Real Analysis Exchange
ISSN:0147-1937
Vol. 31(1), 2005/2006, pp. 309–314

Zbigniew Grande, Institute of Mathematics, Bydgoszcz Academy, Plac Weyssenhoffa 11, 85-072 Bydgoszcz, Poland. email: grande@ab.edu.pl

SOME OBSERVATIONS ON THE SYMMETRICAL QUASICONTINUITY OF PIOTROWSKI AND VALLIN

Abstract

In this article I investigate the measurability of symmetrically quasicontinuous (in the sense of Piotrowski and Vallin [8]) functions and symmetrically cliquish functions $f : \mathbb{R}^2 \to \mathbb{R}$ with respect to the Euclidean topology and with respect to the density topology $T_d \times T_d$.

If (X, T_X) and (Y, T_Y) are topological spaces and (Z, ρ) is a metric space, then a function $f : X \times Y \to Z$ is said to be:

1. quasicontinuous (resp. cliquish) at a point $(x, y) \in X \times Y$, if for every set $U \times V \in T_X \times T_Y$ containing (x, y), and for each positive real η, there are nonempty sets $U' \in T_X$ contained in U and $V' \in T_Y$ contained in V such that $f(U' \times V') \subset K(f(x, y), \eta) = \{t \in Z : \rho(t, f(x, y)) < \eta\}$ (resp. $\operatorname{diam}(f(U' \times V')) = \sup\{\rho(f(t, t'), f(u, u')) : t, u \in U' \text{ and } t', u' \in V'\} < \eta$) ([6, 7]);

2. quasicontinuous at (x, y) with respect to x (alternatively y), if for every set $U \times V \in T_X \times T_Y$ containing (x, y), and for each positive real η, there are nonempty sets $U' \in T_X$ contained in U and $V' \in T_Y$ contained in V such that $x \in U'$ (alternatively $y \in V'$) and $f(U' \times V') \subset K(f(x, y), \eta)$ ([8]);

3. cliquish at (x, y) with respect to x (alternatively y), if for every set $U \times V \in T_X \times T_Y$ containing (x, y), and for each positive real η, there are nonempty sets $U' \in T_X$ contained in U and $V' \in T_Y$ contained in V such that $x \in U'$ (alternatively $y \in V'$) and $\operatorname{diam}(f(U' \times V')) < \eta$;

Key Words: quasicontinuity, cliquishness, separate continuity, symmetrical cliquishness, symmetrical quasicontinuity, measurability.
Mathematical Reviews subject classification: Primary 26B05; Secondary 54C05, 54C30
Received by the editors July 26, 2005
Communicated by: B. S. Thomson

4. symmetrically quasicontinuous (resp. symmetrically cliquish) at (x, y), if it is quasicontinuous (resp. cliquish) at (x, y) with respect to x and simultaneously with respect to y;
5. separately continuous if the sections $f_x(t) = f(x, t)$ and $f^y(u) = f(u, y)$, where $x, u \in X$ and $y, t \in Y$, are continuous.

Evidently, each quasicontinuous function is cliquish, and each symmetrically quasicontinuous function is quasicontinuous and symmetrically cliquish. Moreover, each symmetrically cliquish function is also cliquish. But there are quasicontinuous functions which are not symmetrically cliquish. For example, the function

$$f(x, y) = 0 \text{ for } x < 0 \text{ and } f(x, y) = 1 \text{ otherwise on } \mathbb{R}^2,$$

is quasicontinuous (with respect to the Euclidean topology $T_e \times T_e$), but it is not symmetrically cliquish at any point $(0, y)$, where $y \in \mathbb{R}$.

