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A CHARACTERIZATION OF RINGS OF DENSITY CONTINUOUS FUNCTIONS

Abstract

A density continuous function is defined as a continuous function from a Tychonoff space X into the real numbers with the density topology. The collection of density continuous functions on X is denoted by $C(X, \mathbb{R}_d)$. It is shown that $C(X, \mathbb{R}_d)$ is a ring precisely when each density continuous function is locally constant, and in this case X is defined to be a density P-space. Examples of density P-spaces are given.

1 Introduction.

Historically, the study of the density topology on the real numbers can be traced back to A. Denjoy in 1915. Denjoy defined a function from the real numbers into the real numbers as being *approximately continuous* at a point x if the inverse image M of an open subset of the real numbers is dense about x in terms of Lebesgue measure; that is, if

$$\lim_{h \to 0^+} \frac{1}{2h} |M \cap (x-h, x+h)| = 1 \tag{1}$$

where $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}$. In the 1950s, this idea was used to construct the density topology on the real numbers, denoted $\mathbb{R}_d$, where a set $O \subseteq \mathbb{R}$ is open in the density topology if equation (1) holds for all $x \in O$. An approximately continuous function, as defined by Denjoy, turned out to be a continuous function $f : \mathbb{R}_d \to \mathbb{R}$. Later, mathematicians called a continuous function $f : \mathbb{R}_d \to \mathbb{R}_d$ a *density continuous* function.

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It is known that the density topology is finer than the usual topology on $\mathbb{R}$ and that subsets of $\mathbb{R}$ of measure zero are closed in the density topology. In particular, countable subsets of $\mathbb{R}_d$ are closed; that is, $\mathbb{R}_d$ is a weak P-space. It follows that the only compact subsets of $\mathbb{R}_d$ are the finite ones. Sometimes we will have cause to consider the real numbers with the discrete topology which will be denoted by $\mathbb{R}^d$.

Let X be an arbitrary topological space. We will assume that all spaces are Tychonoff; that is, completely regular and Hausdorff. Given a subset A of X, $\operatorname{int} A$ denotes the interior of A in X, and $\operatorname{cl} A$ denotes the closure of A in X. Given two spaces X and Y, the collection of continuous functions $f : X \to Y$ is denoted by $C(X, Y)$. When $Y = \mathbb{R}$ with the usual topology, we will write $C(X)$ instead, and this set is a ring under pointwise addition and multiplication. Given $f \in C(X)$, the set $Z(f) = \{x \in X : f(x) = 0\}$ is called the zeroset of f, and the set $\operatorname{coz}(f) = \{x \in X : f(x) \neq 0\}$ is called the cozeroset of f. In a Tychonoff space, the cozerosets form a base for the topology on X. It is shown in [6] that $\mathbb{R}_d$ is a Tychonoff space which is not normal.

Previous authors have used the term *density continuous* function to refer specifically to elements of $C(\mathbb{R}_d, \mathbb{R}_d)$. We will generalize this phrase to refer to any member of $C(X, \mathbb{R}_d)$. The density topology is finer than the usual topology, so we have that $C(X, \mathbb{R}_d) \subseteq C(X)$ for any topological space X. We will also use the notation $C(X, \mathbb{R}_d)^+$ to denote the set $\{f \in C(X, \mathbb{R}_d) : f \geq 0\}$.

A function $f : (a, b) \to \mathbb{R}$ is called *(real) analytic* if, at each $x_o \in (a, b)$, f is represented by a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n$$

that converges in some open interval $I_\delta = (x_o - \delta, x_o + \delta)$, $\delta > 0$. It is known that the sum, product, reciprocal, composite, and inverse function of real analytic functions is again real analytic. Polynomials are obviously analytic functions on $\mathbb{R}$. The exponential function $g(x) = e^x$ is analytic on $\mathbb{R}$, so its inverse function $g^{-1}(x) = \ln x$ is analytic on $(0, \infty)$. The following theorem concerning density continuity of real analytic functions was proved independently by [1] and [2].

Theorem 1.1. *Real analytic functions are density continuous.*

2 The Sum of Density Continuous Functions.

For any space X, it is known that $C(X)$ is a ring and a lattice under pointwise operations. It is natural to ask if $C(X, \mathbb{R}_d)$ has the same properties. In [7], it is shown that $C(X, \mathbb{R}_d)$ is a sublattice of $C(X)$. The focus of this paper is the investigation of when $C(X, \mathbb{R}_d)$ is a group or a ring. In the case of $X = \mathbb{R}_d$, as we are about to see in Example 2.1, the set $C(\mathbb{R}_d, \mathbb{R}_d)$ is not closed under addition and hence not a group. This example comes from Theorem 2 of [1] and is the motivation for our characterization of rings of density continuous functions.

