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# DIMENSION OF FAT SERPIŃSKI GASKETS

**Abstract**

In this paper we continue the work started by Broomhead, Montaldi and Sidorov investigating the Hausdorff dimension of fat Sierpiński gaskets. We obtain generic results where the contraction rate $\lambda$ is in a certain region.

## 1 Introduction.

Let $F = \{S_1, \ldots, S_k\}$ be a family of contractions on $\mathbb{R}^d$. It was shown in [6] that there exists a unique non-empty compact set $\Lambda(F)$, called the attractor of $F$, such that,

$$\Lambda(F) = \cup_{i=1}^{k} S_i(\Lambda(F)).$$

In the case where the contractions are similarities and a technical condition called the open set condition (OSC) is satisfied it is a straightforward to calculate the Hausdorff dimension of $\Lambda(F)$ (see [4]). Not satisfying the OSC essentially means that the images $s_i(\Lambda(F))$ overlap in a non trivial manner. In this case calculating the Hausdorff dimension of the attractor of the IFS becomes a much more difficult question. Two approaches have been used to deal with this problem. One is to consider exceptional cases where the overlap is regular ([2], [3], [8] and [10]) and the other is to try and obtain results for generic parameter values ([11], [12], [14] and [15]). We adopt the second approach and study a specific case in $\mathbb{R}^2$.

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The fat Sierpiński gasket was introduced by Simon and Solomyak in [13]. It is defined to be the attractor, $\Lambda(\lambda) \subset \mathbb{R}^2$ of the IFS, $F = \{T_0, T_1, T_2\}$ where,

$$\begin{aligned} T_0(x) &= \lambda x \\ T_1(x) &= \lambda x + (1,0) \\ T_2(x) &= \lambda x + \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \end{aligned}$$

for $\lambda > \frac{1}{2}$. In proposition 3.3 of [13] they show that there exists a dense subset, $A \subset [\frac{1}{2}, \frac{1}{\sqrt{3}}]$, such that for all $\lambda \in A$, $\dim_{\mathcal{H}} \Lambda(\lambda) < -\frac{\log 3}{\log \lambda}$.

A systematic investigation of the Hausdorff dimension of $\Lambda(\lambda)$ was started by Broomhead, Montaldi and Sidorov in [2]. They were able to compute the exact Hausdorff dimension of $\Lambda(\lambda)$ when $\lambda$ is in a special class of algebraic numbers they call the multinacci numbers. These are the positive solutions, $\omega_n$, to the equations $\sum_{k=1}^{n} \lambda^k = 1$. In particular $\omega_2$ is equal to the reciprocal of the golden ratio. They obtain the following result.

**Theorem 1** (Broomhead, Montaldi, Sidorov)**.**

$$\dim_{\mathcal{H}}(\Lambda(\omega_n)) = \frac{\log \tau_n}{\log \omega_n},$$

*where $\tau_m$ is the smallest positive root of the polynomial $3z^{n+1} - 3z + 1$.*

It should be noted that $\frac{\log \tau_n}{\log \omega_n} < -\frac{\log 3}{\log \omega_n}$.

In this paper we continue the investigation into the Hausdorff dimension of $\Lambda(\lambda)$. The following is our main result.

**Theorem 2.** *1. For almost all $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3} \approx 0.529]$,*

$$\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}.$$

*2. For almost all $\lambda \geq 0.5853$,*

$$\dim_{\mathcal{H}} \Lambda(\lambda) = 2.$$

Our methods only enable us to show that $\dim_{\mathcal{H}} \Lambda(\lambda) = 2$ for almost all $\lambda \leq 0.649$. However it is clear that for all $\lambda \geq \frac{2}{3}$, $\dim_{\mathcal{H}} \Lambda(\lambda) = 2$ and in [2] it is shown that for all $\lambda \geq 0.648$, $\Lambda(\lambda)$ has non-empty interior and hence Hausdorff dimension 2. It should be noted that the results in [13] and [2] mean

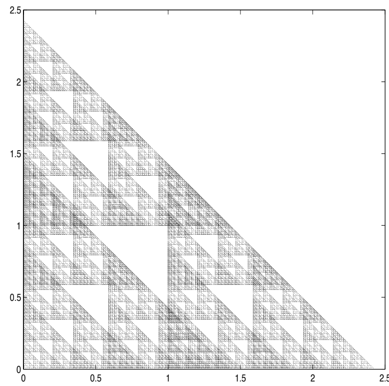


Figure 1: $\Lambda(\lambda)$ for $\lambda = 0.59$. Theorem 2 states that for almost all $\lambda > 0.5853$, $\dim_{\mathcal{H}} \Lambda(\lambda) = 2$.

