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GENERALIZED CONTINUITY AND UNIFORM APPROXIMATION BY STEP FUNCTIONS

Abstract

Given two topological spaces X and Y and a family $\mathcal{O}_*$ of subsets of X, a function $f : X \to Y$ is called $\mathcal{O}_*$-continuous if $f^{-1}(V) \in \mathcal{O}_*$ for every open set $V \subseteq Y$. An $\mathcal{O}_*$-step function is meant to be a function $\varphi : X \to Y$ that is piecewise constant on a partition of X into sets from $\mathcal{O}_*$. Using some technical assumptions on X, Y, and $\mathcal{O}_*$ we give representations of $\mathcal{O}_*$-continuous functions as uniform limits of $\mathcal{O}_*$-step functions. We deal in particular with α-continuous, nearly continuous, almost quasi-continuous, and somewhat continuous functions. The paper is motivated by a corresponding characterization of quasi-continuous functions.

1 Introduction.

Given two topological spaces X and Y, a function $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in X for every open subset $V \subseteq Y$. Replacing in this definition the system $\mathcal{O}$ of open subsets of X by another family $\mathcal{O}_* \subseteq 2^X$ of subsets of X, that does not necessarily satisfy the properties of a topology, one obtains a modified concept of continuity. A function $f : X \to Y$ is called $\mathcal{O}_*$-*continuous* if $f^{-1}(V) \in \mathcal{O}_*$ for every open $V \subseteq Y$. Of course, $\mathcal{O}_*$-continuity generalizes the classical continuity if $\mathcal{O} \subseteq \mathcal{O}_*$.

Various well-known types of generalized continuity fit in with the above scheme. We consider five examples.

Key Words: Generalized continuity, α-continuous, quasi-continuous, nearly continuous, almost continuous, almost quasi-continuous, somewhat continuous, α-set, semi-open set, nearly open set, preopen set, semi-preopen set, step function, uniform limit
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A subset A of a topological space X is called *semi-open* if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$, $\mathrm{cl}(\cdot)$ and $\mathrm{int}(\cdot)$ denoting the closure and the interior operator, respectively. This concept goes back to [12]. In [16] the name *β-set* is used, much later in [24] the name *robust set*. We denote the family of all semi-open subsets of X by $\mathcal{O}_s$. A function f mapping X into a topological space Y is called *quasi-continuous* if $f^{-1}(V) \in \mathcal{O}_s$ for every open set $V \subseteq Y$ (see [11, 16]). The authors of [3, 12, 21] speak of *neighborly*, *semi-continuous*, and *robust* functions, respectively.

A set $A \subseteq X$ is called *nearly open* if $A \subseteq \mathrm{int}(\mathrm{cl}(A))$ (see [23]). Sets of that type already play a role in [6], where they are called *locally dense*. Very often the name *preopen* is used, starting with [13]. We denote the system of all nearly open subsets of X by $\mathcal{O}_n$. A function $f : X \to Y$ is called *nearly continuous* if $f^{-1}(V) \in \mathcal{O}_n$ for every open $V \subseteq Y$. The name was introduced by Pták, who showed that every linear map between two Banach spaces is nearly continuous (see [19]). Already in [4] Blumberg speaks of *densely approached* functions. At present the phrase *almost continuous in the sense of Husain* is widely used, going back to the paper [10]. In [13] and related papers these functions are called *precontinuous*.

Quasi-continuity and near continuity are complementary in so far as a function from a topological space X into a regular space Y is continuous if and only if it is both quasi-continuous and nearly continuous (see [22, 18, 17]).

One easily checks that a set $A \subseteq X$ is both semi-open and nearly open if and only if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$. Njåstad calls sets of that kind *α-sets* (see [16]). We denote the family of all α-sets in X by $\mathcal{O}_\alpha$; that is, $\mathcal{O}_\alpha = \mathcal{O}_s \cap \mathcal{O}_n$. Adjacently, a function $f : X \to Y$ is called *α-continuous* if $f^{-1}(V) \in \mathcal{O}_\alpha$ for every open $V \subseteq Y$ (see [14]). Though the concept of an α-set usually is strictly wider than that of an open set, because α-sets are exactly those sets that can be expressed as the difference of an open and a nowhere dense set (see [16]), the above observation shows that all α-continuous functions $f : X \to Y$ are continuous, provided that Y is regular.

A common generalization of semi-open sets and nearly open sets is given by the system $\mathcal{O}_{sp}$ of all *semi-preopen* subsets $A \subseteq X$ whose characteristic property is $A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ (see [2]). A function $f : X \to Y$ is *almost quasi-continuous* if $f^{-1}(V) \in \mathcal{O}_{sp}$ for every open $V \subseteq Y$ (see [5]). In [1] the authors speak of *β-open sets* and *β-continuous functions*. Although the name "β-open" is older than "semi-preopen", we do not use it to avoid confusions with Njåstad's β-sets mentioned above.

Denoting the class of continuous, α-continuous, quasi-continuous, nearly continuous, and almost quasi-continuous functions from X to Y by $C(X,Y)$, $C_\alpha(X,Y)$, $C_s(X,Y)$, $C_n(X,Y)$, and $C_{sp}(X,Y)$, respectively, we obtain the hierarchies

$$\mathcal{O} \subseteq \mathcal{O}_\alpha = \mathcal{O}_s \cap \mathcal{O}_n \subseteq \left\langle \begin{matrix} \mathcal{O}_s \\ \mathcal{O}_n \end{matrix} \right\rangle \subseteq \mathcal{O}_{sp}$$

and

$$C(X,Y) \subseteq C_\alpha(X,Y) = C_s(X,Y) \cap C_n(X,Y) \subseteq \left\langle \begin{matrix} C_s(X,Y) \\ C_n(X,Y) \end{matrix} \right\rangle \subseteq C_{sp}(X,Y).$$

We finally recall the concept of a *somewhat continuous* function that has been explicitly named in [9] and goes back to Frolík's studies concerning invariance of Baire spaces under mappings (see [8]). A function $f : X \to Y$ is called somewhat continuous if $\operatorname{int}(f^{-1}(V)) \neq \emptyset$ for every open set $V \subseteq Y$ with $f^{-1}(V) \neq \emptyset$. Correspondingly, we shall speak of a *somewhat open* set $A \subseteq X$ if either $A = \emptyset$ or $\operatorname{int}(A) \neq \emptyset$. We use the symbols $\mathcal{O}_{sw}$ and $C_{sw}(X,Y)$ for denoting the classes of all somewhat open subsets of X and of all somewhat continuous functions $f : X \to Y$, respectively. Of course, $\mathcal{O}_s \subseteq \mathcal{O}_{sw}$ and $C_s(X,Y) \subseteq C_{sw}(X,Y)$. However, in contrast with the previous examples somewhat openness and somewhat continuity do not describe local properties of sets or functions, respectively.

The present paper is motivated by a characterization of quasi-continuous functions from [20].

Theorem 1 ([20, Theorems 1 and 2])**.** *Let f be a real-valued quasi-continuous function on a topological space X. Then f can be represented as the uniform limit of a sequence $(\varphi_k)_{k=1}^\infty$ of semi-open step functions which are defined on a chain $K = (\mathcal{P}_k)_{k=1}^\infty$ of semi-open partitions $\mathcal{P}_k = \{P_i^{(k)} : i \in I_k\}$. If f is locally bounded, then there exists a chain K of locally finite partitions with the above property. If f is bounded, then K can be chosen to be a chain of finite partitions.*

If, moreover, X is compact and metrizable, then one can choose K such that in addition every continuous real-valued function $g : X \to \mathbb{R}$ can be attained as the uniform limit of a sequence of semi-open step functions that are defined on K.

In [15] Naimpally proposed a similar study of nearly continuous functions, this way motivating investigations of α-continuous, nearly continuous, almost quasi-continuous, and somewhat continuous functions in the present paper.

Since Theorem 1 serves as a model, we define the involved concepts in detail. A *partition* of X is a cover by mutually disjoint sets. We say that a cover $\{C_l^{(2)} : l \in I_2\}$ *refines* a cover $\{C_k^{(1)} : k \in I_1\}$ if, for every $l \in I_2$, there is $k \in I_1$ such that $C_l^{(2)} \subseteq C_k^{(1)}$. A *chain* of partitions is a sequence

$(\mathcal{P}_k)_{k=1}^{\infty}$ of partitions such that $\mathcal{P}_{k+1}$ refines $\mathcal{P}_k$ for every $k \geq 1$. A partition of X is called *semi-open* if it consists of semi-open sets. A function φ that is piecewise constant on the sets of a semi-open partition is called a *semi-open step function* (see [20]).

Since semi-open step functions are quasi-continuous and quasi-continuity is preserved under uniform limits (see [20] or Section 2 of the present paper), Theorem 1 characterizes quasi-continuous functions as uniform limits of semi-open step functions. Moreover, Theorem 1 gives a structural insight into the class $C_s(X,\mathbb{R})$ that usually is not a linear space. Every chain $K = (\mathcal{P}_k)_{k=1}^{\infty}$ of semi-open partitions induces a complete uniform linear space consisting of all uniform limits of semi-open step functions defined on K. The space induced by K is separable if all partitions $\mathcal{P}_k$ are finite.

The goal of the present paper is to obtain similar theorems related to other concepts of generalized continuity. Before coming back to the particular continuity notions introduced above (see Sections 3-6), we establish some general statements.

