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ON A THEOREM OF VOLKMANN

Abstract

We generalize a theorem of Volkmann concerning the Hausdorff measures on subfields of $\mathbb{R}$. Our short proof is based on a mensural trichotomy law for invariant subsets of a locally compact group.

1 Introduction.

Let $s \in (0, 1)$ and let $\mathcal{H}^s$ denote the s-dimensional Hausdorff outer measure on $\mathbb{R}$. A theorem of Volkmann [8] states that any s-dimensional subfield F of $\mathbb{R}$ is "dimensionslos" in the sense that $\mathcal{H}^s(F) = 0$ or $\mathcal{H}^s(F \cap O) = +\infty$ for every nonvoid open subset O of $\mathbb{R}$. Edgar and Miller [1] have recently shown that there exists no s-dimensional subring of $\mathbb{R}$ that is an analytic set. In contrast to this, according to Foran [2] there exist s-dimensional subgroups G, H of $\mathbb{R}$ that are F_σ-sets with $\mathcal{H}^s(G) = 0$ and $\mathcal{H}^s(H \cap O) = +\infty$ for every nonvoid open set O. In view of these results, it therefore becomes interesting to know whether Volkmann's theorem can be extended to the *entire* class of subgroups of $\mathbb{R}$, and whether it admits a natural analogue in other locally compact groups.

In our note we address this problem. Precisely, in Section 3 we will see that in any locally compact group whose Haar measure is a generalized Hausdorff measure every dense subgroup is dimensionslos (in a sense even stronger than Volkmann's). The key idea of the proof turns out to be the mensural trichotomy law established in Lemma 1.

Throughout, $(G, \cdot)$ stands for a locally compact Hausdorff topological group with e as its identity element. By $\mathcal{B}$ and $\mathcal{B}_0$ we denote, respectively, the Borel

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σ-algebra of G and the Baire σ-algebra of G; i.e., the smallest σ-algebra with respect to which every real valued continuous function on G is measurable. Let $\mathcal{S}$ denote the σ-ideal generated by all compact subsets of G. Then $\mathcal{B} \cap \mathcal{S}$ and $\mathcal{B}_0 \cap \mathcal{S}$ coincide, respectively, with the σ-ring of Borel sets and the σ-ring of Baire sets in the terminology of Halmos.

Let $\lambda : \mathcal{B} \to [0, +\infty]$ be a left Haar measure on G; i.e., a nonzero left invariant σ-additive measure on G which is finite on compact sets and inner regular with respect to the compact sets. Let λ^* and λ_* denote, respectively, the outer and inner measures induced by λ on the power set $\mathcal{P}(G)$ of G. With $\mathcal{M}$ we indicate the σ-algebra of λ^*-measurable subsets of G. We denote the restriction of λ^* to $\mathcal{M}$ by λ again.

If H is a subgroup of G, a set $X \in \mathcal{P}(G)$ is *left H-invariant* if $hX \subseteq X$ for every $h \in H$ (equivalently, $hX = X$ for every $h \in H$).

2 A Mensural Trichotomy Law.

The next lemma is the basic tool for proving Theorem 5.

Lemma 1. *Let H be a dense subgroup of G and let X be a left H-invariant subset of G. Suppose further that $\mu^* : \mathcal{P}(G) \to [0, +\infty]$ is an outer measure such that every $B \in \mathcal{B}_0 \cap \mathcal{S}$ is μ^*-measurable and $\mu^*(hA) = \mu^*(A)$ for every $A \in \mathcal{P}(G)$ and $h \in H$. Then precisely one of the following cases occurs:*

i) $\mu^*(X \cap S) = 0$ *for every* $S \in \mathcal{S}$*;*

ii) $\mu^*(X \cap O) = +\infty$ *for every nonvoid open subset O of G;*

iii) there exists $c \in (0, +\infty)$ such that, for every $M \in \mathcal{M} \cap \mathcal{S}$,

$$\mu^*(X \cap M) = c\lambda^*(X \cap M) = c\lambda(M). \qquad (*)$$

PROOF. Suppose $\mu^*(X \cap S) > 0$ for a certain $S \in \mathcal{S}$ and $\mu^*(X \cap O) < +\infty$ for a certain nonvoid open set O. We prove the existence of $c \in (0, +\infty)$ as stated in *iii)*. First observe that $\nu(B) := \mu^*(X \cap B)$ defines a nonzero σ-additive measure on $\mathcal{B}_0 \cap \mathcal{S}$.