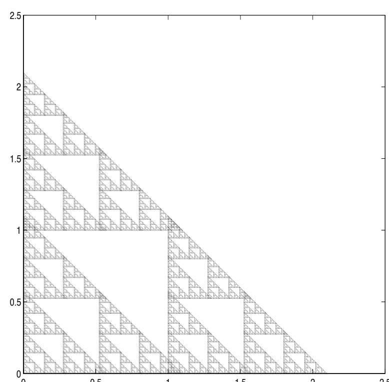


Figure 2: $\Lambda(\lambda)$ for $\lambda = 0.521$. Theorem 2 shows that for almost all $\lambda \in [0.5, 0.529]$ $\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}$.

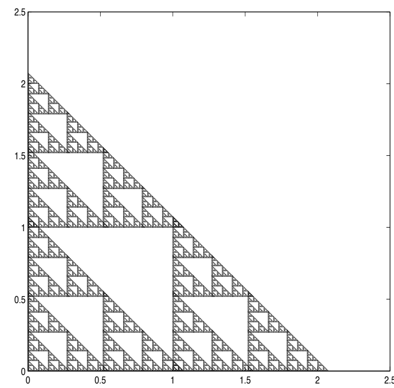


Figure 3: $\Lambda(\lambda)$ for $\lambda = \omega_4 \approx 0.519$. It is shown in [2] that $\dim_{\mathcal{H}} \Lambda(\lambda) = \frac{\log \tau_4}{\log \omega_4} \approx 1.654 < -\frac{\log 3}{\log \omega_4}$. Theorem 2 shows that this is an exceptional value.

that the equality in Theorem 2 certainly does not hold for all $\lambda$. It would be interesting to know whether the region of $\lambda$ for which Theorem 2 is true can be extended to a larger region. However the method used in this paper only provides almost sure lower bounds for $\lambda \in (0.529, 0.5853]$ which are strictly less than $-\frac{\log 3}{\log \lambda}$. Theorem 2 has the following topological analogue.

**Corollary 1.** *1. There exists a residual set $A \subset [\frac{1}{2}, \frac{\sqrt[3]{4}}{2}]$ such that for any $\lambda \in A$,*

$$\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}.$$

*2. There exists a residual set $B \subset [0.5853, 1]$ such that for any $\lambda \in B$*

$$\dim_{\mathcal{H}} \Lambda(\lambda) = 2.$$

Hence the results found in [2] and [13] in the above region were exceptional cases both in a topological and measure theoretic sense.

It is notationally more convenient to look at a slightly different IFS. This is defined by the similarities,

$$T_0(x) = \lambda x$$
$$T_1(x) = \lambda x + (1, 0)$$
$$T_2(x) = \lambda x + (0, 1).$$

However the attractor of this IFS can be obtained by an affine transformation applied to the set $\Lambda(\lambda)$ and hence has the same Hausdorff dimension. There has been a lot of study of overlapping IFS's in one dimension ([15], [12], [11], [14]). Most of this work has used the idea of transversality introduced in [12] to obtain generic results. Typically these results compute the Hausdorff dimension of the attractor for a set of full measure. Various work has been done on lower semi-continuity of the dimension overlapping IFS. This includes unpublished work by Pollicott and Simon-Solomyak as well as the published work by Jonker and Veerman [7]. Using this work it is often possible to compute the Hausdorff dimension for a residual set (a subset which contains a dense countable intersection of open sets). We examine cross sections to enable us to use the method of transversality which has been so effective in the one-dimensional setting.

## 2 Definitions and Technical Lemmas.

For a set $F \subseteq \mathbb{R}^n$ the $s$-dimensional Hausdorff dimension is defined by

$$H^s(F) = \lim_{\epsilon \to 0} \inf \left\{ \sum |u_i|^s \mid \{u_i\}_i \text{ is a finite or countable } \epsilon\text{-cover of } F \right\}.$$

The Hausdorff dimension of $F$ is then defined as,

$$\dim_{\mathcal{H}} F = \inf\{s : H^s(F) = 0\} = \sup\{s : H^s(F) = \infty\}.$$

For a probability measure $\mu$ on $\mathbb{R}^n$ the Hausdorff dimension is defined by,

$$\dim_{\mathcal{H}} \mu = \inf\{\dim_{\mathcal{H}} F : \mu(F) = 1 \text{ and } F \text{ is a Borel set}\}.$$

The mass distribution principle can be used to show the following equality concenting the dimension of a measure.

$$\dim_{\mathcal{H}} \mu = \text{ess-sup} \left\{ \frac{\log \mu(B(x,r))}{\log r} : x \in \mathbb{R}^n \right\}. \tag{1}$$

Here ess-sup means the essential supremum.

We now prove a slight variation of the potential theoretic method for calculating lower bounds of Hausdorff dimension, [4]. For more details and links to generalized dimension see [5].