2 A General Approach.

Let X be a set, $\mathcal{O}_*$ a family of subsets of X, and Y a topological space. We denote the set of all $\mathcal{O}_*$-continuous functions from X into Y by $C_*(X,Y)$. The elements of $\mathcal{O}_*$ will be called *$\mathcal{O}_*$-open sets.*

Since $\mathcal{O}_*$ plays a similar role in the definition of $\mathcal{O}_*$-continuity as a system of open sets does with respect to classical continuity, a fairly natural property of $\mathcal{O}_*$ is *closedness with respect to unions.*

(U) *The union of any subfamily of $\mathcal{O}_*$ is an element of $\mathcal{O}_*$.*

As we shall see now, property **(U)** already implies closedness of $C_*(X,Y)$ with respect to uniform limits and motivates a concept of step functions related to $\mathcal{O}_*$-continuity. (When considering uniform limits of functions with values in Y, we assume Y to be metric, although sometimes a uniform structure on Y would suffice.)

Proposition 2. *Let X be a set and (Y,d_Y) a metric space and let a family $\mathcal{O}_* \subseteq 2^X$ satisfy* **(U)**. *Then $C_*(X,Y)$ is closed with respect to uniform limits.*

Proof. Let f be the uniform limit of a sequence $(f_k)_{k=1}^{\infty} \subseteq C_*(X,Y)$, say $\sup_{x\in X} d_Y(f(x),f_k(x)) < 2^{-k}$. We need to show that $f^{-1}(V) \in \mathcal{O}_*$ for every open set $V \subseteq Y$. Given a fixed V, we write $V = \bigcup_{k=1}^{\infty} V_k$ with open subsets

$$V_k = \left\{ y \in V : \inf_{y' \in Y\setminus V} d_Y(y,y') > 2^{-k} \right\}.$$

One easily checks that $f^{-1}(V) = \bigcup_{k=1}^{\infty} f_k^{-1}(V_k)$. The sets $f_k^{-1}(V_k)$ are in $\mathcal{O}_*$, because $f_k \in C_*(X,Y)$. Thus $f^{-1}(V) \in \mathcal{O}_*$ by **(U)**. □

Proposition 3. *Let X be a set and Y a topological space and let a family $\mathcal{O}_* \subseteq 2^X$ satisfy* **(U)**. *Then a function $\varphi : X \to Y$ with discrete range $\varphi(X) \subseteq Y$ is $\mathcal{O}_*$-continuous if and only if there exists a partition $\mathcal{P} = \{P_i : i \in I\}$ of X into sets $P_i \in \mathcal{O}_*$ such that φ is constant on P_i for every $i \in I$.*

Proof. First suppose $\varphi \in C_*(X,Y)$. Since $\varphi(X)$ is discrete, every $y \in \varphi(X)$ has an open neighborhood V such that $\varphi^{-1}(y) = \varphi^{-1}(V) \in \mathcal{O}_*$. Then the partition $\{\varphi^{-1}(y) : y \in \varphi(X)\}$ of X clearly has the required property.

If φ is piecewise constant on a partition $\mathcal{P} = \{P_i : i \in I\} \subseteq \mathcal{O}_*$, then the inverse image $\varphi^{-1}(V)$ of any open set $V \subseteq Y$ is a union of members of $\mathcal{P}$ and therefore contained in $\mathcal{O}_*$ by **(U)**. Hence φ is $\mathcal{O}_*$-continuous. □

A function with discrete range can be considered as a natural generalization of a classical step function, that has to have a finite range. Proposition 3 suggests the following definition. A function $\varphi : X \to Y$ is called an *$\mathcal{O}_*$-step function* if there exists a partition $\mathcal{P}$ of X into $\mathcal{O}_*$-open sets such that φ is constant on every member of $\mathcal{P}$. Partitions of that kind will be called *$\mathcal{O}_*$-open partitions.*

Note that the range of an $\mathcal{O}_*$-step function need not be discrete. However, every $\mathcal{O}_*$-step function is $\mathcal{O}_*$-continuous, as the second part of the above proof shows (provided that $\mathcal{O}_*$ satisfies **(U)**). Of course, $\mathcal{O}_*$-step functions can be seen as very elementary members of $C_*(X,Y)$.

A characterization of $C_*(X,Y)$ as the set of all uniform limits of $\mathcal{O}_*$-step functions from X into Y would be complete once we had shown that every $f \in C_*(X,Y)$ is representable as the uniform limit of some sequence of $\mathcal{O}_*$-step functions. The following *refinement properties* of $\mathcal{O}_*$, to be chosen in dependence on the structure of Y, yield the desired representation.

(R) *Every cover of X by $\mathcal{O}_*$-open sets can be refined to a partition of X into $\mathcal{O}_*$-open sets.*

($\mathbf{R}_\sigma$) *Every at most countable cover of X by $\mathcal{O}_*$-open sets can be refined to a partition of X into $\mathcal{O}_*$-open sets.*

($\mathbf{R}_f$) *Every finite cover of X by $\mathcal{O}_*$-open sets can be refined to a partition of X into $\mathcal{O}_*$-open sets.*

Proposition 4. *Let X be a set, $\mathcal{O}_* \subseteq 2^X$ a family of subsets of X, and (Y, d_Y) a metric space.*

If $\mathcal{O}_*$ satisfies **(R)**, *then every $f \in C_*(X,Y)$ is the uniform limit of $\mathcal{O}_*$-step functions.*

If (Y, d_Y) is separable and $\mathcal{O}_*$ satisfies **($\mathbf{R}_\sigma$)**, *then every $f \in C_*(X,Y)$ is the uniform limit of $\mathcal{O}_*$-step functions.*

If (Y, d_Y) is totally bounded and $\mathcal{O}_*$ satisfies **($\mathbf{R}_f$)**, *then every $f \in C_*(X,Y)$ is the uniform limit of $\mathcal{O}_*$-step functions.*

PROOF. Let $f \in C_*(X,Y)$ and $\varepsilon > 0$. We fix a cover $\{V_j : j \in J\}$ of Y by open sets of diameter $\operatorname{diam}(V_j) = \sup_{y_1,y_2 \in V_j} d_Y(y_1,y_2) \le \varepsilon$. We choose an at most countable cover of that type if (Y, d_Y) is separable and a finite one if (Y, d_Y) is totally bounded. Then $\{f^{-1}(V_j) : j \in J\}$ is an $\mathcal{O}_*$-open cover of X. The respective refinement property of $\mathcal{O}_*$ yields an $\mathcal{O}_*$-open partition $\{P_i : i \in I\}$ of X refining $\{f^{-1}(V_j) : j \in J\}$. For every $i \in I$, we fix $j(i) \in J$ with $P_i \subseteq f^{-1}(V_{j(i)})$ and a point $y_{j(i)} \in V_{j(i)}$. We define an $\mathcal{O}_*$-step function φ on $\{P_i : i \in I\}$ by $\varphi(P_i) \equiv y_{j(i)}$, $i \in I$. Then

$$\begin{aligned}
&\sup_{x\in X} d_Y(f(x),\varphi(x)) = \sup_{i\in I}\sup_{x\in P_i} d_Y(f(x),\varphi(x))\\
&= \sup_{i\in I}\sup_{x\in P_i} d_Y\big(f(x),y_{j(i)}\big) \le \sup_{i\in I}\sup_{x\in f^{-1}(V_{j(i)})} d_Y\big(f(x),y_{j(i)}\big)\\
&\le \sup_{i\in I}\sup_{y\in V_{j(i)}} d_Y\big(y,y_{j(i)}\big) \le \sup_{i\in I}\operatorname{diam}\big(V_{j(i)}\big) \le \varepsilon.
\end{aligned}$$

This completes the proof. □

We see that $\mathcal{O}_*$-continuous functions from X into a metric space (Y, d_Y) are exactly the uniform limits of $\mathcal{O}_*$-step functions between X and Y, provided that $\mathcal{O}_*$ satisfies **(U)** and **(R)**. The relatively strong assumption **(R)** can be weakened in dependence on the structure of (Y, d_Y). Of course, besides **($\mathbf{R}_\sigma$)** and **($\mathbf{R}_f$)** there exist other weakened versions of **(R)**.

Now we intend to obtain a stronger representation of $\mathcal{O}_*$-continuous functions analogous to Theorem 1. We shall replace **(R)** by a strong refinement property, whose formulation requires some preparation.

Let $f \in C_*(X,Y)$. A set $A \subseteq X$ is called $(f,\mathcal{O}_*)$-*admissible* if $A \cap f^{-1}(V) \in \mathcal{O}_*$ for every open set $V \subseteq Y$. Every $(f,\mathcal{O}_*)$-admissible set A belongs to $\mathcal{O}_*$, because $A = A \cap f^{-1}(Y) \in \mathcal{O}_*$ by definition. The intersection $A \cap f^{-1}(W)$ of an $(f,\mathcal{O}_*)$-admissible set A with the inverse image of an open set $W \subseteq Y$ obviously itself is $(f,\mathcal{O}_*)$-admissible.

An equivalent definition of $(f,\mathcal{O}_*)$-admissibility reads as follows.

Lemma 5. *Let X be a set, $\mathcal{O}_* \subseteq 2^X$ a family satisfying* **(U)**, *Y a topological space, and $f \in C_*(X,Y)$ an $\mathcal{O}_*$-continuous function. Then, given any base $\mathcal{B}$ of the system of open subsets of Y, a subset $A \subseteq X$ is $(f,\mathcal{O}_*)$-admissible if $A \cap f^{-1}(B) \in \mathcal{O}_*$ for every $B \in \mathcal{B}$.*

Proof. Suppose $A \cap f^{-1}(B) \in \mathcal{O}_*$ for all $B \in \mathcal{B}$. Every open set $V \subseteq Y$ has a representation $V = \bigcup_{i \in I} B_i$ with suitable sets $B_i \in \mathcal{B}$. Thus

$$A \cap f^{-1}(V) = \bigcup_{i \in I} \left(A \cap f^{-1}(B_i)\right)$$

is a union of sets $A \cap f^{-1}(B_i) \in \mathcal{O}_*$. Property **(U)** yields $A \cap f^{-1}(V) \in \mathcal{O}_*$. □

Now we formulate the *strong refinement property*. Since the applications to follow in the next sections concern separable spaces Y, we give the corresponding version. Of course, other variants are possible.