Let us show that $\nu(K) < +\infty$ for every compact Baire subset K of G. Since H is dense in G, there are $h_1, h_2, \ldots, h_n \in H$ such that $K \subseteq \bigcup_{i=1}^n h_i O$. Thus

$$\nu(K) \leq \sum_{i=1}^n \mu^*(X \cap h_i O) = \sum_{i=1}^n \mu^*(h_i X \cap h_i O) = n\mu^*(X \cap O) < +\infty.$$

Moreover, it is immediately seen that $\nu(hB) = \nu(B)$ for every $h \in H$ and $B \in \mathcal{B}_0 \cap \mathcal{S}$. Hence, by Theorem 62.G in [4] and the density of H in G, we get $\nu(gB) = \nu(B)$ for every $g \in G$ and $B \in \mathcal{B}_0 \cap \mathcal{S}$. Since both $I_\nu(f) := \int f d\nu$ and $I_\lambda(f) := \int f d\lambda$ define left Haar integrals on the space of all real valued continuous functions on G with compact support, by the uniqueness theorem for the Haar integral there is a constant $c \in (0, +\infty)$ such that $I_\nu = cI_\lambda$. Thus, $\nu(B) = c\lambda(B)$ for every $B \in \mathcal{B}_0 \cap \mathcal{S}$.[1]

To conclude the proof, we appeal to Theorem 64.I in [4]. Given $M \in \mathcal{M} \cap \mathcal{S}$, choose $A, B \in \mathcal{B}_0 \cap \mathcal{S}$ such that

$$X \cap M \subseteq B \text{ and } \lambda^*(X \cap M) = \lambda(B),$$
$$A \subseteq M \text{ and } \lambda(A) = \lambda(M).$$

Then the following chain of inequalities gives $(*)$:

$$\begin{aligned} c\lambda(M) &= c\lambda(A) = \nu(A) = \mu^*(X \cap A) \leq \mu^*(X \cap M) \\ &\leq \mu^*(X \cap B) = \nu(B) = c\lambda(B) = c\lambda^*(X \cap M) \leq c\lambda(M). \end{aligned}$$ □

Remark 2. In the case $G = (\mathbb{R}, +)$ the nonelementary part of the proof above (i.e., the existence of the constant c such that $\nu(B) = c\lambda(B)$ for every $B \in \mathcal{B}_0$ $(= \mathcal{B})$) can be derived in a very basic way. To see this, define $f : [0, +\infty) \to [0, +\infty)$ by $f(x) := \nu([0, x))$. Of course f is increasing and satisfies $f(g+h) = f(g) + f(h)$ for all nonnegative $g, h \in H$. Using the density of H in $\mathbb{R}$, it follows that f is continuous. Hence $f(x) = cx$ for some constant c. One then obtains $\nu([g, h)) = c\lambda([g, h))$ for every $g, h \in H$. Since $\mathcal{B}$ is generated by the semiring $\{[g, h) \mid g, h \in H\}$, we infer that ν and $c\lambda$ coincide on $\mathcal{B}$.

In generalizing Volkmann's theorem we will apply Lemma 1 in the case when μ^* is a generalized Hausdorff outer measure; still, it is worth mentioning that Lemma 1 is of interest also for $\mu^* = \lambda^*$. We recall that a set $A \in \mathcal{P}(G)$ is *completely nonmeasurable* if $A \cap M \notin \mathcal{M}$ whenever $M \in \mathcal{M}$ and $\lambda(M) > 0$ –equivalently, if $\lambda_*(A) = \lambda_*(G \setminus A) = 0$. Completely nonmeasurable sets (also called "saturated nonmeasurable" in the literature) have been studied since the early works of Sierpiński. A detailed account on them can be found in [6] (see also [9]).

Corollary 3. *Let H be a dense subgroup of G and X a left H-invariant subset of G such that $\lambda^*(X) > 0$ and $\lambda^*(G \setminus X) > 0$. Then X is completely nonmeasurable.*

[1] As an alternative argument, one can directly apply Exercise 60.7 in [4].

PROOF. For $\mu^* = \lambda^*$, condition *ii)* of Lemma 1 is never satisfied, while condition *i)* does not hold since $\lambda^*(X) > 0$. From $(*)$ we then infer $\lambda(M) = 0$ for every compact set M disjoint from X, which means $\lambda_*(G \setminus X) = 0$. The same argument for $G \setminus X$ yields $\lambda_*(X) = 0$. It follows that X is completely nonmeasurable. $\square$

3 A Generalization of Volkmann's Theorem.

In this section we present a generalization of Volkmann's theorem quoted in the introduction. It proves to be a consequence of the following proposition, which in turn follows directly from Lemma 1.

Proposition 4. *Besides the assumptions of Lemma 1 suppose that $\mu^*(S) < +\infty$ implies $\lambda^*(S) = 0$ for every $S \in \mathcal{S}$. Then:*

i) $\mu^*(X \cap S) = 0$ *for every* $S \in \mathcal{S}$ *or*

ii) $\mu^*(X \cap O) = +\infty$ *for every nonvoid open subset* O *of* G.