**Lemma 1.** *Let $A \subseteq \mathbb{R}$ be a Borel set and $\alpha, s \in (0,1]$ . If there exists a measure $\mu$ on $A$ such that,*

$$\int \left( \int \frac{d\mu(x)}{|x-y|^s} \right)^\alpha d\mu(y) < \infty, \tag{2}$$

*then $\dim_{\mathcal{H}} A \geq s$ and $\dim_{\mathcal{H}} \mu \geq s$.*

Proof. Let $\phi_\mu(y) = \left(\int \frac{\mathrm{d}\mu(x)}{|x-y|^s}\right)$. If the inequality (2) holds for a measure $\mu$ on a set $A$, then it follows that $(\phi_\mu(y))^\alpha$ is integrable with respect to $\mu$. This means that there exists $M$ such that,

$$A_M = \{y : (\phi_\mu(y))^\alpha \leq M\}$$

satisfies $\mu(A_M) > 0$. Thus we can define a measure $\nu$ simply by the restriction of $\mu$ to $A_M$. Hence for any $x \in A$,

$$M^{\frac{1}{\alpha}} \geq \int_A \frac{\mathrm{d}\nu(x)}{|x-y|^s} \geq \int_{B(x,r)} \frac{\mathrm{d}\nu(y)}{|x-y|^s} \geq \frac{1}{r^s}\nu(B(x,r)).$$

Thus for any $x \in A$, $\nu(B(x,r)) \leq M^{\frac{1}{\alpha}} r^s$ and by the mass distribution principle $\dim_{\mathcal{H}} A \geq s$ and $\dim_{\mathcal{H}} \mu \geq s$. □

We also need a lemma which relies on the idea of transversality of a power series. This idea was first used in [12] and has since been the main tool in investigating IFS with overlaps. A power series $g$ is said to satisfy the $\epsilon$-transversality condition if $g$ crosses any line within $\epsilon$ of the origin with slope at most $-\epsilon$. Consequences of transversality include the absolute continuity of Bernoulli convolutions ([15], [11]) and almost sure results for the dimension of several fractal families ([12], [14]). Consider a power series of the form,

$$g(x) = 1 + \sum_{k=1}^{\infty} g_k x^k, \text{ with } g_k \in \{-1, 0, 1\}. \tag{3}$$

Let

$$b(1) = \inf\{\lambda > 0 : \exists g(x) \text{ of the form (3) such that } g(\lambda) = g'(\lambda) = 0\}.$$

Thus for any $0 < a < c < b(1)$ and any $g$ of the form (3) there exists $\epsilon > 0$ such that for any $\lambda$ where $|g(\lambda)| < \epsilon$, $|g'(\lambda)| \geq \epsilon$. Thus any power series of the form (3) where $\lambda$ takes values less than $c$ for some $c < b(1)$ satisfies $\epsilon$-transversality for some $\epsilon$. Peres and Solomyak have computed values for $b(1)$ (Lemma 5.2 in [11]). They obtain $b(1) \approx 0.649$. This allows us to prove the following Lemma which is almost identical to Lemma 2 in [12].

**Lemma 2.** *For any interval $I = [a, c]$ where $0 < a < c < b(1)$, $s < 1$ and any $\{a_k\}_{k\in\mathbb{N}}$ where $a_0 \neq 0$ and $a_k \in \{0, \pm 1\}$ there exists $K(s)$ such that,*

$$\int_I \frac{d\lambda}{|a_0 + \sum_{n=1}^{\infty} a_n \lambda^n|^s} \leq K(s).$$

PROOF. From above we know there exists $\epsilon > 0$ such that if $|g(\lambda)| \leq \epsilon$, then $|g'(\lambda)| \geq \epsilon$ for any $\lambda \in [a, c]$. This allows exactly the same method of proof as used to proof Lemma 2 in [12]. □

The tool which allows us to use these one-dimensional methods to obtain a result about a subset of $\mathbb{R}^2$ is a generalization of the Marstrand slicing theorem, [9]. It also appears in [4] as Corollary 7.12 and it is stated and proved as Theorem 4.1 in Chapter 3 of [1].