Let $X = Y = Z = \mathbb{R}$, and $T_X = T_Y = T_e$, where T_e denotes the Euclidean topology in $\mathbb{R}$, and let $\rho(z_1, z_2) = |z_1 - z_2|$ for $z_1, z_2 \in Z$. Then each separately continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is symmetrically quasicontinuous. It is well known that each separately continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is of the first class of Baire ([9]), and there are quasicontinuous functions from $\mathbb{R}^2 \to \mathbb{R}$ which are not Lebesgue measurable ([7]). We will prove the following:

Theorem 1. *Let $X = Y = Z = \mathbb{R}$, and $T_X = T_Y = T_e$, where T_e denotes the Euclidean topology in $\mathbb{R}$, and let $\rho(z_1, z_2) = |z_1 - z_2|$ for $z_1, z_2 \in Z$. There is a symmetrically quasicontinuous function $f : \mathbb{R}^2 \to \mathbb{R}$ which is not Lebesgue measurable.*

Proof. Let $C \subset [0, 1]$ be a Cantor set of positive measure such that $0, 1 \in C$. Enumerate all components of the set $[0, 1] \setminus C$ in a sequence (I_n) such that $I_n \cap I_m = \emptyset$ for $n \neq m$. In each interval I_n, we find two nondegenerate closed intervals J_n and K_n, such that $K_n \subset \text{int}(J_n) \subset J_n \subset I_n$ ($\text{int}(J_n)$ denotes the interior of J_n). Now, by induction, we will define two sequences of sets.

Step 1. For $n = 1$, let $M_{1,1} = [0, 1]$, and let $L_{1,1}$ be a closed interval such that $M_{1,1} \subset \text{int}(L_{1,1}) \subset (-\frac{1}{2}, 1 + \frac{1}{2})$. Moreover, let $M_{1,2}$ and $L_{1,2}$ be the unions of two disjoint nondegenerate closed intervals such that

$$C \subset M_{1,2} \subset \text{int}(L_{1,2}) \subset L_{1,2} \subset (-\frac{1}{2}, 1 + \frac{1}{2}) \setminus J_1.$$

Put

$$P_{1,1} = K_1 \times M_{1,1},\ Q_{1,1} = J_1 \times L_{1,1},\ \ P_{1,2} = M_{1,2} \times K_1 \text{ and } Q_{1,2} = L_{1,2} \times J_1.$$

Step 2. For $n = 2$, let $M_{2,1}$ and $L_{2,1}$ be the unions of two disjoint nondegenerate closed intervals such that

$$C \subset M_{2,1} \subset \text{int}(L_{2,1}) \subset L_{2,1} \subset (-\frac{1}{4}, 1 + \frac{1}{4}) \setminus J_1.$$

Similarly, let $M_{2,2}$ and $L_{2,2}$ be the unions of three disjoint nondegenerate closed intervals such that

$$C \subset M_{2,2} \subset \text{int}(L_{2,2}) \subset L_{2,2} \subset (-\frac{1}{4}, 1 + \frac{1}{4}) \setminus (J_1 \cup J_2).$$

Put

$$P_{2,1} = K_2 \times M_{2,1},\ Q_{2,1} = J_2 \times L_{2,1},\ P_{2,2} = M_{2,2} \times K_2 \text{ and } Q_{2,2} = L_{2,2} \times J_2.$$

Step n. Next, for $n > 2$, let $M_{n,1}$ and $L_{n,1}$ be the unions of n disjoint nondegenerate closed intervals such that

$$C \subset M_{n,1} \subset \text{int}(L_{n,1}) \subset L_{n,1} \subset (-\frac{1}{2^n}, 1 + \frac{1}{2^n}) \setminus (J_1 \cup \ldots \cup J_{n-1}).$$

Similarly, let $M_{n,2}$ and $L_{n,2}$ be the unions of $n + 1$ disjoint nondegenerate closed intervals such that

$$C \subset M_{n,2} \subset \text{int}(L_{n,2}) \subset L_{n,2} \subset (-\frac{1}{2^n}, 1 + \frac{1}{2^n}) \setminus (J_1 \cup \ldots \cup J_n).$$

Put

$$P_{n,1} = K_n \times M_{n,1},\ \ Q_{n,1} = J_n \times L_{n,1},\ \ P_{n,2} = M_{n,2} \times K_n, \text{ and } Q_{n,2} = L_{n,2} \times J_n.$$

