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THE SET OF CONTINUOUS PIECEWISE DIFFERENTIABLE FUNCTIONS

Abstract

The purpose of this article is to show that if $-\infty < \alpha < \beta < \infty$, then the set PD_α^β of piecewise differentiable functions in $C([\alpha, \beta], \mathbb{R})$ is $\mathbf{\Pi}_1^1$-complete.

1 Introduction.

Two notions which are introduced in elementary calculus (see, for example, 5.15 on pages 176-179 of [4]) are the notion of a piecewise continuous function and the notion of a piecewise differentiable function of class C^1. By analogy, one can define the notion of a continuous *piecewise differentiable* function, as follows

> Given any real numbers α and β such that $\alpha < \beta$, a continuous function $f : [\alpha, \beta] \to \mathbb{R}$ is said to be piecewise differentiable if there exist $n \in \mathbb{N} \setminus \{0\}$ and points $x_1, \ldots, x_n$ in $[\alpha, \beta]$ such that $x_0 = \alpha < x_1 < \cdots < x_n < \beta = x_{n+1}$ and for any $i \in \{0, \ldots, n\}$, the restriction of f on (x_i, x_{i+1}) is everywhere differentiable.

Our purpose in this article is to prove the following result.

Theorem 1.1. *Given any real numbers α and β such that $\alpha < \beta$, the set PD_α^β of piecewise differentiable functions in $C([\alpha, \beta], \mathbb{R})$ is $\mathbf{\Pi}_1^1$-complete and therefore not Borel.*

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Thus, given any real numbers α and β such that $\alpha < \beta$, there exist no Borel measurable necessary and sufficient conditions on a continuous function $[\alpha, \beta] \to \mathbb{R}$, which can assert that it is piecewise differentiable; in other words, any necessary and sufficient conditions for piecewise differentiability of a continuous function $[\alpha, \beta] \to \mathbb{R}$ cannot be expressed analytically through explicit formulas, simple analytic expressions and the like.

Finally, we should mention that differentiability properties which give rise to non-Borel sets were also given by Mazurkiewicz in [3], Mauldin in [2], and the author in [6].

2 Elements from Descriptive Set Theory.

Descriptive set theory is the study of definable sets in Polish spaces, which are defined as separable completely metrizable spaces. In this theory sets are classified in the Borel and the projective hierarchy according to the complexity of their definition. Given a Polish space X, the first level of the Borel hierarchy that corresponds to X consists of the class of its $\mathbf{\Sigma}^0_1$-sets or G-sets, which are by definition its open sets, and the class of its $\mathbf{\Pi}^0_1$-sets or F-sets, which are by definition its closed sets; the second level consists of the class of its $\mathbf{\Sigma}^0_2$-sets or F_σ-sets, which are defined as countable unions of its $\mathbf{\Pi}^0_1$-sets, and the class of its $\mathbf{\Pi}^0_2$-sets or G_δ-sets, which are defined as countable intersections of its $\mathbf{\Sigma}^0_1$-sets; the third level consists of the class of its $\mathbf{\Sigma}^0_3$-sets or $G_{\delta\sigma}$-sets, which are defined as countable unions of its $\mathbf{\Pi}^0_2$-sets, and the class of its $\mathbf{\Pi}^0_3$-sets or $F_{\sigma\delta}$-sets, which are defined as countable intersections of its $\mathbf{\Sigma}^0_2$-sets, etc. On the other hand, the first level of the projective hierarchy that corresponds to X consists of the class of its analytic or $\mathbf{\Sigma}^1_1$-sets, which are defined as continuous images of Polish spaces, and the class of its co-analytic or $\mathbf{\Pi}^1_1$-sets, which are defined as complements of its $\mathbf{\Sigma}^1_1$-sets; the second level consists of its $\mathbf{\Sigma}^1_2$-sets, which are defined as continuous images of $\mathbf{\Pi}^1_1$-sets, and the class of its $\mathbf{\Pi}^1_2$-sets, which are defined as complements of its $\mathbf{\Sigma}^1_2$-sets, etc. (See, for example, the Introduction, 11.B on pages 68-69, 25.A on pages 196–197, 32.A on pages 242–243, and 37.A on pages 313–315 of [1].)