**Lemma 3.** *Let $F$ be any subset of $\mathbb{R}^2$, and let $E$ be a subset of the $y$-axis. Let $L_y = \{(x, z) \in \mathbb{R}^2 : z = y\}$. If $\dim_{\mathcal{H}}(F \cap L_x) \geq t$ for all $y \in E$, then $\dim_{\mathcal{H}} F \geq t + \dim_{\mathcal{H}} E$.*

## 3 Biased Bernoulli Convolutions.

Let $\lambda \in [0.5, 0.649\ldots]$ and $\underline{p} = (p_0, p_1)$ be a probability vector. We let

$$T_0(x) = \lambda x$$
$$T_1(x) = \lambda x + 1.$$

Let $\nu = \nu_\lambda^{p_0,p_1}$ be the self-similar measure such that for all $J \subset \left[0, \frac{1}{1-\lambda}\right]$,

$$\nu(J) = p_0 \nu(T_0^{-1}(J)) + p_1 \nu(T_1^{-1}(J)).$$

We will also let $\mu = \mu_{p_0,p_1}$ be $(p_0, p_1)$-Bernoulli measure defined on the sequence space, $\{0, 1\}^{\mathbb{N}_0}$ where $\mathbb{N}_0$ denotes the non-negative integers. We let $\Pi_\lambda : \{0, 1\}^{\mathbb{N}_0} \to \mathbb{R}$ be defined by $\Pi_\lambda(\underline{i}) = \sum_{n=0}^{\infty} i_n \lambda^n$. This gives $\nu_\lambda^{(p_0,p_1)} = \mu^{(p_0,p_1)} \circ \Pi_\lambda^{-1}$. We will also use the notation $|\underline{i} \wedge \underline{j}| = \min\{k : i_k \neq j_k\}$, $W_{\omega,k} = \{\tau \in \Omega : \tau_j = \omega_j : j \leq k - 1\}$, $W_k$ consists of all $k$th level cylinders,

$$[i_0, i_1, \ldots, i_{k-1}] = \{\underline{j} : i_r = j_r \text{ for } 0 \leq r \leq k - 1\}$$

and $k_r(\underline{i}) = \text{card}\{0 \leq j \leq k - 1 : x_j = r\}$.

**Proposition 1.** *Fix $(p_0, p_1)$. For almost all $\lambda \in [0.5, 0.649\ldots]$,*

$$\dim_{\mathcal{H}} \nu_\lambda^{(p_0,p_1)} = \min\left(\frac{p_0 \log p_0 + p_1 \log p_1}{\log \lambda}, 1\right).$$

This result could be deduced as a Corollary to Theorem 7.2 in [14]. However in the present simpler setting it is possible to construct a more elementary proof which is based on methods used in [11].

**Proof of Proposition 1.**

The proof of the upper bound is standard. Note that by the strong law of large numbers,

$$\lim_{n\to\infty}\frac{1}{n}\log\mu([i_0,\ldots,i_{n-1}])=p_0\log p_0+p_1\log p_1 \text{ for } \mu\text{-almost all } \underline{i}.$$

Thus for all $\epsilon>0$ there exists $N$ such that for all $n\geq N$

$$\nu_\lambda(B(\Pi_\lambda\underline{i},\lambda^n))\geq n(p_0\log p_0+p_1\log p_1-\epsilon)$$

for $\mu$ almost every $\underline{i}$. However because $\nu_\lambda=\mu\circ\Pi_\lambda^{-1}$

$$\frac{\log(\nu_\lambda(B(x,\lambda^n)))}{\log\lambda^n}\leq\frac{p_0\log p_0+p_1\log p_1-\epsilon}{\log\lambda}$$

for $\nu_\lambda$ almost all $x$. Hence by (1) the proof of the upper bound is complete.

For the lower bound the following lemma is needed. It involves the use of an exponent $\alpha\in(0,1]$. The idea to use this exponent came from [11].

**Lemma 4.** *Fix $(p_0,p_1)$. For all $\alpha\in(0,1]$ we have that for almost all $\lambda\in[0.5,b(1)]$*

$$\dim_{\mathcal{H}}\nu_\lambda^{(p_0,p_1)}\geq\min\left(\frac{\log((p_0^{\alpha+1}+p_1^{\alpha+1})^{\frac{1}{\alpha}})}{\log\lambda},1\right).$$

Proof. Fix $(p_0,p_1)$ and let $\epsilon>0$. For simplicity let $d(\alpha,\epsilon)=(p_0^{\alpha+1}+p_1^{\alpha+1}+\epsilon)^{\frac{1}{\alpha}}$. We let $S_\epsilon(\lambda)=\min\left(\frac{\log(d(\alpha,\epsilon))}{\log\lambda},1-\epsilon\right)$. We use Lemma 1 together with Fubini's theorem and Lemma 2.

$$I=\int_{0.5}^{b(1)}\int\left(\int\frac{\mathrm{d}\nu_\lambda(x)}{|x-y|^{S_\epsilon(\lambda)}}\right)^\alpha\mathrm{d}\nu_\lambda(y)\mathrm{d}\lambda=\int_{0.5}^{b(1)}\int\left(\int\frac{\mathrm{d}\mu(\underline{i})}{|\Pi_\lambda(\underline{i})-\Pi_\lambda(\underline{j})|^{S_\epsilon(y)}}\right)^\alpha\mathrm{d}\mu(\underline{j})\mathrm{d}\lambda$$