Example 2.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ which satisfies

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ 1 - x & \text{if } x \geq \frac{1}{2} \\ \frac{1}{n} - x & \text{if } \frac{1}{2n} \leq x \leq \frac{1}{2(n-1)} - 2^{-n-10} \text{ for some } n \geq 2 \end{cases}$$

and f is linear and continuous on $\left[\frac{1}{2(n-1)} - 2^{-n-10}, \frac{1}{2(n-1)}\right]$ for $n \geq 2$. The function f is a density continuous function. However, adding the identity function $i(x) = x$ to f yields a function $g(x) = f(x) + x$ which is not an element of $C(\mathbb{R}_d, \mathbb{R}_d)$. The function g fails to be density continuous at zero. In particular, the inverse image of the closed set $F = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not a density closed set.

It follows from Example 2.1 that $C(\mathbb{R}_d, \mathbb{R}_d)$ is not a group under addition. Example 2.1 leads us to the following characterization of when $C(X, \mathbb{R}_d)$ is a group (and a ring).

Theorem 2.2. *For a space X, the following are equivalent:*

(1) $C(X, \mathbb{R}_d) = C(X, \mathbb{R}^d)$; that is, every density continuous function is locally constant.

(2) $C(X, \mathbb{R}_d)$ is a ring.

(3) $C(X, \mathbb{R}_d)$ is closed under multiplication.

(4) $C(X, \mathbb{R}_d)$ is a group.

(5) $Z(f)$ is open for each $f \in C(X, \mathbb{R}_d)$.

The proof of Theorem 2.2 will involve a construction in Proposition 2.5 of two density continuous functions whose sum is not density continuous. There was nothing unique about $i(x) = x$ in Example 2.1 or even the domain $\mathbb{R}_d$. Given any space X and any $f \in C(X, \mathbb{R}_d)$ whose zeroset is not open, we can construct a function $g \in C(X, \mathbb{R}_d)$ such that $f + g$ is not an element of $C(X, \mathbb{R}_d)$. We will need the next lemmas for the proof of Theorem 2.2.

Lemma 2.3. *For any topological space X, $C(X,\mathbb{R}^d)$ is a von Neumann regular ring; that is, for each $f \in C(X,\mathbb{R}^d)$ there exists $f_0 \in C(X,\mathbb{R}_d)$ such that $f^2 f_0 = f$.*

Lemma 2.4. *If O is an open neighborhood of zero in $\mathbb{R}_d$, then the set*

$$O' = ((-\infty,0]\cap O) \cup \bigcup_{n=1}^{\infty} \left(\left(\left(\frac{1}{n+1} + \frac{1}{10^n}, \frac{1}{n} \right) \cap O \right) - \frac{1}{10^n} \right)$$

is also an open neighborhood of zero in $\mathbb{R}_d$.

PROOF. The set $V = (-\infty,0] \cup \bigcup_{n=1}^{\infty} (\frac{1}{n+1} + \frac{1}{10^n}, \frac{1}{n})$ is a density open neighborhood of zero, and the density open set $V \cap O$ can be written as

$$V \cap O = ((-\infty,0]\cap O) \cup \bigcup_{n=1}^{\infty} \left(\left(\frac{1}{n+1} + \frac{1}{10^n}, \frac{1}{n} \right) \cap O \right).$$

Take $\epsilon > 0$, and choose a natural number N such that $\frac{1}{N+1} \le \epsilon \le \frac{1}{N}$. Let

$$A = \left(\frac{1}{N+1} + \frac{1}{10^N}, \frac{1}{N} \right).$$

Then

$$|O' \cap (-\epsilon,\epsilon)| = \left| O' \cap \left(-\epsilon, \frac{1}{N+1}\right) \right| + \left| \left((A \cap O) - \frac{1}{10^N} \right) \cap (0,\epsilon) \right|.$$