Given a class $\mathbf{\Gamma}$ of sets in either the Borel or the projective hierarchy, if X and Y are any Polish spaces, then we call a $\mathbf{\Gamma}$-set $B \subseteq Y$ *Wadge reducible* to a set $A \subseteq X$, in symbols $B \leq_W A$, if there exists a continuous mapping $f : Y \to X$ such that $B = f^{-1}[A]$; moreover, we call A $\mathbf{\Gamma}$-*hard*, if for any Polish space Y and for any $\mathbf{\Gamma}$-set $B \subseteq Y$, we have $B \leq_W A$, and, in particular, we call A $\mathbf{\Gamma}$-*complete*, if it also constitutes a $\mathbf{\Gamma}$-set. A powerful technique to find a lower bound for the complexity of a given set is to show that it is $\mathbf{\Gamma}$-hard for some class $\mathbf{\Gamma}$ of sets in either the Borel or the projective hierarchy, usually by proving that another set which is known to be $\mathbf{\Gamma}$-hard is Wadge reducible to it, and by showing that it is $\mathbf{\Gamma}$-complete we compute its exact complexity. (See, for example, 21.13 on page 156, 22.B on pages 169–170, and 26.C on pages 206–207 of [1].)

3 Trees and Continuous Functions.

Trees are basic combinatorial tools in descriptive set theory. A *tree* on $\mathbb{N}$ is a subset T of the set $\mathbb{N}^{<\mathbb{N}} = \cup_{n\in\mathbb{N}}\mathbb{N}^n$ of all finite sequences of natural numbers, which is closed under initial segments, and its *body* is

$$[T] = \left\{\alpha \in \mathbb{N}^{\mathbb{N}} : (\forall n \in \mathbb{N})(\alpha|n \in T)\right\},$$

where $\alpha|n = (\alpha(0), \ldots, \alpha(n-1))$. A tree is usually viewed as an element of $2^{\mathbb{N}^{<\mathbb{N}}}$ by identifying it with its characteristic function, where $2^{\mathbb{N}^{<\mathbb{N}}}$ is equipped with the product topology with $2 = \{0, 1\}$ discrete, making it homeomorphic to the Cantor space, a closed subset of which is the set Tr of all trees on $\mathbb{N}$. Thus, Tr acquires the structure of a Polish space (i.e., a separable completely metrizable space), and it is partitioned into two characteristic subsets, the set

$$IF = \{T \in Tr : [T] \neq \emptyset\}$$

of *ill-founded* trees on $\mathbb{N}$, which is $\mathbf{\Sigma}^1_1$-complete, and the set

$$WF = \{T \in Tr : [T] = \emptyset\}$$

of *well-founded* trees on $\mathbb{N}$, which is $\mathbf{\Pi}^1_1$-complete. (See, for example, 2.A on pages 5–6, 4.32 on pages 27–28, 2.E on page 10, 27.1 on page 209, and 32.B on page 243 of [1].)

What we are about to present below constitutes a refinement of the construction given on pages 248–249 of [1], which is based on the construction given on page 63 of the doctoral dissertation [5] that the author worked out at the California Institute of Technology during the years 1995–1999 under the direction of Alexander S. Kechris. (The reader who cannot find [5] may also look it up on pages 738–739 of [6].)

In what follows let $-\infty < \alpha < \beta < \infty$. We fix any standard coding $\langle\cdot\rangle$ of all the finite sequences of natural numbers by all the natural numbers and given any real numbers a and b such that $a < b$, we denote by $\psi_{[a,b]}$ the function in $C^1(\mathbb{R}, \mathbb{R}_+)$ which is defined by the relation

$$\psi_{[a,b]}(x) = \begin{cases} \frac{16(x-a)^2(x-b)^2}{(b-a)^3} & \text{if } x \in [a, b], \\ 0 & \text{if } x \in \mathbb{R} \setminus [a, b]. \end{cases}$$

It is not difficult to verify that

$$\begin{aligned}&\psi_{[a,b]} = 0 \text{ off } [a,b],\\ &\psi_{[a,b]} \text{ attains its maximum at the point } \frac{a+b}{2},\\ &\psi_{[a,b]}\left(\frac{a+b}{2}\right) = b-a.\end{aligned} \tag{3.1}$$