Apply Fubini's theorem and Hölder's inequality ($\int f^\alpha\leq C(\int f)^\alpha$ for $\alpha\in(0,1]$.) to get

$$\begin{aligned}I&\leq C\int\left(\int_{0.5}^{b(1)}\int\frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{|\Pi_\lambda(\underline{i})-\Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda)}}\right)^\alpha\mathrm{d}\mu(\underline{j})\\&\leq C_1\int\left(\int_{0.5}^{b(1)}\int\frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{|\sum_{n=0}^\infty(i_n-j_n)\lambda^n|^{s_\epsilon(\lambda)}}\right)^\alpha\mathrm{d}\mu(\underline{j})\\&\leq C_1\int\left(\int_{0.5}^{b(1)}\int\frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{\left(\lambda^{|\underline{i}\wedge\underline{j}|}\left|a_0+\sum_{n=1}^\infty a_n\lambda^n\right)\right|^{s_\epsilon(\lambda)}}\right)^\alpha\mathrm{d}\mu(\underline{j})\end{aligned}$$

where $a_n \in \{-1,0,1\}$ for $n \geq 1$ and $a_0 \in \{-1,1\}$. We now use Lemma 2 to continue

$$
\begin{aligned}
I &\leq C_1 \int \left( \int_{0.5}^{b(1)} \int \frac{\mathrm{d}\mu(\underline{i})\mathrm{d}\lambda}{\left(d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}\,|a_0 + \sum_{n=1}^{\infty} a_n\lambda^n|\right)^{s_\epsilon(\lambda)}} \right)^{\alpha} \mathrm{d}\mu(\underline{j}) \\
&\leq C_1 \int \left( \int_{0.5}^{b(1)} \frac{\mathrm{d}\lambda}{|a_0 + \sum_{n=1}^{\infty} a_n\lambda^n|^{s_\epsilon(\lambda)}} \int \frac{\mathrm{d}\mu(\underline{i})}{d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}} \right)^{\alpha} \mathrm{d}\mu(\underline{j}) \\
&\leq C_2 \int \left( \int \frac{\mathrm{d}\mu(\underline{i})}{d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}} \right)^{\alpha} \mathrm{d}\mu(\underline{j}) \leq C_2 \int \left( \sum_{k=0}^{\infty} \frac{\mu(W_{\omega,k})}{d(\alpha,\epsilon)^k} \right)^{\alpha} \mathrm{d}\mu(\omega)
\end{aligned}
$$

We proceed by using the inequality $\left(\sum_i b_i\right)^{\alpha} \leq \sum_i b_i^{\alpha}$ for $b_i > 0$ and $\alpha \in (0,1]$

$$
I \leq C_2 \sum_{k=0}^{\infty} \sum_{w \in W_k} \frac{\mu(W)^{\alpha+1}}{d(\alpha,\epsilon)^{\alpha k}} \leq C_2 \sum_{k=0}^{\infty} d(\alpha,\epsilon)^{-\alpha k} (p_0^{\alpha+1} + p_1^{\alpha+1})^k.
$$

Thus because $d(\alpha,\epsilon)^{\alpha} > p_0^{\alpha+1} + p_1^{\alpha+1}$ we have $I < \infty$. Hence by Lemma 1 $\dim_{\mathcal{H}} \nu_\lambda \geq \min\left(\frac{d(\alpha,\epsilon)}{\log\lambda}, 1-\epsilon\right)$ for almost all $\lambda$. To complete the proof we let $\epsilon = \frac{1}{n}$ for $n \in \mathbb{N}$ and let $n \to \infty$. □

To complete the proof of Proposition 1 we let $\alpha_n = \frac{1}{n}$ for $n \in \mathbb{N}$ and observe that

$$
\lim_{n\to\infty} \frac{\log(p_0^{\alpha_n+1} + p_1^{\alpha_n+1})}{\alpha_n \log\lambda} = \frac{p_0\log p_0 + p_1 \log p_1}{\log\lambda}.
$$