Lebesgue measure is translation invariant, so

$$\left| O' \cap \left(-\epsilon, \frac{1}{N+1}\right) \right| = \left| O \cap V \cap \left(-\epsilon, \frac{1}{N+1}\right) \right|$$

implies that

$$|O' \cap (-\epsilon,\epsilon)| = \left| O \cap V \cap \left(-\epsilon, \frac{1}{N+1}\right) \right| + \left| \left((A \cap O) - \frac{1}{10^N} \right) \cap (0,\epsilon) \right|.$$

Notice also that

$$\left| \left((A \cap O) - \frac{1}{10^N} \right) \cap (0,\epsilon) \right| \ge |A \cap O \cap (0,\epsilon)|,$$

and thus

$$\begin{aligned}|O'\cap(-\epsilon,\epsilon)| \geq& \left|O\cap V\cap\left(-\epsilon,\frac{1}{N+1}\right)\right| + |A\cap O\cap(0,\epsilon)| \\ =&|O\cap V\cap(-\epsilon,\epsilon)|\end{aligned}$$

From the previous inequality it follows that

$$1 \geq \frac{|O'\cap(-\epsilon,\epsilon)|}{2\epsilon} \geq \frac{|O\cap V\cap(-\epsilon,\epsilon)|}{2\epsilon}.$$

However, $\lim\limits_{\epsilon\to 0^+} \frac{|O\cap V\cap(-\epsilon,\epsilon)|}{2\epsilon} = 1$ since $O\cap V$ is a density open set. Hence $\lim\limits_{\epsilon\to 0^+} \frac{|O'\cap(-\epsilon,\epsilon)|}{2\epsilon} = 1$; that is, O' is an open subset of $\mathbb{R}_d$. □

Proposition 2.5. *Let $f \in C(X,\mathbb{R}_d)^+$ with $Z(f)$ not open. For each $n \geq 2$ let $g_n : \mathbb{R}_d \to \mathbb{R}_d$ be the linear mapping of $[\frac{1}{n} - \frac{1}{10^n}, \frac{1}{n}]$ onto $[\frac{1}{n}, \frac{1}{n} + \frac{1}{10^{n-1}}]$. Define a function $g : X \to \mathbb{R}_d$ by*

$$g(x) = \begin{cases} \frac{1}{10} + f(x) & \text{if } x \in f^{-1}([\frac{1}{2},\infty)) \\ \frac{1}{10^n} + f(x) & \text{if } x \in f^{-1}([\frac{1}{n+1}, \frac{1}{n} - \frac{1}{10^n}]) \text{ for some } n \geq 2 \\ (g_n\circ f)(x) & \text{if } x \in f^{-1}([\frac{1}{n} - \frac{1}{10^n}, \frac{1}{n}]) \text{ for some } n \geq 2 \\ 0 & \text{if } x \in Z(f). \end{cases}$$

Then g is a density continuous function. If $h : X \to \mathbb{R}_d$ is defined by

$$h(x) = \begin{cases} \frac{1}{10} & \text{if } x \in f^{-1}([\frac{1}{2},\infty)) \\ \frac{1}{10^n} & \text{if } x \in f^{-1}([\frac{1}{n+1}, \frac{1}{n} - \frac{1}{10^n}]) \text{ for some } n \geq 2 \\ h_n(x) & \text{if } x \in f^{-1}([\frac{1}{n} - \frac{1}{10^n}, \frac{1}{n}]) \text{ for some } n \geq 2 \\ 0 & \text{if } x \in Z(f). \end{cases}$$

where each h_n is continuous and h is well-defined, then h is not density continuous on $Z(f)\setminus \operatorname{int} Z(f)$. In particular, $g-f : X \to \mathbb{R}$ is not density continuous on $Z(f)\setminus \operatorname{int} Z(f)$.

PROOF. It is straightforward to verify that g is well-defined on X. Since the pieces of g are either linear or the composition of a linear and a density continuous function, g is density continuous on $X\setminus Z(f)$. Clearly g is constant on $\operatorname{int} Z(f)$; we need only show that g is density continuous on $Z(f)\setminus \operatorname{int} Z(f)$.