($\mathbf{SR}_\sigma$) *For every $f \in C_*(X,Y)$, every at most countable cover $\mathcal{C} = \{C_i : i \in I\}$ of X by $(f, \mathcal{O}_*)$-admissible sets C_i can be refined to a partition $\mathcal{P} = \{P_i : i \in I\}$ of X into $(f, \mathcal{O}_*)$-admissible sets.*

We would like to point out that condition **($\mathbf{SR}_\sigma$)** is stronger than **($\mathbf{R}_\sigma$)**.

Lemma 6. *Let X be a set, $\mathcal{O}_*$ a family of subsets of X, and Y a topological space. If $C_*(X,Y) \neq \emptyset$ (that is, if $\{\emptyset, X\} \subseteq \mathcal{O}_*$), then* **($\mathbf{SR}_\sigma$)** *implies* **($\mathbf{R}_\sigma$)**.

Proof. Let $f_0 \in C_*(X,Y)$. Then $\emptyset = f_0^{-1}(\emptyset)$, $X = f_0^{-1}(Y) \in \mathcal{O}_*$. We fix a constant function $\varphi \equiv y_0$, which in turn belongs to $C_*(X,Y)$, since $\varphi^{-1}(V) \in \{\emptyset, X\}$ for every $V \subseteq Y$. Thus every set $A \in \mathcal{O}_*$ is $(\varphi, \mathcal{O}_*)$-admissible, because $A \cap \varphi^{-1}(V) \in \{\emptyset, A\} \subseteq \mathcal{O}_*$.

Now application of **($\mathbf{SR}_\sigma$)** to $f = \varphi$ shows that every at most countable $\mathcal{O}_*$-open cover $\mathcal{C} = \{C_i : i \in I\}$ of X can be refined to a partition $\mathcal{P} = \{P_i : i \in I\}$ of X into $\mathcal{O}_*$-open sets. This proves **($\mathbf{R}_\sigma$)**. □

Note that, in contrast to the refinement properties **(R)**, **($\mathbf{R}_\sigma$)**, and **($\mathbf{R}_f$)**, the strong refinement property **($\mathbf{SR}_\sigma$)** explicitly bounds the cardinality of the refining partition $\mathcal{P}$, because $\mathcal{P}$ has the same index set I as the given cover $\mathcal{C}$.

The second part of Theorem 1 involves continuous functions on X. Then one has to dispose of a topology and a system $\mathcal{O}$ of open subsets of X. The following *closedness of $\mathcal{O}_*$ with respect to intersections with open sets* will help us to relate continuity and $\mathcal{O}_*$-continuity.

(IO) *For every $A \in \mathcal{O}_*$ and every $G \in \mathcal{O}$, $A \cap G \in \mathcal{O}_*$.*

Condition **(IO)** obviously implies $\mathcal{O} \subseteq \mathcal{O}_*$ if $X \in \mathcal{O}_*$. Another simple consequence is the following.

Lemma 7. *Let X be a topological space, $\mathcal{O}_* \subseteq 2^X$ a family satisfying* **(IO)**, *Y a topological space, and $f \in C_*(X,Y)$ an $\mathcal{O}_*$-continuous function. If $A \subseteq X$ is an $(f, \mathcal{O}_*)$-admissible set and $G \subseteq X$ is open, then $A \cap G$ is $(f, \mathcal{O}_*)$-admissible, too.*

Now we come to the abstract counterpart of Theorem 1.

Theorem 8. *Let X be a set, (Y, d_Y) a separable metric space, $\mathcal{O}_* \subseteq 2^X$ a family satisfying* **(SR$_\sigma$)**, *and $f \in C_*(X,Y)$. Then there exists a chain $(\mathcal{P}_k)_{k=1}^\infty$ of at most countable partitions $\mathcal{P}_k = \{P_i^{(k)} : i \in I_k\} \subseteq \mathcal{O}_*$ of X and a sequence $(\varphi_k)_{k=1}^\infty$ of $\mathcal{O}_*$-step functions φ_k defined on the partitions $\mathcal{P}_k$ such that f is the uniform limit of $(\varphi_k)_{k=1}^\infty$.*

If (Y, d_Y) is totally bounded, then this is possible with finite partitions $\mathcal{P}_k$.

If, in addition, X is a compact metrizable space and $\mathcal{O}_*$ satisfies **(IO)**, *then one can choose the chain $(\mathcal{P}_k)_{k=1}^\infty$ such that, given any continuous function $g \in C(X,Y)$, there is a sequence $(\psi_k)_{k=1}^\infty$ of $\mathcal{O}_*$-step functions ψ_k defined on the partitions $\mathcal{P}_k$ such that g is the uniform limit of $(\psi_k)_{k=1}^\infty$.*

Proof. We shall define the chain $(\mathcal{P}_k)_{k=1}^\infty$ inductively starting with the trivial partition $\mathcal{P}_0 = \{X\}$. Note that the set X is $(f, \mathcal{O}_*)$-admissible, because $X \cap f^{-1}(V) = f^{-1}(V) \in \mathcal{O}_*$ for every open set $V \subseteq Y$.

Given $\mathcal{P}_{k-1}$, $k \geq 1$, the partition $\mathcal{P}_k$ will be defined subject to the following conditions (i)-(iv):

(i) $\mathcal{P}_k$ is a refinement of $\mathcal{P}_{k-1}$ and consists of $(f, \mathcal{O}_*)$-admissible sets.

(ii) $\mathcal{P}_k$ is countable and even finite if (Y, d_Y) is totally bounded.

(iii) There is a step function φ_k defined on $\mathcal{P}_k$ such that

$$\sup_{x \in X} d_Y(f(x), \varphi_k(x)) \leq 2^{-k}.$$

(iv) If X is compact with metric d_X and if $\mathcal{O}_*$ satisfies **(IO)**, then, for any $g \in C(X,Y)$, there is an $\mathcal{O}_*$-step function ψ_k defined on $\mathcal{P}_k$ such that

$$\sup_{x \in X} d_Y(g(x), \psi_k(x)) \leq \omega\big(g; 2^{-k}\big),$$

$\omega\big(g; 2^{-k}\big) = \sup\big\{d_Y(g(x_1), g(x_2)) : x_1, x_2 \in X,\, d_X(x_1, x_2) \leq 2^{-k}\big\}$ denoting the modulus of continuity of g.

Suppose k and $\mathcal{P}_{k-1} = \big\{P_i^{(k-1)} : i \in I_{k-1}\big\}$ to be fixed. Let $\{V_j : j \in J\}$ be a cover of Y by at most countably many open sets of diameter $\operatorname{diam}(V_j) \leq 2^{-k}$. We assume J to be finite if Y is totally bounded. If X and $\mathcal{O}_*$ do not satisfy the topological assumptions of statement (iv) we define a cover $\mathcal{C}$ of X by

$$\mathcal{C} = \big\{P_i^{(k-1)} \cap f^{-1}(V_j) : i \in I_{k-1},\, j \in J\big\}.$$

Otherwise we fix an open cover $\{G_1, \ldots, G_m\}$ of X with $\operatorname{diam}(G_l) \leq 2^{-k}$, $1 \leq l \leq m$, before setting

$$\mathcal{C} = \big\{P_i^{(k-1)} \cap f^{-1}(V_j) \cap G_l : i \in I_{k-1},\, j \in J,\, 1 \leq l \leq m\big\}.$$

The sets $P_i^{(k-1)} \cap f^{-1}(V_j)$ are $(f, \mathcal{O}_*)$-admissible, because the sets $P_i^{(k-1)}$ are so according to the induction hypothesis. If $\mathcal{O}_*$ satisfies **(IO)**, Lemma 7 shows that the sets $P_i^{(k-1)} \cap f^{-1}(V_j) \cap G_l$ are $(f, \mathcal{O}_*)$-admissible, too. Hence in any case $\mathcal{C}$ is a cover of X by $(f, \mathcal{O}_*)$-admissible sets. Now we use property $(\mathbf{SR}_\sigma)$ to obtain a partition $\mathcal{P}_k = \big\{P_i^{(k)} : i \in I_k\big\}$ of X into $(f, \mathcal{O}_*)$-admissible sets that refines $\mathcal{C}$. The index set I_k of $\mathcal{P}_k$ coincides with that of $\mathcal{C}$; that is,

$$I_k = I_{k-1} \times J \text{ or } I_k = I_{k-1} \times J \times \{1, \ldots, m\}, \tag{1}$$

respectively.

Since $\mathcal{C}$ is a refinement of $\mathcal{P}_{k-1}$, the partition $\mathcal{P}_k$ obtained by the aid of $(\mathbf{SR}_\sigma)$ satisfies (i). Property (ii) is a consequence of (1), the corresponding induction hypothesis on $\mathcal{P}_{k-1}$, and the choice of J. Claim (iii) follows, because $\mathcal{P}_k$ is via $\mathcal{C}$ a refinement of the cover $\{f^{-1}(V_j) : j \in J\}$. For every $i \in I_k$, we fix a $j(i) \in J$ such that $P_i^{(k)} \subseteq f^{-1}(V_{j(i)})$ and choose a value $y_{j(i)} \in V_{j(i)}$. Then we define a step function $\varphi_k : X \to Y$ on $\mathcal{P}_k$ by $\varphi_k\big(P_i^{(k)}\big) \equiv y_{j(i)}$. We obtain

$$\sup_{x \in X} d_Y(f(x), \varphi_k(x)) \leq \sup_{i \in I_k} \operatorname{diam}\big(V_{j(i)}\big) \leq 2^{-k}$$

as in the proof of Proposition 4. This proves (iii). Similarly, property (iv) is satisfied, because $\mathcal{P}_k$ is a refinement of $\{G_1, \ldots, G_m\}$. Given $i \in I_k$, we fix $l(i) \in \{1, \ldots, m\}$ such that $P_i^{(k)} \subseteq G_{l(i)}$ and pick a point $x_{l(i)} \in G_{l(i)}$. We consider the step function $\psi_k\big(P_i^{(k)}\big) \equiv g(x_{l(i)})$, $i \in I_k$, on $\mathcal{P}_k$. Then

$$\begin{aligned}
&\sup_{x \in X} d_Y(g(x), \psi_k(x)) \;=\; \sup_{i \in I_k} \sup_{x \in P_i^{(k)}} d_Y(g(x), \psi_k(x)) \\
&= \sup_{i \in I_k} \sup_{x \in P_i^{(k)}} d_Y(g(x), g(x_{l(i)})) \;\leq\; \sup_{i \in I_k} \sup_{x \in G_{l(i)}} d_Y(g(x), g(x_{l(i)})) \\
&\leq \sup_{i \in I_k} \omega(g; \operatorname{diam}(G_{l(i)})) \;\leq\; \omega(g; 2^{-k}).
\end{aligned}$$

Hence conditions (i)-(iv) are confirmed.