## 4 Cross Sections of Fat Gaskets.

Consider a sequence $\{i_n\} \in \{0,1,2\}^{\mathbb{N}_0}$ we can then represent each point in $\Lambda(\lambda)$ using the expansion $\sum_{n=0}^{\infty} a_{i_n}\lambda^n$, where $a_0 = (0,0)$, $a_1 = (1,0)$ and $a_2 = (0,1)$. It should be noted that for $\lambda > \frac{1}{2}$ this expansion is not unique. Consider a sequence $\underline{x} \in \{0,1\}^{\mathbb{N}_0}$. Intuitively we think of the case when $x_n = 0$ as corresponding to the bottom two triangles in the gasket and $x_n$=1 corresponding to the top triangle. We then define a complementary sequence $\underline{j} \in \{0,1\}^{\mathbb{N}_0}$ such that $j_n = 0$ whenever $x_n = 1$. The idea of this sequence is to determine a horizontal point on the gasket corresponding to the sequence $\underline{x}$. Thus whenever $x_n = 0$ there are two choices either 0 or 1 corresponding to the bottom two triangles in the gasket. However when $x_n = 1$ there is just the one choice and $j_n$ must equal 0. This means if we define $\underline{i} \in \{0,1,2\}^{\mathbb{N}_0}$ such that

$$i_n = \begin{cases} 0 \text{ if } x_n = 0, j_n = 0 \\ 1 \text{ if } x_n = 0, j_n = 1 \\ 2 \text{ if } x_n = 1, j_n = 0 \end{cases},$$

then

$$\left(\sum_{n=0}^{\infty} j_n \lambda^n, \sum_{n=0}^{\infty} x_n \lambda^n\right) = \left(\sum_{n=0}^{\infty} a_{i_n} \lambda^n\right) \in \Lambda(\lambda).$$

Thus if we let

$$L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda)) = \{z \in \mathbb{R} : (z, \Pi_\lambda(\underline{x})) \in \Lambda(\lambda)\},$$

then for any sequence $\underline{i} \in \{0,1\}^{\mathbb{N}_0}$ such that $i_n = 0$ if $x_n = 1$ we have that $\Pi_\lambda(\underline{i}) \in L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda))$.

We look at the dimension of the set $L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda))$. We will fix $(p_0, p_1)$. Let $\mu = \mu_{p_0,p_1}$ be $(p_0, p_1)$-Bernoulli measure on $\{0,1\}^{\mathbb{N}}$. We can define another measure $\tilde{\mu}_{\underline{x}}$ on $\{0,1\}^{\mathbb{N}_0}$ such that

$$\tilde{\mu}_{\underline{x}}(\{\underline{i} : i_n = 0\}) = \begin{cases} 1 \text{ if } x_n = 1 \\ \frac{1}{2} \text{ if } x_n = 0 \end{cases}.$$

This means for $k$th level cylinders,

$$\tilde{\mu}_{\underline{x}}([i_0, \ldots, i_{k-1}]) = \begin{cases} 0 \text{ if } \exists j \text{ such that } i_j = x_j = 1 \\ 2^{-k_0}(\underline{x}) \text{ if for all } x_j = 1 \text{ we have } i_j = 0 \end{cases}.$$

Intuitively this means whenever $x_k = 0$ this corresponds to the bottom two triangles in the gasket and we have a choice of the two triangles but whenever $x_k = 1$ we are in the top triangle in the gasket so there is only one choice.

Let $\tilde{\nu}_{\lambda,\underline{x}} = \tilde{\mu}_{\underline{x}} \circ \Pi_\lambda^{-1}$ and note that it is supported on a subset of $L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda))$.

**Lemma 5.** *For almost all $\lambda \in [0.5, 0.649\ldots]$, and for $\nu_\lambda$ almost all $y \in \mathbb{R}$*

$$\dim_{\mathcal{H}} L_y(\Lambda(\lambda)) \geq \min\left(-\frac{p_0 \log 2}{\log \lambda}, 1\right).$$

PROOF. We shall show that for all $\alpha \in (0,1]$,

$$\dim_{\mathcal{H}} L_{\Pi_\lambda(\underline{x})}(\Lambda(\lambda)) \geq -\frac{\log(1 - p_0(1 - 2^{-\alpha}))}{\alpha \log \lambda},$$

for almost all $\lambda$ and $\mu$ almost all $\underline{x} \in \{0,1\}^{\mathbb{N}_0}$. The result then follows because if we let $\alpha_n = \frac{1}{n}$ for $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \frac{\log(1-p_0(1-2^{-\alpha_n}))}{\alpha_n \log\lambda} = -\frac{p_0 \log 2}{\log\lambda}$$

and if $\dim_{\mathcal{H}} L_{\Pi_\lambda \underline{x}}(\Lambda(\lambda)) \geq s$ for $\mu$-almost all $\underline{x} \in \{0,1\}^{\mathbb{N}_0}$, then $\dim_{\mathcal{H}} L_y(\Lambda(\lambda)) \geq s$ for $\nu_\lambda$ almost all $y \in \mathbb{R}$. Let $\epsilon > 0$ and for simplicity let $d(\alpha,\epsilon) = (1-p_0(1-2^{-\alpha})+\epsilon)^{\frac{1}{\alpha}}$ and $s_\epsilon(\lambda) = \min\left(-\frac{\log d(\alpha,\epsilon)}{\log\lambda}, 1-\epsilon\right)$. We use the measure $\tilde{\nu}_{\lambda,\underline{x}}$. Using the potential theoretic method for calculating Hausdorff dimension it suffices to show that

$$I = \int_{0.5}^{b(1)} \int \left( \int\int \frac{\mathrm{d}\tilde{\nu}_{\lambda,\underline{x}}(y)\mathrm{d}\tilde{\nu}_{\lambda,\underline{x}}(z)}{|z-y|^{s_\epsilon(\lambda)}} \right)^\alpha \mathrm{d}\mu(\underline{x})\mathrm{d}\lambda < \infty.$$