Let $a \in Z(f) \setminus \operatorname{int} Z(f)$ and let O be a density open neighborhood of zero. The set O' as defined in Lemma 2.4 is also a density open neighborhood of zero. Thus by continuity of f there exists an open neighborhood V of a such that $f(V) \subseteq O'$. To see that $g(V) \subseteq O$, take $x \in V$. If $x \in Z(f)$, then $g(x) = 0 \in O$ as needed. If $x \notin Z(f)$, then $f(x)$ being an element of O' means also that $f(x) \in ((\frac{1}{N+1} + \frac{1}{10^N}, \frac{1}{N}) \cap O) - \frac{1}{10^N}$ for some $N \geq 1$. It follows that

$$g(x) = f(x) + \frac{1}{10^N} \in \left(\frac{1}{N+1}, \frac{1}{N} - \frac{1}{10^N}\right) \cap O \subseteq O$$

and thus $g(V) \subseteq O$. Consequently, g is an element of $C(X, \mathbb{R}_d)$.

The function h is not density continuous on $Z(f) \setminus \operatorname{int} Z(f)$. Let U be the density open set $U = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n} - \frac{1}{10^n})$, let $C = h^{-1}(\{\frac{1}{10^n} : n \in \mathbb{N}\})$, and let $a \in Z(f) \setminus \operatorname{int} Z(f)$. If $X \setminus C$ is open, then the set

$$W = (X \setminus C) \cap f^{-1}(U) = (X \setminus C) \cap \left(Z(f) \cup \bigcup_{n=1}^{\infty} f^{-1}\left(\left(\frac{1}{n+1}, \frac{1}{n} - \frac{1}{10^n}\right)\right)\right)$$

is an open neighborhood of a. Moreover, we claim that $W \subseteq Z(f)$. If $x \in W$, then $x \in X \setminus C$ where C contains the set $\bigcup_{n=1}^{\infty} f^{-1}((\frac{1}{n+1}, \frac{1}{n} - \frac{1}{10^n}))$ which implies $x \notin \bigcup_{n=1}^{\infty} f^{-1}(\frac{1}{n+1}, \frac{1}{n} - \frac{1}{10^n})$ and so $x \in Z(f)$. But $a \notin \operatorname{int} Z(f)$ means $W \cap \operatorname{coz}(f)$ is nonempty, which is a contradiction. Consequently, $X \setminus C$ is not open in X; that is, $h^{-1}(\{\frac{1}{10^n} : n \in \mathbb{N}\})$ is not closed where $\{\frac{1}{10^n} : n \in \mathbb{N}\}$ is a closed subspace of the weak P-space $\mathbb{R}_d$. As a result, h is not density continuous on $Z(f) \setminus \operatorname{int} Z(f)$. □

PROOF OF THEOREM 2.2. Clearly (1) and (5) are equivalent and (2) implies (3). Suppose (1) holds. By Lemma 2.3, $C(X, \mathbb{R}^d)$ is a ring. It follows that $C(X, \mathbb{R}_d) = C(X, \mathbb{R}^d)$ is a ring, so (1) implies (2).

Assume $C(X, \mathbb{R}_d)$ is closed under multiplication. To show $C(X, \mathbb{R}_d)$ is a group, it suffices to show it is closed under addition. Recall from Theorem 1.1 that the exponential function $h(x) = e^x$ is density continuous. The functions $(h \circ f)(x) = e^{f(x)}$ and $(h \circ g)(x) = e^{g(x)}$ are compositions of density continuous functions and hence are density continuous. The product $e^{f(x)}e^{g(x)} = e^{f(x)+g(x)}$ is density continuous by assumption. The natural logarithm is a density continuous function on $(0, \infty)$ again by Theorem 1.1, so the function $\ln(e^{f(x)+g(x)}) = f(x) + g(x)$ is density continuous; i.e. $C(X, \mathbb{R}_d)$ is closed under addition.

Now suppose $C(X,\mathbb{R}_d) \neq C(X,\mathbb{R}^d)$; that is, there is a function f in $C(X,\mathbb{R}_d)$ with $Z(f)$ not open. Since $Z(f) = Z(|f|)$ and $|x|$ is density continuous, without loss of generality we may assume $f \geq 0$. Let $g : X \to \mathbb{R}_d$ be the density continuous function defined in Proposition 2.5. By the same proposition, $g - f$ is the sum of two density continuous functions which is not density continuous on $Z(f) \setminus \operatorname{int} Z(f)$. We conclude that $C(X,\mathbb{R}_d)$ is not a group, and so (4) implies (1). □

Definition 2.6. We will call a space that satisfies the conditions of Theorem 2.2 a *density P-space*. Observe that a space X is a density P-space if and only if every point of X has an open neighborhood which is a density P-space. Let us compare the definition of a density P-space to that of a P-space. A point $p \in X$ is called a P-point if $Z(f)$ contains an open neighborhood of p for all $f \in C(X)$. A space X is a P-space if each point of X is a P-point. In fact, X is a P-space if and only if $Z(f)$ is open for every $f \in C(X)$ if and only if $C(X)$ is a von Neumann regular ring. It should be obvious that every P-space is a density P-space, but the converse fails. The compact interval $[0,1]$ is a density P-space, as we will soon see, but it is not a P-space. Examples of spaces which are not density P-spaces include $\mathbb{R}_d$ and the following example.