Note that the above definition of $(\mathcal{P}_k)_{k=1}^\infty$ proves the theorem. Indeed, (i) shows that we have obtained a chain of partitions of X, the partition sets $P_i^{(k)}$ belonging to $\mathcal{O}_*$, because every $(f, \mathcal{O}_*)$-admissible set is in $\mathcal{O}_*$. Condition (ii) gives the claim concerning the cardinality of the partitions $\mathcal{P}_k$ in dependence on the structure of Y. By (iii), f is the uniform limit of the sequence $(\varphi_k)_{k=1}^\infty$. Finally, (iv) shows that $(\psi_k)_{k=1}^\infty$ uniformly tends to g, because $\lim_{k \to \infty} \omega(g; 2^{-k}) = 0$ by continuity of g and compactness of X. $\square$

In the following we shall apply the above abstract results to particular generalizations of continuity.

3 Nearly Continuous Functions.

The family $\mathcal{O}_n$ of nearly open subsets of a topological space X obviously satisfies condition **(U)**. Hence the set $C_n(X,Y)$ of nearly continuous maps from X into a metric space (Y,d_Y) is closed under uniform limits by Proposition 2. Proposition 3 leads to the corresponding concept of *nearly open step functions*. These are piecewise constant functions on nearly open partitions.

We want to characterize nearly continuous functions as uniform limits of nearly open step functions in the strong sense of Theorem 8. Condition **(IO)** applies to $\mathcal{O}_n$. For the case of a perfect metrizable space X and a separable metric space Y we can prove **($\mathbf{SR}_\sigma$)**. We recall that a topological space is called perfect if it does not contain isolated points.

Lemma 9. *Let X be a perfect metrizable space and (Y,d_Y) a separable metric space. Then $\mathcal{O}_n$ satisfies* **($\mathbf{SR}_\sigma$)**.

Proof. Let $f\in C_n(X,Y)$ and let $\mathcal{C}=\{C_i : i\in I\}$ be an at most countable cover of X consisting of $(f,\mathcal{O}_n)$-admissible sets. For the construction of the required refinement $\mathcal{P}=\{P_i : i\in I\}$ we assume X to be metrized by d_X and fix a countable base $\mathcal{B}=\{B_1,B_2,\ldots\}$ of the system of open subsets of Y.

We start by defining pairwise disjoint and locally finite sets $P(i,j,k)\subseteq X$, $(i,j,k)\in I\times\{1,2,\ldots\}^2$, subject to

(i) $P(i,j,k)\subseteq C_i\cap f^{-1}(B_j)$ and,

(ii) for every $x\in C_i\cap f^{-1}(B_j)$, there is $x'\in P(i,j,k)$ with $d_X(x,x')\le 2^{-k}$.

We suppose the index set $I\times\{1,2,\ldots\}^2$ to be ordered by $\preceq$ such that $(I\times\{1,2,\ldots\}^2,\preceq)$ is isomorphic to $(\{1,2,\ldots\},\le)$. We proceed by induction with respect to $\preceq$. Let us assume that $P(i,j,k)$ is already defined for $(i,j,k)\prec(i_0,j_0,k_0)$. Now we construct $P(i_0,j_0,k_0)$ as follows.

By paracompactness of X (see [7, p. 300]), there exists a locally finite open cover $\mathcal{U}=\{U_l : l\in L\}$ of X such that $\operatorname{diam}(U_l)\le 2^{-k_0}$. Let

$$\mathcal{U}'=\bigl\{U_l\cap\bigl(C_{i_0}\cap f^{-1}(B_{j_0})\bigr) : l\in L\bigr\}\setminus\{\emptyset\}=\{U'_l : l\in L'\},$$

where $U'_l=U_l\cap\bigl(C_{i_0}\cap f^{-1}(B_{j_0})\bigr)$ and $L'=\{l\in L : U'_l\ne\emptyset\}$. The sets U'_l are nearly open, because C_{i_0} is $(f,\mathcal{O}_n)$-admissible and $\mathcal{O}_n$ satisfies **(IO)**. For every $l\in L'$, we fix a point $x_l\in U'_l\subseteq\operatorname{int}(\operatorname{cl}(U'_l))$. Then x_l is the limit of a sequence of elements of $U'_l \setminus \{x_l\}$, because X is perfect. Thus we can pick a point

$$x'_l \in U'_l \setminus \bigcup_{(i,j,k)\prec(i_0,j_0,k_0)} P(i,j,k), \tag{2}$$

for $\bigcup_{(i,j,k)\prec(i_0,j_0,k_0)} P(i,j,k)$ is locally finite according to the induction hypothesis.

We define

$$P(i_0,j_0,k_0) = \{x'_l : l \in L'\}.$$

The set $P(i_0,j_0,k_0)$ is locally finite, because $\mathcal{U}'$ is, and disjoint with $P(i,j,k)$, $(i,j,k) \prec (i_0,j_0,k_0)$, by (2). Property (i) comes from $x'_l \in U'_l \subseteq C_{i_0} \cap f^{-1}(B_{j_0})$. For proving (ii) we consider $x \in C_{i_0} \cap f^{-1}(B_{j_0})$. We choose $l \in L$ such that $x \in U_l$. Then $x \in U'_l = U_l \cap \big(C_{i_0} \cap f^{-1}(B_{j_0})\big)$, in particular $l \in L'$. The corresponding point x'_l belongs to $P(i_0,j_0,k_0)$. The inclusion (2) together with $U'_l \subseteq U_l$ yields $d_X(x,x'_l) \le \operatorname{diam}(U_l) \le 2^{-k_0}$. This confirms (ii) and completes the inductive definition of the sets $P(i,j,k)$.

Now we come to the partition $\mathcal{P}$. By setting $P_i^{(1)} = \bigcup_{j,k\ge 1} P(i,j,k)$ we obtain a partition $\big\{P_i^{(1)} : i \in I\big\}$ of $X_1 = \bigcup_{(i,j,k)\in I\times\{1,2,\ldots\}^2} P(i,j,k)$. Since I is at most countable, we can assume $I = \{1,\ldots,n\}$ or $I = \{1,2,\ldots\}$. Then the remainder $X_2 = X \setminus X_1$ of X admits the partition $\big\{P_i^{(2)} : i \in I\big\}$ into the sets $P_i^{(2)} = (C_i \setminus (C_1 \cup \ldots \cup C_{i-1})) \cap X_2$. We finally define the partition $\mathcal{P} = \{P_i : i \in I\}$ of X by $P_i = P_i^{(1)} \cup P_i^{(2)}$. Of course, $\mathcal{P}$ refines $\mathcal{C}$, because $P_i = P_i^{(1)} \cup P_i^{(2)} \subseteq (C_i \cap X_1) \cup (C_i \cap X_2) = C_i$ by (i).

It remains to show that the sets P_i are $(f,\mathcal{O}_n)$-admissible. By Lemma 5, this amounts to $P_i \cap f^{-1}(B_j) \in \mathcal{O}_n$ for all $j \ge 1$. Let i and j be fixed. Properties (i) and (ii) yield

$$\operatorname{cl}\big(C_i \cap f^{-1}(B_j)\big) = \operatorname{cl}\big(\textstyle\bigcup_{k\ge 1} P(i,j,k)\big) \subseteq \operatorname{cl}\big(P_i^{(1)} \cap f^{-1}(B_j)\big) \subseteq \operatorname{cl}\big(P_i \cap f^{-1}(B_j)\big).$$

Using this and the $(f,\mathcal{O}_n)$-admissibility of C_i we obtain

$$\begin{aligned} P_i \cap f^{-1}(B_j) \subseteq C_i \cap f^{-1}(B_j) &\subseteq \operatorname{int}\big(\operatorname{cl}\big(C_i \cap f^{-1}(B_j)\big)\big) \\ &\subseteq \operatorname{int}\big(\operatorname{cl}\big(P_i \cap f^{-1}(B_j)\big)\big), \end{aligned}$$

which shows that $P_i \cap f^{-1}(B_j) \in \mathcal{O}_n$. This completes the proof. □

Now Theorem 8 gives a representation of nearly continuous functions.

Theorem 10. *Let X be a perfect metrizable space, (Y,d_Y) a separable metric space, and $f : X \to Y$ a nearly continuous function. Then there exists a chain $(\mathcal{P}_k)_{k=1}^{\infty}$ of at most countable and nearly open partitions $\mathcal{P}_k = \big\{P_i^{(k)} : i \in I_k\big\}$*

of X and a sequence $(\varphi_k)_{k=1}^\infty$ of nearly open step functions φ_k defined on the partitions $\mathcal{P}_k$ such that f is the uniform limit of $(\varphi_k)_{k=1}^\infty$.

If (Y, d_Y) is totally bounded, then this is possible with finite partitions $\mathcal{P}_k$.