Let n be an arbitrary but fixed natural number. We set[1]

$$a_{\emptyset}^{(n)} = \alpha + \sum_{m=1}^{n} \frac{\beta-\alpha}{2^m},\ b_{\emptyset}^{(n)} = \alpha + \sum_{m=1}^{n+1} \frac{\beta-\alpha}{2^m},\ \text{and } J_{\emptyset}^{(n)} = \left(a_{\emptyset}^{(n)}, b_{\emptyset}^{(n)}\right),$$

and

$$\begin{aligned}c_{\emptyset}^{(n)} &= \frac{a_{\emptyset}^{(n)}+b_{\emptyset}^{(n)}}{2} - \frac{b_{\emptyset}^{(n)}-a_{\emptyset}^{(n)}}{2\cdot(1+2^{\langle\emptyset\rangle})},\\ d_{\emptyset}^{(n)} &= \frac{a_{\emptyset}^{(n)}+b_{\emptyset}^{(n)}}{2} + \frac{b_{\emptyset}^{(n)}-a_{\emptyset}^{(n)}}{2\cdot(1+2^{\langle\emptyset\rangle})},\\ K_{\emptyset}^{(n)} &= \left[c_{\emptyset}^{(n)}, d_{\emptyset}^{(n)}\right].\end{aligned}$$

Now, let $s \in \mathbb{N}^{<\mathbb{N}}$ be such that $J_s^{(n)} = \left(a_s^{(n)}, b_s^{(n)}\right)$ and $K_s^{(n)} = \left[c_s^{(n)}, d_s^{(n)}\right]$ are already defined and let $i \in \mathbb{N}$. If, for convenience, we put $c = c_s^{(n)}$ and $d = \frac{c_s^{(n)}+d_s^{(n)}}{2}$, then we set

$$a_{s^\frown i}^{(n)} = c + \sum_{j=1}^{2i+1} \frac{d-c}{2^j},\ b_{s^\frown i}^{(n)} = c + \sum_{j=1}^{2i+2} \frac{d-c}{2^j}, J_{s^\frown i}^{(n)} = \left(a_{s^\frown i}^{(n)}, b_{s^\frown i}^{(n)}\right),$$

and if, for convenience, we put $J_{s^\frown i}^{(n)} = (a,b)$, then we set

$$\begin{aligned}c_{s^\frown i}^{(n)} &= \frac{a+b}{2} - \frac{b-a}{2\cdot(1+2^{\langle s^\frown i\rangle})},\\ d_{s^\frown i}^{(n)} &= \frac{a+b}{2} + \frac{b-a}{2\cdot(1+2^{\langle s^\frown i\rangle})},\\ K_{s^\frown i}^{(n)} &= \left[c_{s^\frown i}^{(n)}, d_{s^\frown i}^{(n)}\right].\end{aligned}$$

[1]For the sake of symmetry, it is understood that $a_{\emptyset}^{(0)} = \alpha + \sum_{m=1}^{0} \frac{\beta-\alpha}{2^m} = \alpha$.

If for any $s \in \mathbb{N}^{<\mathbb{N}}$, we denote by $LK_s^{(n)}$ the left half of $K_s^{(n)}$ and by $RK_s^{(n)}$ the right half of $K_s^{(n)}$, then it is not difficult to see that all the $RK_s^{(n)}$ are pairwise disjoint, and if $|I|$ stands for the length of an arbitrary interval I in the real line, then $|J_\emptyset^{(n)}| = \frac{\beta-\alpha}{2^{n+1}}$, $|K_s^{(n)}| = \frac{1}{1+2^{\langle s\rangle}} \cdot |J_s^{(n)}|$ and

$$|RK_s^{(n)}| \leq \frac{1}{1+2^{\langle s\rangle}} \cdot \frac{\beta-\alpha}{2^{n+2}}. \tag{3.2}$$