We start by lifting to the sequence space, using Fubini's theorem and Hölder's inequality, $\int f^\alpha \leq C\left(\int f\right)^\alpha$ for $\alpha \in (0,1]$.

$$\begin{aligned}
I &= \int_{0.5}^{b(1)} \int \left( \int\int \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})}{|\Pi_\lambda(\underline{i}) - \Pi_\lambda(\underline{j})|^{s_\epsilon(\lambda,\alpha)}} \right)^\alpha \mathrm{d}\mu(\underline{x})\mathrm{d}\lambda \\
&\leq C \int \left( \int_{0.5}^{b(1)} \int\int \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})\mathrm{d}\lambda}{|\sum_{n=0}^\infty (i_n - j_n)\lambda^n|^{s_\epsilon(\lambda,\alpha)}} \right)^\alpha \mathrm{d}\mu(\underline{x}) \\
&\leq C_1 \int \left( \int_{0.5}^{b(1)} \int\int \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})\mathrm{d}\lambda}{|a_0 + \sum_{n=1}^\infty a_n\lambda^n|^{s_\epsilon(\lambda,\alpha)} \lambda^{|\underline{i}\wedge\underline{j}|s_\epsilon(\lambda,\alpha)}} \right)^\alpha \mathrm{d}\mu(\underline{x}) \\
&\leq C_1 \int \left( \left( \int_{0.5}^{b(1)} \frac{\mathrm{d}\lambda}{|a_0 + \sum_{n=1}^\infty a_n\lambda^n|^{s_\epsilon(\lambda,\alpha)}} \right) \left( \int\int \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})}{d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}} \right) \right)^\alpha \mathrm{d}\mu(\underline{x}),
\end{aligned}$$

where $a_0 \in \{-1,1\}$ and $a_n \in \{-1,0,1\}$ for $n \geq 1$. This means we can apply Lemma 2. Hence

$$I \leq C_2 \int \left( \int\int \frac{\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{i})\mathrm{d}\tilde{\mu}_{\underline{x}}(\underline{j})}{d(\alpha,\epsilon)^{|\underline{i}\wedge\underline{j}|}} \right)^\alpha \mathrm{d}\mu(\underline{x}) \leq C_2 \int \left( \sum_{k=0}^\infty \frac{2^{-k_0(\underline{x})}}{d(\alpha,\epsilon)^k} \right)^\alpha \mathrm{d}\mu(\underline{x}).$$

As in the proof of Lemma 4 we use the inequality $\left(\sum_i b_i\right)^\alpha \leq \sum_i b_i^\alpha$ for $b_i > 0$.

We get

$$\begin{aligned} I &\leq C_2 \sum_{k=0}^{\infty} d(\alpha,\epsilon)^{-\alpha k} \int 2^{-k_0(\underline{x})\alpha} \mathrm{d}\mu(\underline{x}) \\ &\leq C_2 \sum_{k=0}^{\infty} d(\alpha,\epsilon)^{-\alpha k} \sum_{[i_0,\ldots,i_{k-1}]\in W_k} 2^{-k_0([i_0,\ldots,i_{k-1}])\alpha} \mu(W_k) \\ &\leq C_1 \sum_{k=0}^{\infty} \frac{(p_0 2^{-\alpha} + p_1)^k}{(d^{\alpha,\epsilon})^{\alpha k}}. \end{aligned}$$

We can now see that $I < \infty$ because $p_0 2^{-\alpha} + p_1 = 1 - p_0(1 - 2^{-\alpha}) < d(\alpha, \epsilon)$. To finish the proof let $\epsilon = \frac{1}{n}$ for $n \in \mathbb{N}$ and let $n \to \infty$. □