For $n \geq 1$, let $f_{n,1} : Q_{n,1} \to [0, 1]$ be a continuous function such that

$$f_{n,1}(P_{n,1}) = 1, \text{ and } f_{n,1}(Q_{n,1} \setminus \text{int}(Q_{n,1})) = 0,$$

and let $g_{n,2} : Q_{n,2} \to [0, 1]$ be a continuous function such that

$$g_{n,2}(P_{n,2}) = 1, \text{ and } g_{n,2}(Q_{n,2} \setminus \text{int}(Q_{n,2})) = 0.$$

Let $E \subset C \times C$ be a nonmeasurable (in the sense of Lebesgue) set. Put

$$f(x) = \begin{cases} f_{n,1}(x) & \text{for } x \in Q_{n,1} \ \ n \geq 1 \\ g_{n,2}(x) & \text{for } x \in Q_{n,2} \ \ n \geq 1 \\ 1 & \text{for } x \in E \\ 0 & \text{otherwise on } \mathbb{R}^2. \end{cases}$$

□

Since f is continuous at all points $(x,y) \in \mathbb{R}^2 \setminus (C \times C)$, it is also symmetrically quasicontinuous there. Fix a point $(x,y) \in C \times C$, and open sets $U \ni x$ and $V \ni y$, and a positive real η. Assume that $(x,y) \in E$. There are indices n,k with $I_n \subset U$ and $I_k \subset V$. Since $x \in M_{k,2}$, the points $(x,v) \in P_{k,2}$ for $v \in K_k$. Fix a point $v \in K_k$, and observe that, from the continuity of f at (x,v), there are open intervals $V' \subset J_k \subset V$ and $U' \subset L_{k,2} \cap U$ such that $x \in U'$ and $f(U' \times V') \subset (1-\eta, 1]$. This proves that in this case f is quasicontinuous at (x,y) with respect to x. Similarly, we can prove that in this case f is quasicontinuous at (x,y) with respect to y. In the case $(x,y) \in (C \times C) \setminus E$ the reasoning is similar. So f is symmetrically quasicontinuous. Since $f^{-1}(1) \cap (C \times C) = E$, and E is a nonmeasurable set, the function f is nonmeasurable in the sense of Lebesgue. This finishes the proof.

Now we will consider the case of the density topology T_d in $\mathbb{R}$. Let μ denote the Lebesgue measure in $\mathbb{R}$. Recall that a point $x \in \mathbb{R}$ is a density point of a Lebesgue measurable set $A \subset \mathbb{R}$ if

$$\lim_{h \to 0^+} \frac{\mu(A \cap [x-h, x+h])}{2h} = 1,$$

and that the family T_d of all Lebesgue measurable subsets $A \subset \mathbb{R}$ for which the implication

$$x \in A \Longrightarrow x \text{ is a density point of } A$$

is true, is a topology called the density topology ([2, 10]).

In [11], W. Wilczyński proves that there is a Lebesgue nonmeasurable $(T_d \times T_d)$-quasicontinuous function $f : \mathbb{R}^2 \to \mathbb{R}$. We will prove the following:

Theorem 2. *Let $X = Y = Z = \mathbb{R}$, $T_X = T_Y = T_d$, and $\rho(z_1, z_2) = |z_1 - z_2|$ for $z_1, z_2 \in Z$. If at each point (x,y) a function $f : \mathbb{R}^2 \to \mathbb{R}$ is cliquish with respect to x or with respect to y, then f is Lebesgue measurable.*

PROOF. By a Lemma of Davies from [3] and [4], it suffices to prove that, for each real $\eta > 0$ and for each Lebesgue measurable subset $A \subset \mathbb{R}^2$ of positive measure, there is a Lebesgue measurable subset $B \subset A$ of positive measure with $\operatorname{diam}(f(B)) < \eta$. Fix a real $\eta > 0$ and a measurable set $A \subset \mathbb{R}^2$ of positive measure. There is a nonempty Lebesgue measurable subset $E \subset A$ such that the sections

