal{F} = \{x \in [\omega]^\omega : g \subseteq x\}$. It is well-known that a filter is nonprincipal if and only if every descending chain is infinite, and the latter condition implies that nonprincipal filters have no isolated points. Sierpiński showed that nonprincipal ultrafilters are neither $\mathcal{L}$ nor $B_w(E)$. Thus there is a set in $((s)(EL) \cap CR_0) \setminus (\mathcal{L} \cup B_w(E))$ which is perfect.

In [1] Aniszczyk, Frankiewicz and Plewik showed there is a set which is completely Ramsey null but not $(s)(E)$. Their example is an *arbitrary* non-$(s)(E)$ subset of a carefully chosen completely Ramsey null, $E-$perfect set.

The general character of this example makes it impossible to say whether it could possibly have a simple descriptive structure relative to EL.

Example 3.9. *There is a set in* $((s)(EL) \cap CR_0) \setminus (s)(E)$ *which is perfect.*

PROOF. Construct a binary tree T by setting $n <_T 2n+1$ and $n <_T 2n+2$ for all $n < \omega$, and let $\widehat{T}$ denote the collection of branches of T. $\widehat{T}$ is $E-$perfect and an almost disjoint family. Choose $M \subseteq \widehat{T}$ which is $E-$Bernstein relative to $\widehat{T}$, and write $M = \{m_\alpha : \alpha < 2^\omega\}$. For each α, let C_α denote the subsets of m_α whose difference with m_α is finite, and observe that C_α is a closed copy of the rationals. Define $P_M = \bigcup_{\alpha<2^\omega} C_\alpha$. Then $P_M \cap \widehat{T} = M$ and hence P_M is not in $(s)(E)$. In addition, it is not hard to see that P_M is CR_0 by noticing that any collection formed by taking one point from each C_α is almost disjoint, and that almost disjoint sets are CR_0. Since P_M has no isolated points, it only remains to show it is closed.

Assume x is a limit point of P_M. Further assume there exist $j < k \in x$ which are incomparable in the tree order $<_T$. Then no point of $\widehat{T}$ and thus no point of P_M is contained in $[x \cap (k+1), x]$. But this is impossible as x is a limit point. Hence x is a subset of some branch of T, and since it is a limit point of P_M, must be a subset of m_α for some α. It follows that x is a limit point of C_α, which is closed. Therefore $x \in P_M$. $\square$

A set $M \subseteq [\omega]^\omega$ is *uniformly completely Ramsey* (UCR) if and only if for every continuous $f \colon 2^\omega \to 2^\omega$, $f^{-1}(M)$ is Ramsey, where $[\omega]^\omega$ is conflated with 2^ω via the characteristic function. Darji [5] showed that $UCR \subset (s)(E)$ and asked whether the UCR sets could be characterized as either $B_r(EL)$ or $(s)(EL)$. Example 3.9 shows that even some perfect sets fail to be UCR.

Marczewski showed in [11] that $B_r(\mathcal{T}) \subseteq (s)(\mathcal{T})$ whenever $\mathcal{T}$ is a complete separable metric topology. Although EL is far from satisfying this hypothesis, the conclusion holds anyway.

Theorem 3.10. $B_r(EL) \subset (s)(EL)$ *and* $H((s)(EL)) \not\subseteq B_r(EL)$.

PROOF. Assume $M \in B_r(EL)$ and $P \in \mathcal{G}$. Then $M \cap P$ is in $B_w(EL|_P)$. Every $S-$open set can be written as the union of a classical open set and a countable set. Therefore $B_w(EL|_P) = B_w(E|_P)$ and by Lemma 2.1, $M \cap P \in B_w(E|_{\mathcal{C}l_E(P)})$. One of $M \cap P$ or $\mathcal{C}l_E(P) \setminus (M \cap P)$ must contain an $E-$perfect subset, and since $\mathcal{C}l_E(P) \setminus (M \cap P)$ differs from $(M^c \cap P)$ by a countable set, one of $M \cap P$ or $M^c \cap P$ contains an $E-$perfect set, say Q. Clearly Q is linear so $Q \setminus \{x \in Q : (\exists n < \omega)([x \cap n, x] \cap Q \text{ is countable})\}$ is perfect and thus belongs to $\mathcal{G}$. Theorem 3.4 implies $M \in (s)(EL)$, and the second part of the proposition follows from Corollary 3.6. $\square$

Example 3.11. *There is a subset of* $[\omega]^\omega$ *which is* $AFC(EL)$ *but not* $(s)(E)$.

Proof. Let M be constructed as in Example 3.9. Assume P is perfect, $x \in P$, and $n < \omega$. Suppose there is a point $z \in [x \cap n, x] \cap P \cap M$. Then $[\emptyset, z] \cap M = \{z\}$, and it follows that M is nowhere dense relative to P, hence $M \in AFC(EL) \setminus (s)(E)$. $\square$

Note that the technique used in the preceding proof shows that *every* almost disjoint family is $AFC(EL)$.