Moreover, for any $x \in \mathbb{N}^{\mathbb{N}}$, it is not difficult to verify that there exists a unique $\gamma \in (\alpha, \beta)$ such that

$$\bigcap_{i\in\mathbb{N}} J_{x|i}^{(n)} = \bigcap_{i\in\mathbb{N}} K_{x|i}^{(n)} = \bigcap_{i\in\mathbb{N}} LK_{x|i}^{(n)} = \{\gamma\}.$$

Now, given any $T \in Tr$, we set $F_T = \sum_{n=0}^{\infty} \sum_{s\in T} \psi_{RK_s^{(n)}}$ and given any $n \in \mathbb{N}$, we set

$$G_T^{(n)} = \bigcap_{k\in\mathbb{N}} \bigcup_{s\in T\cap\mathbf{N}^k} J_s^{(n)}.$$

Then, we have the following result.

Theorem 3.1. *The mapping*

$$Tr \ni T \mapsto F_T \in C([\alpha,\beta],\mathbb{R}) \tag{$\star$}$$

is well-defined and continuous, while for any $T \in Tr$ and for any $n \in \mathbb{N}$, we have

$$T \in IF \iff G_T^{(n)} \neq \emptyset. \tag{$\star\star$}$$

PROOF. Since the double series

$$\sum_{n=0}^{\infty} \sum_{s\in\mathbb{N}^{<\mathbb{N}}} \left(\frac{1}{1+2^{\langle s\rangle}} \cdot \frac{\beta-\alpha}{2^{n+2}}\right)$$

is easily seen to converge absolutely, by virtue of (3.1) and (3.2), if one makes use of the Weierstrass M-test, it is not difficult to see that F_T is continuous. So ($\star$) is well-defined. Moreover, since for any $(m,n) \in \mathbb{N}^2$ and for any $(s,t) \in \left(\mathbb{N}^{<\mathbb{N}}\right)^2$, we have

$$(m \neq n \ \vee \ s \neq t) \Rightarrow RK_s^{(m)} \cap RK_t^{(n)} = \emptyset,$$

it is not difficult to see that for any $n \in \mathbb{N}$, we have

$$F_T = \begin{cases} \sum_{s \in T} \psi_{RK_s^{(n)}} & \text{on } K_\emptyset^{(n)}, \\ 0 & \text{off } \bigcup_{n \in \mathbf{N}} K_\emptyset^{(n)}. \end{cases}$$

Now, let $\epsilon > 0$ and let k be any natural number such that $\frac{\beta-\alpha}{2^{k+1}} < \epsilon$. If S, T are any trees on $\mathbb{N}$ which agree for any $s \in \mathbb{N}^{<\mathbb{N}}$ for which $\langle s \rangle < k$, then for any $x \in [\alpha, \beta)$, there exists a unique $n \in \mathbb{N}$ such that $x \in \left[a_\emptyset^{(n)}, b_\emptyset^{(n)}\right)$. Hence

$$F_S(x) - F_T(x) = \sum_{s \in S; \langle s \rangle \geq k} \psi_{RK_s^{(n)}}(x) - \sum_{s \in T; \langle s \rangle \geq k} \psi_{RK_s^{(n)}}(x),$$

which implies that

$$\begin{aligned} |F_S(x) - F_T(x)| &\leq \sum_{\langle s \rangle \geq k} \psi_{RK_s^{(n)}}(x) \leq \sum_{\langle s \rangle \geq k} |RK_s^{(n)}| \leq \sum_{\langle s \rangle \geq k} \frac{1}{1 + 2^{\langle s \rangle}} \frac{\beta - \alpha}{2^{n+2}} \\ &\leq \frac{\beta - \alpha}{2^{n+2}} \cdot \sum_{j=k}^{\infty} 2^{-j} = \frac{\beta - \alpha}{2^{n+k+1}} \leq \frac{\beta - \alpha}{2^{k+1}}, \end{aligned}$$

which implies[2] in its turn that $\|F_S - F_T\|_\infty < \epsilon$. So $(\star)$ is continuous.