Example 2.7. Let τ be any nondiscrete topology on $\mathbb{R}$ which is finer than the density topology, and let $\mathbb{R}_\tau$ denote the real numbers with the topology τ. Since τ is not discrete, there is a non-isolated point $r \in \mathbb{R}_\tau$. The space $\mathbb{R}_\tau$ is not a density P-space because the function $i - \mathbf{r} : \mathbb{R}_\tau \to \mathbb{R}_d$ is density continuous but its zeroset $\{z\}$ is not open.

3 Examples of Density P-Spaces.

In this section we explore which topological spaces are density P-spaces. More information on pseudocompact spaces, totally ordered spaces, and other spaces mentioned below can be found in [4].

Definition 3.1. Recall that a space X is *pseudocompact* if for each $f \in C(X)$ there exists a real number M such that $|f(x)| \leq M$ for all $x \in X$. In other words, every real-valued continuous function on X is bounded. An open cover $\mathcal{O}$ of X is *locally finite* if for each $x \in X$ there is an open set O containing x which intersects only finitely many elements of $\mathcal{O}$. Since we are assuming X is Tychonoff, pseudocompactness is equivalent to the condition that every locally finite open cover of X is finite.

Theorem 3.2. *If X is a pseudocompact space, then X is a density P-space. Furthermore, if $f \in C(X, \mathbb{R}_d)$, then $f(X)$ is a finite subset of $\mathbb{R}_d$.*

PROOF. Suppose X is not a density P-space. We will construct a locally finite open cover which is infinite. Select a nonnegative density continuous function f whose zeroset is not open. Because $Z(f)$ is not open, we can find a sequence of distinct positive real numbers $A = \{r_n\}_{n=1}^{\infty} \subseteq f(X)$ which decrease to zero. Next define sets $O_1 = \mathbb{R}_d \setminus A$ and $O_i = (r_i - \frac{r_i - r_{i+1}}{10^i}, r_i + \frac{r_{i-1} - r_i}{10^i})$ for each natural number $i \geq 2$. The set O_1 is cocountable, and each O_i is open in the usual topology for $i \geq 2$, thus each O_i is open in the density topology. It follows that the family $\mathcal{F} = \{f^{-1}(O_i)\}_{i=1}^{\infty}$ is a family of nonempty open sets which covers X.

Next we will show that this family is locally finite. To see this, take $y \in X$. First consider the case when $f(y) = 0$. The set

$$U = (-\infty, 0] \cup \bigcup_{i=2}^{\infty} \left(r_i + \frac{r_{i-1} - r_i}{10^i}, r_{i-1} - \frac{r_{i-1} - r_i}{10^{i-1}} \right)$$

is a density open neighborhood of zero, so the set $f^{-1}(U)$ is an open neighborhood of y in X. Furthermore, $f^{-1}(U) \cap O_i$ is empty except when $i = 1$, so U is an open neighborhood of y such that U intersects exactly one member of $\mathcal{F}$. Next consider the case where $f(y) \neq 0$. If $y \in f^{-1}([r_1, \infty))$, then y can only be an element of O_1. If $y \in f^{-1}((r_{N+2}, r_N))$ for some $N \in \mathbb{N}$, then $f^{-1}((r_{N+1}, r_N))$ only meets O_i for $i \leq N + 3$. Hence $\mathcal{F}$ is a locally finite open cover of X. But $\mathcal{F}$ is not a finite family. Indeed, if a point $x \in f^{-1}(\{r_n\})$ for some n, then $x \in f^{-1}(O_{N+1})$ but $x \notin f^{-1}(O_i)$ for every $i \neq n + 1$. The existence of a locally finite open cover of X which is not finite implies that X is not pseudocompact.