In arbitrary topological spaces $\langle X, \mathcal{T}\rangle$ it is well-known that $B(\mathcal{T}) \subseteq C(\mathcal{T}) \subseteq B_r(\mathcal{T}) \subseteq B_w(\mathcal{T})$. Starting with the Borel sets $B(\mathcal{T})$, form a hierarchy by closing alternately under the Souslin operation $\mathcal{A}$ and complementation. $C(\mathcal{T})$ is the first σ-algebra in this hierarchy closed under the Souslin operation.

A nonprincipal ultrafilter over ω shows $C(E) \subset C(EL)$. If M is a closed linear copy of $\mathcal{S}$, then $C(EL|_M) \subset B_r(EL|_M) \subset B_w(EL|_M)$. This derelativizes to the whole space so $C(E) \subset C(EL) \subset B_r(EL) \subset CR$. It is optimal in the following sense. Let $D(E)$ denote a class in the hierarchy strictly lower than $C(E)$ and M a closed linear copy of $\mathcal{S}$. $C(E|_M) \not\subseteq D(E|_M) = D(EL|_M)$. This also derelativizes to the whole space and thus $C(E) \not\subseteq D(EL)$. Compare this with Ellentuck's result [6] that $C(E) \subseteq CR$. A minor variation of this argument produces a set in $AFC(E) \setminus C(EL)$. This yields the following diagram.

$$\begin{tikzcd}
 & & & B_w(EL) = CR \\
B(EL) \arrow[r] & C(EL) \arrow[r] & B_r(EL) \arrow[r] \arrow[ur] & (s)(EL) \\
B(E) \arrow[u] \arrow[r] & C(E) \arrow[u] \arrow[r] & B_r(E) \arrow[r] \arrow[dr] & (s)(E) \arrow[u] \\
 & & & B_w(E)
\end{tikzcd}$$

With the exception of $B_r(E) \not\subseteq B_r(EL)$, all possible inclusions are indicated above and can be proved in ZFC. $AFC(E) \not\subseteq B_r(EL)$ is consistent with ZFC (see [3] or [5]), but whether $B_r(E) \not\subseteq B_r(EL)$ can be proved in ZFC alone is left as an open question. Note however that if $B_r^*(EL)$ denotes the collection of sets which have the Baire property relative to every subspace $\langle P, EL|_P\rangle$, where P is a closed linear copy of $\mathcal{S}$ or a perfect set which is first category relative to itself, then $B_r(E) \subseteq B_r^*(EL)$.

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References

[1] Bohdan Aniszczyk, Ryszard Frankiewicz, and Szymon Plewik, *Remarks on (s) and Ramsey-Measurable Functions*, Bull. Polish Acad. Sci. Math., **35** (1987), 479–485.

[2] Jorg Brendle, *Strolling Through Paradise*, Fund. Math., **148** (1995), 1–25.

[3] Jack B. Brown, *The Ramsey Sets and Related Sigma Algebras and Ideals*, Fund. Math., **136** (1990), 179–185.

[4] Jack B. Brown and Hussain Elalaoui-Talibi, *Marczewski-Burstin-like Characterizations of σ-Algebras, Ideals, and Measurable Functions*, Colloq. Math., **82, no. 2** (1999), 277–286.

[5] Udayan B. Darji, *Uniformly Completely Ramsey Sets*, Colloq. Math., **64** (1993), no. 2, 163–171.

[6] Eric Ellentuck, *A New Proof that Analytic Sets are Ramsey*, J. Symbolic Logic, **39** (1974), 163–165.

[7] Fred Galvin and Karel Prikry, *Borel Sets and Ramsey's Theorem*, J. Symbolic Logic, **38** (1973), 193–198.

[8] M. Ismail, Sz. Plewik, A. Szymanski, *On Subspaces of Exp(N)*, Rend. Circ. Mat. Palermo, Serie II, Tomo, **XLIX** (2000), 397–414.

[9] Alain Louveau, *Une Démonstration Topologique de Théorèmes de Silver et Mathias*, Bull. Sci. Math., **98** (1974), 97–102.

[10] A. R. D. Mathias, *On a Generalization of Ramsey's Theorem*, Notices AMS, **15** (1968), 931. Abstract 68T-E19.

[11] Edward Marczewski, *Sur une Classe de Fonctions de M. Sierpiński et la Classe Correspondante d'Ensembles*, Fund. Math., **24** (1935), 17–34.

[12] E. Michael, *Topologies on Spaces of Subsets*, TAMS, **71** (1951), 152–182.

[13] V. Popov, *On the Subspaces of* exp X, Colloquia Mathematica Societatis Janos Bolyai, **23**, Topology, Budapest (1978), 977–984.

[14] Patrick Reardon, *Ramsey, Lebesgue, and Marczewski Sets and the Baire Property*, Fund. Math., **149** (1996), 191–203.

[15] Jack Silver, *Every Analytic Set is Ramsey*, J. Symbolic Logic, **35** (1970), 60–64.

[16] Eric K. van Douwen, *The Pixley-Roy Topology on Spaces of Subsets*, Set Theoretic Topology, ed. G. M. Reed, Academic Press, New York, San Francisco, London (1977), 235–258.

[17] L. Vietoris, *Bereiche Zweiter Ordnung*, Monatshefte für Mathematik und Physik, **32** (1922), 258–280.