What is left to show is that given any $T \in Tr$ and any $n \in \mathbb{N}$, $(\star\star)$ is also true. Indeed, if $T \in IF$ and $x \in [T]$, then $G_T^{(n)} \supseteq \cap_{k \in \mathbb{N}} J_{x|k}^{(n)} \neq \emptyset$, while if $G_T^{(n)} \neq \emptyset$ and $x \in G_T^{(n)}$, then for any $k \in \mathbb{N}$, there exists $s_k \in T \cap \mathbb{N}^k$ such that $x \in J_{s_k}^{(n)}$ and since for any non-empty incompatible finite sequences s and t of natural numbers, we have $J_s^{(n)} \cap J_t^{(n)} = \emptyset$, it follows that $s_0 \subset s_1 \subset s_2 \subset \ldots$. Hence $\bigcup_{k \in \mathbb{N}} s_k \in [T]$ and consequently $T \in IF$. $\square$

4 Non-Borel Sets and Piecewise Differentiable Functions.

Theorem 4.1. *Given any real numbers α and β such that $\alpha < \beta$, the set PD_α^β of piecewise differentiable functions in $C([\alpha, \beta], \mathbb{R})$ is $\mathbf{\Pi}_1^1$-complete and therefore not Borel.*

PROOF. To show that PD_α^β is $\mathbf{\Pi}_1^1$, it is enough to show that $C([\alpha, \beta], \mathbb{R}) \setminus PD_\alpha^\beta$ is $\mathbf{\Sigma}_1^1$. But this follows from the fact that for any $f \in C([\alpha, \beta], \mathbb{R})$, we have $f \in C([\alpha, \beta], \mathbb{R}) \setminus PD_\alpha^\beta$, if and only if there exists $(x_n)_{n \in \mathbb{N}} \in (\alpha, \beta)^{\mathbb{N}}$ such that the x_n are pairwise distinct and for any index n, the limit $\lim_{h \to 0} \frac{f(x_n+h) - f(x_n)}{h}$

[2]Notice that for any $T \in Tr$, we have $F_T(\beta) = 0$.

does not exist. So, since for any $(T, n) \in Tr \times \mathbb{N}$, we have $G_T^{(n)} \subseteq J_\emptyset^{(n)}$ and the $J_\emptyset^{(n)}$ are pairwise disjoint, and since it is not difficult to see that at the endpoints of the $J_\emptyset^{(n)}$ the function F_T has a derivative equal to zero, by virtue of Theorem 3.1, given any $(T, n) \in Tr \times \mathbb{N}$ and given any $x \in J_\emptyset^{(n)}$, it is enough to show that $x \notin G_T^{(n)}$ if and only if $F_T'(x)$ exists.

If $x \in G_T^{(n)}$, then the proof of Theorem 3.1 shows that there exists $y \in [T]$ such that for any $k \in \mathbb{N}$, we have $x \in J_{y|(k+1)}^{(n)}$ and hence $x \in \operatorname{Int}\left(LK_{y|k}^{(n)}\right)$. For any $k \in \mathbb{N}$, we denote by ξ_k the midpoint of $RK_{y|k}^{(n)}$ and we set $\zeta_k = 2^{-1} \cdot \left|RK_{y|k}^{(n)}\right|$. Since for any $k \in \mathbb{N}$, we have $x \in \operatorname{Int}\left(LK_{y|k}^{(n)}\right)$ and all the $RK_s^{(n)}$ are pairwise disjoint, it follows that for any $s \in \mathbb{N}^{<\mathbb{N}}$, we have $x \notin RK_s^{(n)}$ and consequently $F_T(x) = 0$. Moreover, for any $k \in \mathbb{N}$, the points ξ_k and $\xi_k + \zeta_k$ lie in $RK_{y|k}^{(n)}$, hence since all the $RK_s^{(n)}$ are pairwise disjoint, it follows that

$$F_T(\xi_k) = \psi_{RK_{y|k}^{(n)}}(\xi_k) \text{ and } F_T(\xi_k + \zeta_k) = \psi_{RK_{y|k}^{(n)}}(\xi_k + \zeta_k).$$