$$E_x = \{v \in \mathbb{R}; (x,v) \in E\}$$

and

$$E^y = \{u \in \mathbb{R}; (u,y) \in E\},$$

for $x, y \in \mathbb{R}$, belong to T_d ([4]). Fix a point $(x, y) \in E$. Let $U = E^y \cap (x - 1, x + 1)$ and $V = E_x \cap (y - 1, y + 1)$. Assume that f is cliquish at (x, y) with respect to x. Then there are nonempty open sets $U', V' \in T_d$ such that $x \in U' \subset U$, $V' \subset V$ and $\text{diam}(f(U' \times V')) < \eta$. Since $x \in U'$, for each $v \in V'$ the point $(x, v) \in E$, and consequently, $x \in E^v \cap U'$. But $E^v \cap U' \in T_d$, so $\mu(E^v \cap U') > 0$ for $v \in V'$. Thus, by Fubini's theorem, the measurable set $E \cap (U' \times V')$ is of positive measure. Since $E \cap (U' \times V') \subset U' \times V'$, it follows that

$$\text{diam}(f(E \cap (U' \times V'))) < \eta.$$

If f is cliquish at (x, y) only with respect to y, the reasoning is analogous. This completes the proof. □

Corollary 1. *Let $X = Y = Z = \mathbb{R}$, $T_X = T_Y = T_d$, and $\rho(z_1, z_2) = |z_1 - z_2|$ for $z_1, z_2 \in Z$. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a symmetrically cliquish function, then f is Lebesgue measurable.*

Remark 1. *In the diagram in [8], the authors write that the separate continuity of functions $f : X \times Y \to Z$ implies the symmetrical quasicontinuity. This is true for $X = Y = Z = \mathbb{R}$ and the Euclidean topologies, but is not true for general cases. For example, in [5] (Example 4), it is proved that there exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ for which all sections f_x and f^y are T_d-continuous, and which is not $(T_d \times T_d)$-quasicontinuous (so it is not also symmetrically $(T_d \times T_d)$-quasicontinuous).*

It is well known that the limit of a pointwise convergent sequence (f_n) of quasicontinuous functions from $\mathbb{R}^2$ to $\mathbb{R}$ is a cliquish function ([7, 1]). This is not true for the symmetrical quasicontinuity and cliquishness. For example, the function

$$f(x, y) = 0, \text{ if } x < 0, \text{ and } f(x, y) = 1 \text{ otherwise on } \mathbb{R}^2,$$

is of the first class of Baire (so it is the limit of a pointwise convergent sequence of continuous functions), but it is not cliquish at $(0, y)$ with respect to x for $y \in \mathbb{R}$.

References

[1] J. Borsík, *Quasiuniform limits of quasicontinuous functions*, Math. Slovaca, **42** (1992), 269–274.

[2] A. M. Bruckner, *Differentiation of Real Functions*, Springer-Verlag, Berlin, 1978.

[3] R. O. Davies, *Separate approximate continuity implies measurability*, Proc. Cambridge Philos. Soc., **73** (1973), 461–465.

[4] Z. Grande, *La mesurabilité des fonctions de deux variables et de la superposition* $F(x, f(x))$, Dissertationes Math., **159** (1978), 1–50.

[5] Z. Grande, T. Natkaniec, *On some topologies of O'Malley type on* $\mathbb{R}^2$, Real Anal. Exchange, **18(1)** (1992/93), 241–248.

[6] S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math., **19**(1932), 184–197.

[7] T. Neubrunn, *Quasi-continuity*, Real Anal. Exchange, **14(2)** (1988/89), 259–306.

[8] Z. Piotrowski, R. W. Vallin, *Conditions which imply continuity*, Real Anal. Exchange, **29(1)** (2003/04), 211–217.

[9] R. Sikorski, *Real Functions I* (in Polish), PWN, Warsaw, 1957.

[10] F. D. Tall, *The density topology*, Pacific J. Math., **62** (1976), 275–284.

[11] W. Wilczyński, *A non-measurable* $d \times d$ *quasi-continuous function*, Bull. Soc. Sci. Lett. Łódź, **40.9** (1990), 147–150.