Now suppose $f \in C(X, \mathbb{R}_d) = C(X, \mathbb{R}^d)$ with $f(X)$ infinite, say $\{r_n\}_{n \in \mathbb{N}} \subseteq f(X)$. Then for each n, we have $f^{-1}(\{r_n\})$ is a clopen subset of X. Define $g : X \to \mathbb{R}$ by

$$g(x) = \begin{cases} n & \text{if } x \in f^{-1}(\{r_n\}) \\ 0 & \text{otherwise,} \end{cases}$$

then g is an unbounded element of $C(X)$, which is a contradiction. Therefore $f(X)$ is finite. □

Definition 3.3. A space X is *locally compact* if for each $x \in X$ there is a compact subspace K of X containing x. A space X is called *sequentially compact* if every sequence of points of X has a convergent subsequence, and it is said to be *locally sequentially compact* if, for every $x \in X$ and any open neighborhood O of x, there exists an open neighborhood U of x such that $\operatorname{cl} U \subseteq O$ and $\operatorname{cl} U$ is sequentially compact.

Corollary 3.4. *Locally compact and locally sequentially compact spaces are density P-spaces.*

Proof. Compact and sequentially compact spaces are pseudocompact. So if each point of the space has an open neighborhood which is pseudocompact and hence a density P-space, then the entire space is a density P-space. □

Definition 3.5. A set X is called *totally ordered* if there exists a partial order $\leq$ on X such that for any two elements $x, y \in X$, either $x \leq y$ or $y \leq x$. A totally ordered set can be equipped with the interval (or order) topology in which open intervals of the form $(a, b) = \{x \in X : a < x < b\}$ form a base for the topology, where we allow for $a = -\infty$ and $b = \infty$. Such a space is called a *totally ordered space.*

Proposition 3.6. *If X is a totally ordered space, then X is a density P-space.*

Proof. Let $f \in C(X, \mathbb{R}_d)$ and fix a point $z \in X$. If z is not the limit of an increasing or decreasing sequence, then by 5.O.1 in [5] we know that z is a P-point; i.e., every element g of $C(X)$ is constant on an open neighborhood of z. Hence f is constant on an open neighborhood of z.

Now suppose z is the limit of an increasing or decreasing sequence; without loss of generality, say z is the limit of the increasing sequence $\{x_n\}_{n=1}^{\infty}$. Suppose, by means of contradiction, that f is not constant on any neighborhood of z. Let $y_1 = x_1$, then inductively define a sequence $\{y_n\}_{n=1}^{\infty}$ such that $x_n < y_n < z$ and $f(y_n) \neq f(z)$ for all $n \in \mathbb{N}$. Notice that the sequence $\{y_n\}$ converges to z because $\{x_n\}$ converges to z and $y_n > x_n$ for all n. The set $A = \{f(y_n) : n \in \mathbb{N}\}$ is a countable subset of the weak P-space $\mathbb{R}_d$ and thus closed in $\mathbb{R}_d$. But then $f^{-1}(\mathbb{R}_d \setminus A)$ is an open neighborhood of z disjoint from the sequence $\{y_n\}$ which converges to z. This is a contradiction. As a result, f is constant on a neighborhood of z, and we conclude that X is a density P-space. □

Proposition 3.7. *Let X be a separable space; that is, a space which contains a countable, dense subset. Then X is a density P-space. Moreover, $|f(X)| = \aleph_0$ for all $f \in C(X, \mathbb{R}_d)$.*

Proof. Let $f \in C(X, \mathbb{R}_d)$, and let A be a countable dense subset of X. The set $f(A)$ is a countable subset of the weak P-space $\mathbb{R}_d$ and thus closed in $\mathbb{R}_d$. Continuity of f yields that $f^{-1}(f(A))$ is a closed subset of X containing A, and since A is dense we must have $f^{-1}(f(A)) = X$; that is, $f(X) = f(A)$. Since $f(A)$ is a countable subset of $\mathbb{R}_d$, it is a discrete subspace. It follows that $f^{-1}(\{r\})$ is a clopen subset of X for all $r \in f(X)$; i.e., f is a locally constant function. □

Definition 3.8. A space X is said to be *locally connected* if for each $x \in X$ and each open set O containing x there is a connected set $C \subseteq O$ containing x.