PROOF. The additional assumption on μ^* and *iii)* of Lemma 1 are incompatible, as can be seen by taking a compact neighborhood of e for M in $(*)$. $\square$

Proposition 4 immediately yields for $G = (\mathbb{R}, +)$ that *any subgroup* of the reals of Hausdorff dimension $s \in (0, 1)$ is dimensionslos in the sense of Volkmann, by taking for μ^* the s-dimensional Hausdorff outer measure $\mathcal{H}^s$. More generally, one obtains such a result for subgroups of a locally compact group G whose left Haar measure λ is the n-dimensional Hausdorff measure for some $n > 0$. This leads in a natural way to the question when the left Haar measure λ on G coincides with a Hausdorff measure. It is well known that this is the case if G is a locally compact linear space over a locally compact field (i.e., a finite dimensional linear space over $\mathbb{R}$ or over a p-adic number field $\mathbb{Q}_p$ or over a field $F((X))$ of formal power series in one indeterminate with coefficients in a finite field). As proved by Goetz [3], this is also the case when G is an n-dimensional Lie group. Another result in this direction has been given by Kahnert [5]. He proved that for a much larger class of locally compact groups (namely for separable locally compact groups of finite topological dimension) any left Haar measure is a *generalized* Hausdorff measure (possibly different from the Hausdorff measures in the usual sense). We will generalize Volkmann's theorem in this situation.

First we recall the notion of a generalized Hausdorff outer measure. Suppose that G is metrizable and let d be a left invariant metric inducing the topology of G. Let $\mathcal{F}$ be the class of all continuous, increasing functions $f : [0, +\infty) \to [0, +\infty)$ such that $f(t) > 0$ for every $t > 0$ and $f(0) = 0$. For $f \in \mathcal{F}$ and $A \in \mathcal{P}(G)$ we define

$$\mu_f^*(A) := \sup_{t>0} \inf \left\{ \sum_{i=1}^{+\infty} f(\delta(A_i)) \;\middle|\; A \subseteq \bigcup_{i=1}^{+\infty} A_i,\ \delta(A_i) \le t \text{ for each } i \in \mathbb{N} \right\},$$

where $\delta(A_i)$ denotes the diameter of A_i with respect to d. (If $f(t) = t^s$ for some $s > 0$, then μ_f^* is the usual s-dimensional Hausdorff outer measure.)

It is well known that $\mu_f^* : \mathcal{P}(G) \to [0, +\infty]$ is a left invariant outer measure and that every Borel set of G is μ_f^*-measurable (see [7], Theorem 27). Therefore, if $\lambda(K) = \mu_f^*(K)$ for some compact neighborhood of e, then μ_f^* is a Haar measure on $\mathcal{B} \cap \mathcal{S}$ and so coincides on $\mathcal{B} \cap \mathcal{S}$ with λ, by the uniqueness theorem for Haar measures. (Recall that on $\mathcal{B} \cap \mathcal{S}$ the regularity condition required in the definition of a Haar measure is automatically satisfied, by Halmos's Theorem 64.I [4].)[2] It then follows that $\mu_f^*(S) = \lambda^*(S)$ for all $S \in \mathcal{S}$, as for every $A \in \mathcal{P}(G)$ it holds $\mu_f^*(A) = \inf\{\mu_f^*(B) : A \subseteq B,\ B \in \mathcal{B}\}$ ([7], Theorem 27).

For $f, g \in \mathcal{F}$ we write $g \prec f$ if $\lim_{t \to 0^+} f(t)/g(t) = 0$. If $g \prec f$, then $\mu_g^*(A) < +\infty$ implies $\mu_f^*(A) = 0$ for every $A \in \mathcal{P}(G)$ ([7], Theorem 40).

From these facts on Hausdorff measures and Proposition 4 we immediately obtain the announced generalization of Volkmann's theorem.

Theorem 5. *Let H be a dense subgroup of G and X be a left H-invariant subset of G. Suppose that the topology of G is induced by a left invariant metric d and that, for some $f \in \mathcal{F}$ and some compact neighborhood K of e, $\mu_f^*(K) \in (0, +\infty)$. Let $g \in \mathcal{F}$ with $g \prec f$. Then:*

i) $\mu_g^*(X \cap S) = 0$ *for every* $S \in \mathcal{S}$ *or*

ii) $\mu_g^*(X \cap O) = +\infty$ *for every nonvoid open subset* O *of* G.

(Of course, μ_f^* *and* μ_g^* *are defined with respect to* d*.)*

Remark 6. Even in the situation considered by Volkmann (where $G = (\mathbb{R}, +)$ and $g(t) = t^s$ and thus $\mu_g^* = \mathcal{H}^s$), Theorem 5 cannot be sharpened in the sense that $\mathcal{H}^s(X \cap B)$ admits only the values 0 or $+\infty$ for every Borel set B. To see this, let H be a subgroup of $\mathbb{R}$ that is a Borel set with Hausdorff dimension $s \in (0, 1)$ and $\mathcal{H}^s(H) = +\infty$ (as mentioned in the introduction, such a group does exist [2]). By Theorem 57 in [7], H contains a compact set K with $\mathcal{H}^s(K) \in (0, +\infty)$. Therefore, for $X = H$ we have $\mathcal{H}^s(X \cap K) \in (0, +\infty)$.

[2] Of course, this conclusion can also be drawn from Lemma 1 for $\mu^* = \mu_f^*$ and $X = G$.

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