If, in addition, X is a compact metrizable space, then one can choose the chain $(\mathcal{P}_k)_{k=1}^\infty$ such that, given any continuous function $g \in C(X,Y)$, there is a sequence $(\psi_k)_{k=1}^\infty$ of nearly open step functions ψ_k defined on the partitions $\mathcal{P}_k$ such that g is the uniform limit of $(\psi_k)_{k=1}^\infty$.

As Theorem 1 does for quasi-continuous functions, Theorem 10 does not only characterize nearly continuous functions as uniform limits of nearly open step functions, but also explains substructures of $C_n(X,Y)$. The set of all uniform limits of nearly open step functions defined on a fixed chain $K = (\mathcal{P}_k)_{k=1}^\infty$ of nearly open partitions is closed with respect to uniform limits. If the partitions are finite, then K defines a complete separable metric space with the distance $d(f_1, f_2) = \sup_{x\in X} d_Y(f_1(x), f_2(x))$. If Y is a linear space, then every K gives rise to a linear subspace of $C_n(X,Y)$, although $C_n(X,Y)$ usually is not closed under linear operations.

We want to end this section by showing that the assumption of perfectness of X cannot be dropped in Theorem 10. We give an example of a non-perfect compact metric subspace X of the Euclidean plane $\mathbb{R}^2$ and of a continuous function $f : X \to [0,1]$ such that f cannot be represented as the uniform limit of a sequence of nearly open step functions.

Let $X = X_\infty \cup \bigcup_{i=0}^\infty X_i$, where $X_i = \{(m\,2^{-i}, 2^{-i}) : m = 0, \ldots, 2^i\}$ and $X_\infty = [0,1] \times \{0\}$, and let $f(\xi_1, \xi_2) = \xi_1$. Assume that there exists a nearly open step function $\varphi : X \to [0,1]$ such that $\sup_{x\in X} |f(x) - \varphi(x)| < \frac{1}{2}$. That is

$$\sup_{(\xi_1,\xi_2)\in X} |\xi_1 - \varphi(\xi_1,\xi_2)| < \tfrac{1}{2}. \tag{3}$$

We infer a contradiction.

φ is piecewise constant on a nearly open partition $\mathcal{P}$. Say $(0,0) \in P_0 \in \mathcal{P}$. Since $\varphi(P_0) \equiv \alpha$ is constant, inequality (3) yields

$$\xi = \sup\{\xi_1 : (\xi_1, 0) \in P_0\} < 1,$$

because $\xi \le |0 - \alpha| + |\xi - \alpha| \le 2 \sup_{(\xi_1,0)\in P_0} |\xi_1 - \varphi(\xi_1, 0)| < 2 \cdot \frac{1}{2} = 1$. Now we consider the point $x_0 = (\xi, 0)$.

Case 1: $x_0 \in P_0$. We obtain $x_0 \in \operatorname{int}(\operatorname{cl}(P_0))$, because $P_0 \in \mathcal{O}_n$. Thus there is a positive radius $\varepsilon_0 > 0$ such that $B(x_0, \varepsilon) \subseteq \operatorname{cl}(P_0)$, $B(x_0, \varepsilon_0)$ denoting the ball $\{x \in X : d_X(x, x_0) \le \varepsilon_0\}$. Since an isolated point $x \in X \setminus X_\infty$ belongs to P_0 if it is an element of $\operatorname{cl}(P_0)$, we have

$$B(x_0, \varepsilon_0) \setminus X_\infty \subseteq P_0. \tag{4}$$

Now we fix $x_1 = (\xi+\delta, 0) \in X_\infty$ with $\delta > 0$ and $x_1 \in B(x_0, \varepsilon_0)$. Then $x_1 \notin P_0$ by the choice of ξ, say $x_1 \in P_1$. Therefore $x_1 \in \operatorname{int}(\operatorname{cl}(P_1))$ and, as above, there exists $\varepsilon_1 > 0$ such that

$$B(x_1, \varepsilon_1) \setminus X_\infty \subseteq P_1. \tag{5}$$

By $x_1 \in B(x_0, \varepsilon_0)$, there is an overlap $(B(x_0, \varepsilon_0) \cap B(x_1, \varepsilon_1)) \setminus X_\infty \neq \emptyset$. But then inclusions (4) and (5) yield $P_0 \cap P_1 \neq \emptyset$, a contradiction.

Case 2: $x_0 \notin P_0$, say $x_0 \in P_1$. Now we proceed in a similar way. We find $\varepsilon_0 > 0$ such that $B(x_0, \varepsilon_0) \setminus X_\infty \subseteq P_1$. Then we can pick $x_1 = (\xi - \delta, 0) \in P_0$ with $x_1 \in B(x_0, \varepsilon_0)$ according to the choice of ξ. We find $\varepsilon_1 > 0$ such that $B(x_1, \varepsilon_1) \setminus X_\infty \subseteq P_0$. Again $\emptyset \neq ((B(x_0, \varepsilon_0) \cap B(x_1, \varepsilon_1)) \setminus X_\infty \subseteq P_1 \cap P_0$. This contradiction completes the example.

4 Almost Quasi-Continuous Functions.

The family $\mathcal{O}_{sp}$ of semi-preopen subsets of a topological space X clearly satisfies **(U)** and **(IO)**. Proposition 2 shows that the set $C_{sp}(X, Y)$ of almost quasi-continuous functions into a metric space (Y, d_Y) is closed under uniform limits. Proposition 3 motivates the concept of a *semi-preopen step function*, that has to be piecewise constant on the sets of a partition of X into semi-preopen subsets. Condition $(\mathbf{SR}_\sigma)$ is not trivial.

Lemma 11. *Let X be a metrizable space and (Y, d_Y) a separable metric space. Then $\mathcal{O}_{sp}$ satisfies $(\mathbf{SR}_\sigma)$.*

We prepare the proof of Lemma 11 by a technical statement.

Lemma 12. *Let X be a metrizable space, let X' denote the set of non-isolated points of X, let (Y, d_Y) be a separable metric space, and let $f \in C_{sp}(X, Y)$. If A is an $(f, \mathcal{O}_{sp})$-admissible set and x_0 is a point from $A \cap X'$, then one of the following claims applies:*

(α) there is a sequence $(\tilde{x}_m)_{m=1}^\infty \subseteq A \setminus X'$ such that $\lim_{m\to\infty} \tilde{x}_m = x_0$ and $\lim_{m\to\infty} f(\tilde{x}_m) = f(x_0)$ or

(β) $x_0 \in \operatorname{cl}\big(\operatorname{int}\big(\operatorname{cl}\big(A \cap f^{-1}(V) \cap X'\big)\big)\big)$ for every open set $V \subseteq Y$ with $x_0 \in f^{-1}(V)$.

Proof. Let A and x_0 be fixed and suppose that (β) is not the case. That is, there exists an open set $V_0 \subseteq Y$ such that

$$x_0 \in f^{-1}(V_0) \text{ and } x_0 \notin \operatorname{cl}\big(\operatorname{int}\big(\operatorname{cl}\big(A \cap f^{-1}(V_0) \cap X'\big)\big)\big). \tag{6}$$

We assume X to be metrized by d_X. We shall show that, for every $m \geq 1$, there exists $\tilde{x}_m \in A \backslash X'$ with $d_X(x_0, \tilde{x}_m) \leq 2^{-m}$ and $d_Y(f(x_0), f(\tilde{x}_m)) \leq 2^{-m}$. This then obviously implies (α).

We fix an open set $V_m \subseteq Y$ such that $f(x_0) \in V_m \subseteq V_0$ and $\mathrm{diam}(V_m) \leq 2^{-m}$. Since A is $(f, \mathcal{O}_{sp})$-admissible, we have

$$x_0 \in A \cap f^{-1}(V_m) \subseteq \mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big(A \cap f^{-1}(V_m)\big)\big)\big).$$

We abbreviate $H = A \cap f^{-1}(V_m)$. Since X splits into the two open sets $X \setminus X'$ and $\mathrm{int}(X')$ and the nowhere dense set $\mathrm{bd}(X')$, we obtain

$$\begin{aligned}
x_0 &\in \mathrm{cl}(\mathrm{int}(\mathrm{cl}(H))) \\
&= \mathrm{cl}(\mathrm{int}(\mathrm{cl}(H)) \cap (X \setminus X')) \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(H)) \cap \mathrm{int}(X')) \\
&\subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(H \cap (X \setminus X')))) \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(H \cap \mathrm{int}(X')))) \\
&= \mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big(\big(A \cap f^{-1}(V_m)\big) \cap (X \setminus X')\big)\big)\big) \\
&\qquad \cup \mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big(\big(A \cap f^{-1}(V_m)\big) \cap \mathrm{int}(X')\big)\big)\big) \\
&\subseteq \mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big((A \setminus X') \cap f^{-1}(V_m)\big)\big)\big) \cup \mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big(A \cap f^{-1}(V_0) \cap X'\big)\big)\big).
\end{aligned}$$

Now the second part of (6) yields $x_0 \in \mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big((A \setminus X') \cap f^{-1}(V_m)\big)\big)\big)$. Thus we find $\tilde{x}_m \in (A \setminus X') \cap f^{-1}(V_m)$ such that $d_X(x_0, \tilde{x}_m) \leq 2^{-m}$. The inclusion $x_0 \in f^{-1}(V_m)$ gives the estimate $d_Y(f(x_0), f(\tilde{x}_m)) \leq \mathrm{diam}(V_m) \leq 2^{-m}$. □

Proof of Lemma 11. Let a function $f \in C_{sp}(X, Y)$ and an at most countable cover $\mathcal{C} = \{C_i : i \in I\}$ of X by $(f, \mathcal{O}_{sp})$-admissible sets be given ($I = \{1, \ldots, n\}$ or $I = \{1, 2, \ldots\}$). We assume X to be metrized by a metric d_X and fix a countable base $\mathcal{B} = \{B_1, B_2, \ldots\}$ of the system of open subsets of Y. Let X' denote the set of all non-isolated points of X.