Proposition 3.9. *Any locally connected space is a density P-space.*

Proof. First suppose X is a locally connected topological space, $x \in X$, and $f \in C(X, \mathbb{R}_d)$. Note that $O = (\mathbb{R}_d \setminus \mathbb{Q}) \cup \{f(x)\}$ is an open neighborhood of $f(x)$ in $\mathbb{R}_d$. By continuity of f there exists an open neighborhood U of x so that $f(U) \subseteq O$. Moreover, using local connectedness of X we can choose U to be connected, and thus $f(U)$ is connected. However, we claim that no subspace of O containing $f(x)$ is connected except $\{f(x)\}$. This is true because a connected subspace of $\mathbb{R}_d$ is necessarily connected in $\mathbb{R}$, and no subspace of O as a subspace of $\mathbb{R}$ is connected except for singleton sets. Hence $f(U) = \{f(x)\}$; i.e., f is constant on U. □

Remark 3.10. It is not possible to generalize the above argument to connected spaces. The space $\mathbb{R}_d$ is connected but not a density P-space, so it is not locally connected.

Definition 3.11. Given a point $x \in X$, we say X is *countably tight at x* if, for any subset A of X, $x \in \operatorname{cl} A$ implies $x \in \operatorname{cl} C$ for some countable set $C \subseteq A$. If X is countably tight at all points of X, then X is said to be a *countably tight* space.

Proposition 3.12. *Let x be a countably tight point in a topological space X and let $f \in C(X, \mathbb{R}_d)$, then there is a neighborhood of x on which f is constant. In particular, a countably tight space is a density P-space.*

PROOF. Let $A = f^{-1}(\mathbb{R}_d \setminus \{f(x)\})$ and note that A is open in X by continuity of f. Assume, to get a contradiction, that $x \in \operatorname{cl} A$. Then because x is a countably tight point, there exists a countable subset C of A such that $x \in \operatorname{cl} C$. The set $f(C)$ is a countable subspace of the weak P-space $\mathbb{R}_d$ and hence closed. So $f^{-1}(\mathbb{R}_d \setminus f(C))$ is an open neighborhood of x disjoint from C, which is a contradiction. Hence $x \notin \operatorname{cl} A$, and $X \setminus \operatorname{cl} A$ is an open neighborhood of x on which the function f is constant. □

Corollary 3.13. *First countable spaces are density P-spaces.*

Remark 3.14. We now have that each of the following types of spaces are density P-spaces:

(1) pseudocompact

(2) totally ordered

(3) separable

(4) locally connected

(5) locally compact

(6) locally sequentially compact

(7) P-space

(8) countably tight

A topological sum is a density P-space if and only if each summand is a density P-space. Examples of non-density P-spaces include the spaces mentioned in Example 2.7 as well as a topological sum of a non-density P-space with any other space. The product of an arbitrary space with a non-density P-space is a non-density P-space. We do not know if the product of density P-spaces is again a density P-space.

Recall that every subspace of a P-space is again a P-space. This does not hold for density P-spaces. One example is $\mathbb{R}_d$ as a subspace of its Stone-Čech compactification $\beta\mathbb{R}_d$. We have already seen that $\mathbb{R}_d$ is not a density P-space, but $\beta\mathbb{R}_d$ is a density P-space because it is compact. However, open subspaces of density P-spaces are density P-spaces in certain situations. To explore this, we need the following definition.

Definition 3.15. A space X is called *density Tychonoff* if for each closed set $A \subset X$ and for each point $x \in X$ not in A there exists $f \in C(X, \mathbb{R}_d)$ such that $f(x) = 0$ and $f(A) = 1$. In this case, the cozerosets of density continuous functions form a base for the topology on X.

Zero-dimensional spaces; that is, spaces in which clopen sets form a base for the topology, are common examples of density Tychonoff spaces. When X is a density P-space it is easy to see that density Tychonoff and zero-dimensional are equivalent. We do not, at this time, have an example of a density Tychonoff space which is not zero-dimensional.

Proposition 3.16. *Suppose X is a density Tychonoff, density P-space. Then every open subspace of X is a density P-space.*

PROOF. Let O be an open subset of X, $f \in C(O, \mathbb{R}_d)$, and $x \in O$. Since X is density Tychonoff, there exists $g \in C(X, \mathbb{R}_d)$ such that $\mathrm{coz}(g) = C$ contains x and $C \subseteq O$. This means C is a clopen subset of X and hence also a density P-space. Now let $\overline{f} = f|_C$, then $\overline{f} \in C(C, \mathbb{R}_d)$ means $\overline{f}$ is constant on some open neighborhood of x in C. It follows that f is constant on some open neighborhood of x in O. $\square$

Question 3.17. Can we drop the assumption that X be a density Tychonoff space in Proposition 3.16?

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