Therefore, the very definition of ξ_k and ζ_k imply that

$$\left|\frac{F_T(\xi_k) - F_T(x)}{\xi_k - x}\right| = \frac{\psi_{RK_{y|k}^{(n)}}(\xi_k)}{\xi_k - x} = \frac{2\zeta_k}{\xi_k - x} \geq \frac{2\zeta_k}{3\zeta_k} = \frac{2}{3} \tag{4.1}$$

and

$$\left|\frac{F_T(\xi_k + \zeta_k) - F_T(x)}{(\xi_k + \zeta_k) - x}\right| = \frac{\psi_{RK_{y|k}^{(n)}}(\xi_k + \zeta_k)}{(\xi_k + \zeta_k) - x} = \frac{0}{(\xi_k + \zeta_k) - x} = 0. \tag{4.2}$$

But $\bigcap_{k \in \mathbb{N}} J_{y|(k+1)}^{(n)} = \{x\}$ and for any $k \in \mathbb{N}$, we have $J_{y|(k+1)}^{(n)} \supseteq K_{y|(k+1)}^{(n)}$, hence $\lim_{k \to \infty} \xi_k = \lim_{k \to \infty} (\xi_k + \zeta_k) = x$ and consequently (4.1) and (4.2) imply that $F_T'(x)$ does not exist.

Now, assume that $x \notin G_T^{(n)}$. Then there exists a least $k \in \mathbb{N} \setminus \{0\}$ such that $x \notin \bigcup_{s \in T \cap \mathbb{N}^k} J_s^{(n)}$. Hence there exists a least $j \in \mathbb{N} \setminus \{0\}$ such that for any $s \in \mathbb{N}^{<\mathbb{N}}$ for which $\langle s \rangle \geq j$, we have $x \notin J_s^{(n)}$ and therefore $x \notin K_s^{(n)}$. We remark that if $s \in \mathbb{N}^{<\mathbb{N}}$ and $\langle s \rangle \geq j$, then for any $h \in \mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned}
\left| \frac{\psi_{RK_s^{(n)}}(x+h) - \psi_{RK_s^{(n)}}(x)}{h} \right| &= \frac{\psi_{RK_s^{(n)}}(x+h)}{|h|} \leq \frac{|RK_s^{(n)}|}{\operatorname{dist}(x, RK_s^{(n)})} \\
&\leq \frac{|RK_s^{(n)}|}{\operatorname{dist}(x, K_s^{(n)})} \leq \frac{|RK_s^{(n)}|}{\frac{|J_s^{(n)}| - |K_s^{(n)}|}{2}} \qquad (4.3) \\
&= \frac{|K_s^{(n)}|}{|J_s^{(n)}| - |K_s^{(n)}|} = \frac{1}{2^{\langle s \rangle}}.
\end{aligned}$$

Thus, given any integer $i \geq j$, if we set $f_i = \sum_{s \in T; \langle s \rangle \leq i} \psi_{RK_s^{(n)}}$, then (4.3) implies that for any $h \in \mathbb{R} \setminus \{0\}$ such that $x + h \in J_\emptyset^{(n)}$, we have

$$\left| \frac{F_T(x+h) - F_T(x)}{h} - \frac{f_i(x+h) - f_i(x)}{h} \right| \leq \sum_{\langle s \rangle > i} \frac{1}{2^{\langle s \rangle}} = \frac{1}{2^i}.$$

In other words, we have

$$-\frac{1}{2^i} \leq \frac{F_T(x+h) - F_T(x)}{h} - \frac{f_i(x+h) - f_i(x)}{h} \leq \frac{1}{2^i} \qquad (4.4)$$

and if we take in (4.4) the lim inf and the lim sup as $h \to 0$, then since f_i is obviously C^1, we have

$$-\frac{1}{2^i} \leq \liminf_{h \to 0} \frac{F_T(x+h) - F_T(x)}{h} - f_i'(x) \leq \frac{1}{2^i}$$

and

$$-\frac{1}{2^i} \leq \limsup_{h \to 0} \frac{F_T(x+h) - F_T(x)}{h} - f_i'(x) \leq \frac{1}{2^i}.$$

So

$$0 \leq \limsup_{h \to 0} \frac{F_T(x+h) - F_T(x)}{h} - \liminf_{h \to 0} \frac{F_T(x+h) - F_T(x)}{h} \leq \frac{1}{2^{i-1}} \qquad (4.5)$$

and by taking in (4.5) the limit as $i \to \infty$, the claim follows. □

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