The construction of the refinement $\mathcal{P}$ of $\mathcal{C}$ required in $(\mathbf{SR}_\sigma)$ starts with the definition of pairwise disjoint and locally finite sets $P(i, j, k) \subseteq X$, $(i, j, k) \in I \times \{1, 2, \ldots\}^2$, that satisfy

(i) $P(i, j, k) \subseteq C_i \cap f^{-1}(B_j)$,

(ii) for every $x_0 \in C_i \cap f^{-1}(B_j) \cap X'$, there exists $x' \in P(i, j, k)$ with $d_X(x_0, x') \leq 2^{-k}$, and

(iii) for every $x_0 \in C_i \cap f^{-1}(B_j) \cap X'$, if there exists a sequence $(\tilde{x}_m)_{m=1}^\infty \subseteq C_i \setminus X'$ such that $\lim_{m\to\infty} \tilde{x}_m = x_0$ and $\lim_{m\to\infty} f(\tilde{x}_m) = f(x_0)$, then there is $x' \in P(i, j, k) \setminus X'$ with $d_X(x_0, x') \leq 2^{-k}$.

As in the proof of Lemma 9 we assume $I \times \{1, 2, \ldots\}^2$ to be ordered by $\preceq$. We proceed by induction and suppose in the sense of an induction hypothesis that $P(i, j, k)$ is already defined for $(i, j, k) \prec (i_0, j_0, k_0)$. The construction of $P(i_0, j_0, k_0)$ is as follows.

We fix a locally finite open cover $\mathcal{U} = \{U_l : l \in L\}$ of X with $\operatorname{diam}(U_l) \le 2^{-k_0}$. Let

$$\mathcal{U}' = \big\{U_l \cap \big(C_{i_0} \cap f^{-1}(B_{j_0})\big) \cap X' : l \in L\big\} \setminus \{\emptyset\} = \{U'_l \cap X' : l \in L'\},$$

where $U'_l = U_l \cap \big(C_{i_0} \cap f^{-1}(B_{j_0})\big)$ and $L' = \{l \in L : U'_l \cap X' \neq \emptyset\}$. Let $l \in L'$ be fixed. Then $U'_l \cap X' \neq \emptyset$ and the set U'_l is $(f, \mathcal{O}_{sp})$-admissible by Lemma 7. We apply Lemma 12 to $A = U'_l$.

Case α: There is a point $x_0 \in U'_l \cap X'$ satisfying property (α). Then we can pick a point

$$x_l \in (U'_l \setminus X') \setminus \textstyle\bigcup_{(i,j,k) \prec (i_0,j_0,k_0)} P(i, j, k),$$

because $\bigcup_{(i,j,k) \prec (i_0,j_0,k_0)} P(i, j, k)$ is locally finite by the induction hypothesis.

Case β: No point of $U'_l \cap X'$ satisfies (α). Lemma 12 provides us with a point $x_0 \in U'_l \cap X'$ satisfying (β), in particular $x_0 \in \operatorname{cl}(\operatorname{int}(\operatorname{cl}(U'_l)))$. Since x_0 is not isolated in X, x_0 is the limit of other points of U'_l. Therefore we can choose a point

$$x_l \in U'_l \setminus \textstyle\bigcup_{(i,j,k) \prec (i_0,j_0,k_0)} P(i, j, k),$$

because $\bigcup_{(i,j,k) \prec (i_0,j_0,k_0)} P(i, j, k)$ is locally finite.

Now we define $P(i_0, j_0, k_0)$ by

$$P(i_0, j_0, k_0) = \{x_l : l \in L'\}.$$

Clearly, $P(i_0, j_0, k_0)$ is locally finite, because $x_l \in U'_l \subseteq U_l$ and $\mathcal{U}$ is locally finite, and $P(i_0, j_0, k_0)$ is disjoint with $P(i, j, k)$ for $(i, j, k) \prec (i_0, j_0, k_0)$. Condition (i) applies, since $x_l \in U'_l \subseteq C_{i_0} \cap f^{-1}(B_{j_0})$.

To verify (ii) and (iii) we consider $x_0 \in C_{i_0} \cap f^{-1}(B_{j_0}) \cap X'$. There exists $l \in L$ with $x_0 \in U_l$. Then $x_0 \in U_l \cap C_{i_0} \cap f^{-1}(B_{j_0}) \cap X' = U'_l \cap X'$ and $l \in L'$. Thus we find $x' = x_l \in P(i_0, j_0, k_0)$ and obtain $d_X(x_0, x') \le \operatorname{diam}(U_l) \le 2^{-k_0}$, because $x_0 \in U_l$ and $x' = x_l \in U'_l \subseteq U_l$. This confirms (ii). Now we suppose in addition that there is a sequence $(\tilde{x}_m)_{m=1}^{\infty} \subseteq C_{i_0} \setminus X'$ with $\lim_{m\to\infty} \tilde{x}_m = x_0$ and $\lim_{m\to\infty} f(\tilde{x}_m) = f(x_0)$. Since U_l and B_{j_0} are neighborhoods of x_0 and $f(x_0)$, respectively, we can assume that $(\tilde{x}_m)_{m=1}^{\infty} \subseteq U_l \cap f^{-1}(B_{j_0})$ and obtain $(\tilde{x}_m)_{m=1}^{\infty} \subseteq \big(U_l \cap f^{-1}(B_{j_0})\big) \cap (C_{i_0} \setminus X') = U'_l \setminus X'$. Hence the point $x' = x_l$ is chosen according to Case α, in particular $x' \in X \setminus X'$. This proves (iii).

Now, having successfully constructed the sets $P(i,j,k)$, we define the partition $\mathcal{P} = \{P_i : i \in I\}$ as in the proof of Lemma 9. $P_i = P_i^{(1)} \cup P_i^{(2)}$, where $\{P_i^{(1)} : i \in I\}$ with $P_i^{(1)} = \bigcup_{j,k\geq 1} P(i,j,k)$ is a partition of $X_1 = \bigcup_{(i,j,k)\in I\times\{1,2,\ldots\}^2} P(i,j,k)$ and $\{P_i^{(2)} : i \in I\}$ with $P_i^{(2)} = (C_i \setminus (C_1 \cup \ldots \cup C_{i-1})) \setminus X_1$ is a partition of $X \setminus X_1$. Property (i) shows that $\mathcal{P}$ refines $\mathcal{C}$.

We finally have to show that the sets P_i are $(f, \mathcal{O}_{sp})$-admissible. That is, for every $i \in I$ and every $j \in \{1,2,\ldots\}$,

$$P_i \cap f^{-1}(B_j) \subseteq \mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big(P_i \cap f^{-1}(B_j)\big)\big)\big).$$

We consider fixed i_0 and j_0 and a fixed point $x_0 \in P_{i_0} \cap f^{-1}(B_{j_0})$ for proving this inclusion.

Case 1: $x_0 \in X \setminus X'$. Then x_0 is isolated and

$$x_0 \in \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\{x_0\}))) \subseteq \mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big(P_{i_0} \cap f^{-1}(B_{j_0})\big)\big)\big).$$

Case 2: $x_0 \in X'$. Then $x_0 \in P_{i_0} \cap X' \subseteq C_{i_0} \cap X'$, the set $C_{i_0} \in \mathcal{C}$ being $(f, \mathcal{O}_{sp})$-admissible. Application of Lemma 12 to $A = C_{i_0}$ gives the following two subcases.

Case 2.1: $x_0 \in \mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big(C_{i_0} \cap f^{-1}(B_{j_0}) \cap X'\big)\big)\big)$. Here property (ii) yields

$$\mathrm{cl}\big(C_{i_0} \cap f^{-1}(B_{j_0}) \cap X'\big) \subseteq \mathrm{cl}\big(\textstyle\bigcup_{k\geq 1} P(i_0,j_0,k)\big) \subseteq \mathrm{cl}\big(P_{i_0} \cap f^{-1}(B_{j_0})\big)$$

and therefore

$$x_0 \in \mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big(C_{i_0} \cap f^{-1}(B_{j_0}) \cap X'\big)\big)\big) \subseteq \mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big(P_{i_0} \cap f^{-1}(B_{j_0})\big)\big)\big).$$

Case 2.2: There is a sequence $(\tilde{x}_m)_{m=1}^{\infty} \subseteq C_{i_0} \setminus X'$ *with* $\lim_{m\to\infty} \tilde{x}_m = x_0$ *and* $\lim_{m\to\infty} f(\tilde{x}_m) = f(x_0)$. Since $x_0 \in C_{i_0} \cap f^{-1}(B_{j_0}) \cap X'$, property (iii) shows that, for every $k \geq 1$, there exists $x'_k \in P(i_0,j_0,k) \setminus X'$ with $d_X(x_0, x'_k) \leq 2^{-k}$. This yields

$$\begin{aligned} x_0 &\in \mathrm{cl}(\{x'_k : k = 1,2,\ldots\}) = \mathrm{cl}\Big(\bigcup_{k\geq 1}\{x'_k\}\Big) \\ &= \mathrm{cl}\Big(\bigcup_{k\geq 1} \mathrm{int}\big(\mathrm{cl}\big(\{x'_k\}\big)\big)\Big) \subseteq \mathrm{cl}\Big(\mathrm{int}\Big(\mathrm{cl}\Big(\bigcup_{k\geq 1}\{x'_k\}\Big)\Big)\Big) \\ &\subseteq \mathrm{cl}\Big(\mathrm{int}\Big(\mathrm{cl}\Big(\bigcup_{k\geq 1} P(i_0,j_0,k)\Big)\Big)\Big) \subseteq \mathrm{cl}\big(\mathrm{int}\big(\mathrm{cl}\big(P_{i_0} \cap f^{-1}(B_{j_0})\big)\big)\big). \qquad \square \end{aligned}$$

Theorem 8 now gives a characterization of almost quasi-continuous functions.

Theorem 13. *Let X be a metrizable space, (Y, d_Y) a separable metric space, and $f : X \to Y$ an almost quasi-continuous function. Then there exists a chain $(\mathcal{P}_k)_{k=1}^{\infty}$ of at most countable and semi-preopen partitions $\mathcal{P}_k = \{P_i^{(k)} : i \in I_k\}$ of X and a sequence $(\varphi_k)_{k=1}^{\infty}$ of semi-preopen step functions φ_k defined on the partitions $\mathcal{P}_k$ such that f is the uniform limit of $(\varphi_k)_{k=1}^{\infty}$.*

If (Y, d_Y) is totally bounded, then this is possible with finite partitions $\mathcal{P}_k$.

If, in addition, X is a compact metrizable space, then one can choose the chain $(\mathcal{P}_k)_{k=1}^{\infty}$ such that, given any continuous function $g \in C(X, Y)$, there is a sequence $(\psi_k)_{k=1}^{\infty}$ of semi-preopen step functions ψ_k defined on the partitions $\mathcal{P}_k$ such that g is the uniform limit of $(\psi_k)_{k=1}^{\infty}$.

The remarks on the role of Theorem 10 concerning nearly continuous functions apply analogously to Theorem 13 with respect to almost quasi-continuous functions. However, it is remarkable that Theorem 13 covers all metrizable spaces X, whereas the restriction of Theorem 10 to perfect spaces is essential.

5 α-Continuous Functions.

The family $\mathcal{O}_\alpha$ of α-sets in a topological space X again satisfies **(U)** and **(IO)**. Hence the set $C_\alpha(X, Y)$ of all α-continuous functions from X into a metric space (Y, d_Y) is closed under uniform limits by Proposition 2. Proposition 3 leads to the concept of *α-step functions*. These are piecewise constant functions on partitions of X into α-sets.

In contrast with $\mathcal{O}_s$, $\mathcal{O}_n$, $\mathcal{O}_{sp}$, and $\mathcal{O}_{sw}$, $\mathcal{O}_\alpha$ is a topology (see [16]). We have already mentioned in the introduction that α-sets are very close to open sets, though $\mathcal{O}_\alpha$ usually is strictly larger than $\mathcal{O}$. A strong relation between $\mathcal{O}_\alpha$ and $\mathcal{O}$ is the following.

Proposition 14. *If a topological space X is partitioned into α-sets A_i, $i \in I$, then the sets A_i are open.*

Proof. We equip I with the discrete metric d_I. Then the map $\varphi : X \to I$, $\varphi(A_i) \equiv i$, is α-continuous by Proposition 3. An α-continuous function into a metric space is continuous, because it is quasi-continuous as well as nearly continuous (see [17]). Thus $A_i = \varphi^{-1}(\{i\})$ is open, since it is the inverse image of the open set $\{i\}$ under the continuous map φ. □

Proposition 14 shows that α-step functions on a space X in fact are based on open partitions of X. We do not use the name “open step function” in order to avoid confusion with the concept of open functions.

A space X clearly cannot be connected if it possesses non-constant α-step functions. We characterize the spaces X that admit an analogue of Theorem 8 within the class of all paracompact spaces. We recall that a topological space X is *paracompact* if it is a Hausdorff space and every open cover of X has a locally finite open refinement. Paracompact spaces are normal. A normal space is *strongly zero-dimensional* if, for every pair A, B of completely separated subsets of X, there is an open and closed set $G \subseteq X$ such that $A \subseteq G \subseteq X \setminus B$. Here A, B are called *completely separated* if there is a continuous function f from X into the closed interval $[0, 1]$ such that $f(A) \equiv 0$ and $f(B) \equiv 1$ (see [7, pp. 42, 299, 300, 361]).

As already mentioned in the introduction, a map from a topological space into a metric space is α-continuous if and only if it is continuous. For that reason we formulate the following theorem for continuous functions.

Theorem 15. *Let X be a paracompact space. The following are equivalent:*

(i) Every continuous function $f : X \to [0, 1]$ is the uniform limit of α-step functions $\varphi_k : X \to [0, 1]$, $k = 1, 2, \dots$

(ii) Let (Y, d_Y) be a separable metric space and let $f \in C(X, Y)$. Then there exists a chain $(\mathcal{P}_k)_{k=1}^\infty$ of at most countable and open partitions $\mathcal{P}_k = \{P_i^{(k)} : i \in I_k\}$ of X and a sequence $(\varphi_k)_{k=1}^\infty$ of α-step functions φ_k defined on the partitions $\mathcal{P}_k$ such that f is the uniform limit of $(\varphi_k)_{k=1}^\infty$. If (Y, d_Y) is totally bounded, then this is possible with finite partitions $\mathcal{P}_k$. If, in addition, X is compact and metrizable, then one can choose the chain $(\mathcal{P}_k)_{k=1}^\infty$ such that, given any continuous function $g \in C(X, Y)$, there is a sequence $(\psi_k)_{k=1}^\infty$ of α-step functions ψ_k defined on the partitions $\mathcal{P}_k$ such that g is the uniform limit of $(\psi_k)_{k=1}^\infty$.

(iii) X is strongly zero-dimensional.

PROOF. *(ii)⇒(i)* is trivial.

(i)⇒(iii). Let $A, B \subseteq X$ be completely separated by a function $f \in C(X, [0, 1])$. By (i), there exists an α-step function $\varphi : X \to [0, 1]$ such that $\sup_{x \in X} d_Y(f(x), \varphi(x)) < \frac{1}{2}$. Let $G = \varphi^{-1}\big([0, \frac{1}{2})\big)$. Then G is open and closed, since both $G = \varphi^{-1}\big([0, \frac{1}{2})\big)$ and $X \setminus G = \varphi^{-1}\big([\frac{1}{2}, 1]\big)$ are unions of sets from the underlying open partition of φ. Condition $\sup_{x \in X} d_Y(f(x), \varphi(x)) < \frac{1}{2}$ yields

$$A \subseteq \varphi^{-1}\big([0, \tfrac{1}{2})\big) = G \subseteq X \setminus \varphi^{-1}\big((\tfrac{1}{2}, 1]\big) \subseteq X \setminus B.$$

(iii)⇒(ii). We want to apply Theorem 8. We have to verify the conditions **(SR$_\sigma$)** and **(IO)** for the family $\mathcal{O}$ of open subsets of X, because the claim

concerns continuous functions and step functions on open partitions. **(IO)** is trivial. Since $\{A \subseteq X : A \text{ is } (f,\mathcal{O})\text{-admissible}\} = \mathcal{O}$ for every $f \in C(X,Y)$, condition **(SR$_\sigma$)** amounts to: *Every at most countable open cover $\mathcal{C} = \{C_i : i \in I\}$ of X can be refined to an open partition $\mathcal{P} = \{P_i : i \in I\}$.*

We suppose $I = \{1,2,\dots\}$. (If I is finite we proceed analogously.) By paracompactness of X, there exists a locally finite open refinement $\mathcal{C}' = \{C'_j : j \in J\}$ of $\mathcal{C}$. We obtain a locally finite open refinement $\tilde{\mathcal{C}} = \{\tilde{C}_i : i \in I\}$ of $\mathcal{C}$ by choosing an index $i(j) \in I$ with $C'_j \subseteq C_{i(j)}$ for every $j \in J$ and setting $\tilde{C}_i = \bigcup\{C'_j : j \in J, i(j) = i\}$. Now we define the required open partition $\mathcal{P} = \{P_i : i \in I\}$ that refines $\mathcal{C}$ by inductively constructing pairwise disjoint open and closed sets $P_i \subseteq \tilde{C}_i$ such that $\{P_1,\dots,P_i,\tilde{C}_{i+1},\tilde{C}_{i+2},\dots\}$ covers X, $i = 1,2,\dots$ (The covering property of $\mathcal{P}$ follows from the local finiteness of $\tilde{\mathcal{C}}$.)

Let i_0 be fixed and assume $P_1,\dots,P_{i_0-1}$ to be already defined. Let

$$A = X \setminus \left(\bigcup_{i<i_0} P_i \cup \bigcup_{i>i_0} \tilde{C}_i\right) \text{ and } B = X \setminus \tilde{C}_{i_0}.$$

The sets A and B are closed and disjoint. Hence they are completely separated, since X is normal. By (iii), there exists an open and closed set $G \subseteq X$ such that $A \subseteq G \subseteq X \setminus B$. We define $P_{i_0} = G \setminus \bigcup_{i<i_0} P_i$. This set is open and closed by the induction hypothesis. It is disjoint with P_i, $i < i_0$, and satisfies $P_{i_0} \subseteq G \subseteq X \setminus B = \tilde{C}_{i_0}$. Finally, $\{P_1,\dots,P_{i_0},\tilde{C}_{i_0+1},\tilde{C}_{i_0+2},\dots\}$ covers X, because

$$P_{i_0} = G \setminus \bigcup_{i<i_0} P_i \supseteq A \setminus \bigcup_{i<i_0} P_i = A$$

and therefore

$$P_{i_0} \cup \bigcup_{i<i_0} P_i \cup \bigcup_{i>i_0} \tilde{C}_i \supseteq A \cup \bigcup_{i<i_0} P_i \cup \bigcup_{i>i_0} \tilde{C}_i = X. \qquad \square$$

6 Somewhat Continuous Functions.

The family $\mathcal{O}_{sw}$ of somewhat open subsets of a topological space X trivially satisfies **(U)**. Proposition 2 ensures that the set $C_{sw}(X,Y)$ of somewhat continuous functions from X into a metric space (Y,d_Y) is closed under uniform limits. We call a function $\varphi : X \to Y$ a *somewhat open step function* if φ is piecewise constant on a partition of X into somewhat open sets. Somewhat open step functions represent a basic type of somewhat continuous functions (see Proposition 3).

However, simple examples on $X = \mathbb{R}$ show that $\mathcal{O}_{sw}$ does not satisfy **(IO)**. Moreover, already the weakest refinement property **(R$_f$)** fails on the compact space $X = [0,1] \cup \{2\} \subseteq \mathbb{R}$. The cover $\{\{0,2\},\{1,2\},(0,1)\}$ is somewhat open, but cannot be refined to a partition of X into somewhat open sets, because neither $\{0,2\}$ nor $\{1,2\}$ can be subdivided into two non-empty somewhat open subsets.

Nevertheless, one can prove an analogue of the main part of Theorem 8.

Theorem 16. *Let X be a topological space and (Y, d_Y) a metric space and let $f \in C_{sw}(X, Y)$. Then there exists a chain $(\mathcal{P}_k)_{k=1}^{\infty}$ of somewhat open partitions $\mathcal{P}_k = \{P_i^{(k)} : i \in I_k\}$ of X and a sequence $(\varphi_k)_{k=1}^{\infty}$ of somewhat open step functions φ_k defined on the partitions $\mathcal{P}_k$ such that f is the uniform limit of $(\varphi_k)_{k=1}^{\infty}$. This is possible with at most countable partitions $\mathcal{P}_k$ if (Y, d_Y) is separable and with finite partitions $\mathcal{P}_k$ if (Y, d_Y) is totally bounded.*

The proof is based on the following claim.

Lemma 17. *Let X be a topological space, (Y, d_Y) a metric space, $f : X \to Y$ a somewhat continuous function, and $\varepsilon > 0$. Then every $(f, \mathcal{O}_{sw})$-admissible set $A \subseteq X$ can be partitioned into $(f, \mathcal{O}_{sw})$-admissible subsets A_j, $j \in J$, such that $\operatorname{diam}(f(A_j)) \leq \varepsilon$. This is possible with an at most countable set J if (Y, d_Y) is separable and with finite J if (Y, d_Y) is totally bounded.*

PROOF. We consider $f(A)$ as a metric subspace of (Y, d_Y). We shall denote the closure operator in $f(A)$ by $\operatorname{cl}_{f(A)}(\cdot)$ in contrast with the operators $\operatorname{cl}(\cdot)$ and $\operatorname{int}(\cdot)$ in Y. Using paracompactness of $f(A)$ (see [7, p. 300]) we obtain a locally finite open cover $\mathcal{W} = \{W_j : j \in J\}$ of $f(A)$ with $\operatorname{diam}(W_j) \leq \varepsilon$. We can assume J to be at most countable if (Y, d_Y) is separable and to be finite if (Y, d_Y) is totally bounded.

We suppose J to be totally ordered and define

$$W_j' = W_j \setminus \bigcup_{i<j} \operatorname{cl}_{f(A)}(W_i) = W_j \setminus \operatorname{cl}_{f(A)}\big(\bigcup_{i<j} W_i\big),$$

the second equation being a consequence of the local finiteness of $\mathcal{W}$. The sets W_j' are disjoint and open in $f(A)$. We obtain $f(A) = \bigcup_{j \in J} \operatorname{cl}_{f(A)}(W_j')$. Hence the sets

$$W_j'' = \operatorname{cl}_{f(A)}(W_j') \setminus \bigcup_{i<j} \operatorname{cl}_{f(A)}(W_i'),$$

$j \in J$, form a partition of $f(A)$. Consequently,

$$A_j = A \cap f^{-1}(W_j''),\ j \in J,$$

defines a partition of A. Clearly,

$$\operatorname{diam}(f(A_j)) \leq \operatorname{diam}(W_j'') \leq \operatorname{diam}(\operatorname{cl}_{f(A)}(W_j')) = \operatorname{diam}(W_j') \leq \operatorname{diam}(W_j) \leq \varepsilon.$$

Now it remains to prove that the sets A_j are $(f, \mathcal{O}_{sw})$-admissible; that is, for every $j \in J$ and every open set $V \subseteq Y$, $A_j \cap f^{-1}(V) \in \mathcal{O}_{sw}$. Let j and V be fixed. We can assume $A_j \cap f^{-1}(V) \neq \emptyset$, because otherwise trivially $A_j \cap f^{-1}(V) = \emptyset \in \mathcal{O}_{sw}$.

Since W'_j is open in $f(A)$, there exists an open set $\tilde{W}_j \subseteq Y$ such that $W'_j = f(A) \cap \tilde{W}_j$. Next we show that

$$\operatorname{int}\big(A \cap f^{-1}\big(\tilde{W}_j \cap V\big)\big) \neq \emptyset. \tag{7}$$

We have

$$\emptyset \neq f\big(A_j \cap f^{-1}(V)\big) \subseteq f(A_j) \cap V \subseteq W''_j \cap V \subseteq \operatorname{cl}_{f(A)}(W'_j) \cap V \subseteq \operatorname{cl}(W'_j) \cap V.$$

This yields $W'_j \cap V \neq \emptyset$, because V is open. Since $W'_j \cap V \subseteq W_j \subseteq f(A)$, we obtain

$$\begin{aligned}\emptyset \neq A \cap f^{-1}(W'_j \cap V) &= A \cap f^{-1}\big(f(A) \cap \tilde{W}_j \cap V\big)\\ &= A \cap f^{-1}(f(A)) \cap f^{-1}\big(\tilde{W}_j \cap V\big) = A \cap f^{-1}\big(\tilde{W}_j \cap V\big).\end{aligned}$$

This yields (7), because the last set is somewhat open, for A is $(f, \mathcal{O}_{sw})$-admissible and $\tilde{W}_j \cap V$ is open.

Since the sets W'_i, $i \in J$, are disjoint and open in $f(A)$, we obtain

$$f(A) \cap \tilde{W}_j = W'_j = W'_j \setminus \textstyle\bigcup_{i<j} \operatorname{cl}_{f(A)}(W'_i) \subseteq W''_j.$$

This yields

$$A \cap f^{-1}\big(\tilde{W}_j \cap V\big) = A \cap f^{-1}\big(f(A) \cap \tilde{W}_j \cap V\big) \subseteq A \cap f^{-1}(W''_j \cap V) = A_j \cap f^{-1}(V).$$

Combining this inclusion with (7) we arrive at $\operatorname{int}\big(A_j \cap f^{-1}(V)\big) \neq \emptyset$. This gives $A_j \cap f^{-1}(V) \in \mathcal{O}_{sw}$ and completes the proof. □

PROOF OF THEOREM 16. We apply Lemma 17 to the $(f, \mathcal{O}_{sw})$-admissible set X and $\varepsilon_1 = 2^{-1}$ and obtain a partition $\mathcal{P}_1 = \big\{P_i^{(1)} : i \in I_1\big\}$ of X into $(f, \mathcal{O}_{sw})$-admissible sets with $\operatorname{diam}\big(f\big(P_i^{(1)}\big)\big) \leq 2^{-1}$. Given the partition $\mathcal{P}_{k-1}$, application of Lemma 17 to all elements of $\mathcal{P}_{k-1}$ and $\varepsilon_k = 2^{-k}$ yields a partition $\mathcal{P}_k = \big\{P_i^{(k)} : i \in I_k\big\}$ into $(f, \mathcal{O}_{sw})$-admissible sets with $\operatorname{diam}\big(f\big(P_i^{(k)}\big)\big) \leq 2^{-k}$. This defines a chain $(\mathcal{P}_k)_{k=1}^{\infty}$ of somewhat open partitions. The lemma gives at most countable or even finite partitions $\mathcal{P}_k$ if (Y, d_Y) is separable or totally bounded, respectively. Since $\operatorname{diam}\big(f\big(P_i^{(k)}\big)\big) \leq 2^{-k}$ for $i \in I_k$, there is a somewhat open step function φ_k on $\mathcal{P}_k$ with $\sup_{x \in X} d_Y(f(x), \varphi_k(x)) \leq 2^{-k}$. □

As previous theorems did for other concepts of generalized continuity, Theorem 16 illuminates the structure of $C_{sw}(X,Y)$. Every chain $(\mathcal{P}_k)_{k=1}^{\infty}$ of somewhat open partitions defines a subset of $C_{sw}(X,Y)$ that is closed under uniform limits. It is a metric space if the partitions $\mathcal{P}_k$ all are finite. The space is separable if (Y,d_Y) in addition is separable. For the case of a linear space Y the subset of $C_{sw}(X,Y)$ defined by a chain $(\mathcal{P}_k)_{k=1}^{\infty}$ is a linear space, though $C_{sw}(X,Y)$ itself usually is not.

The following example illustrates that, in contrast with Theorems 1, 8, 10, 13, and 15, the final passage on "universal chains" of partitions on compact metrizable spaces X does not apply to somewhat continuous functions. Let $X = Y = [0,2]$ be equipped with the usual distance from $\mathbb{R}$. Then the functions

$$f(x) = \begin{cases} 1 & \text{for } x \in \{0\} \cup [1,2] \\ 0 & \text{for } x \in (0,1) \end{cases}, \text{ and } g(x) = x$$

are somewhat continuous and continuous, respectively. However, there does not exist a somewhat open partition $\mathcal{P}$ of X and somewhat open step functions φ and ψ on $\mathcal{P}$ such that $\sup_{x\in X} |f(x)-\varphi(x)| < \frac{1}{2}$ and $\sup_{x\in X} |g(x)-\psi(x)| < \frac{1}{2}$. Indeed, otherwise the set $P \in \mathcal{P}$ with $0 \in P$ would satisfy $P \subseteq \{0\} \cup [1,2]$ by the first estimate and $P \subseteq [0,1)$ by the second estimate. But this yields $P = \{0\}$, a contradiction with $P \in \mathcal{O}_{sw}$.

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