## 5 Proof of Theorem 2.

It is a standard result that $\dim_{\mathcal{H}} \Lambda(\lambda) \leq -\frac{\log 3}{\log \lambda}$ for all $\lambda$, (see, for example, [4]). Let $\underline{p} = (\frac{2}{3}, \frac{1}{3})$, let $\mu_{\underline{p}}$ be the standard $\underline{p}$-Bernoulli measure on $\{0,1\}^{\mathbb{N}_0}$ and let $\nu_\lambda = \mu_{\underline{p}} \circ \Pi_\lambda^{-1}$. We know from Proposition 1 that for almost all $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$,

$$\dim_{\mathcal{H}} \nu_\lambda = \frac{\frac{1}{3}\log\left(\frac{1}{3}\right) + \frac{2}{3}\log\left(\frac{2}{3}\right)}{\log \lambda}$$

and by Lemma 5 that for almost all $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$ and $\nu_\lambda$ almost all $y \in \mathbb{R}$

$$\dim_{\mathcal{H}} L_y(\Lambda(\lambda)) \geq -\frac{\frac{2}{3}\log 2}{\log \lambda}.$$

Thus using Lemma 3 we have that

$$\dim_{\mathcal{H}} \Lambda(\lambda) \geq \frac{\frac{1}{3}\log\left(\frac{1}{3}\right) + \frac{2}{3}\log\left(\frac{2}{3}\right)}{\log \lambda} - \frac{\frac{2}{3}\log 2}{\log \lambda} = -\frac{\log 3}{\log \lambda}$$

for almost all $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$.

To prove part 2 of Theorem 2 we need to take an alternative choice of probability vector. For example if we choose $\underline{p} = (0.7729, 0.2271)$, then

$$\frac{0.7729 \log 0.7729 + 0.2271 \log 0.2271}{\log 0.5853} \geq 1 \text{ and } -\frac{0.7729 \log 2}{\log 0.5853} \geq 1.$$

Thus by letting $\nu_\lambda = \mu_{\underline{p}} \circ \Pi_\lambda^{-1}$ and applying Proposition 1, Lemma 5 and Lemma 3 we have that $\dim_{\mathcal{H}} \Lambda(\lambda) = 2$ for almost all $\lambda \in [0.5853, b(1)]$. It is shown in [2] that $\Lambda(\lambda)$ has non-empty interior for all $\lambda \geq 0.648\ldots < b(1)$. Thus $\dim_{\mathcal{H}} \Lambda(\lambda) = 2$ for almost all $\lambda \geq 0.5853$.

## 6 Proof of Corollary 1.

We shall only prove part 1. of Corollary 1 because the proof of part 2 can be done using exactly the same method. The method is similar to the proof of Theorem 2.3 in [13]. From Theorem 2 we know there exists a dense set of $\lambda \in [\frac{1}{2}, \frac{\sqrt[3]{4}}{3}]$ such that $\dim_{\mathcal{H}} \Lambda(\lambda) = -\frac{\log 3}{\log \lambda}$. Let $F$ be the set of all IFS's, $\{S_i\}_{i=0}^2$ in $\mathbb{R}^2$ such that $S_i(x) = \lambda x + b_i$ for $b_i \in \mathbb{R}^2$. We define a topology on $F$ by the natural bijection from $F$ to $[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}] \times \mathbb{R}^6$.

From Theorem B in [7] we know that the function $\alpha(F) = \dim_{\mathcal{H}}(\Lambda(F))$ is lower semi-continuous. However since for a fixed $\lambda$, $\dim_{\mathcal{H}}(\Lambda(F))$ is constant. The function, $\overline{\alpha}(\lambda) = \dim_{\mathcal{H}}(\Lambda(\lambda))$ is also lower semi-continuous. If we let $\beta(\lambda) = -\frac{\log 3}{\log \lambda}$, then we have that $\beta$ is continuous and $\overline{\alpha}(\lambda) \leq \beta(\lambda)$. We now show that

$$\left\{\lambda \in \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right] : \overline{\alpha}(\lambda) = \beta(\lambda)\right\} = \left\{\lambda \in \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right] : \overline{\alpha} \text{ is continuous at } \lambda\right\}. \quad (4)$$

Firstly consider $\lambda \in \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right]$ such that $\overline{\alpha}(\lambda) = \beta(\lambda)$. We know that $\beta$ is continuous and $\overline{\alpha} \leq \beta$ is lower semi continuous: thus $\overline{\alpha}$ is continuous at $\lambda$. On the other hand $\overline{\alpha}$ cannot be continuous at $\lambda$ if $\overline{\alpha}(\lambda) \neq \beta(\lambda)$ because $\overline{\alpha}(\lambda) = \beta(\lambda)$ for a.e. $\lambda \in \left[\frac{1}{2}, \frac{\sqrt[3]{4}}{3}\right]$. This completes the proof of (4).

The set of continuity points for any function is a $G_\delta$ set. Hence the set of points where $\overline{\alpha}(\lambda) = \beta(\lambda)$ contains a dense $G_\delta$ set and since a residual set is a dense $G_\delta$ the proof is complete. □

## References

[1] C. J. Bishop, *Topics in real analysis: Hausdorff dimension and fractals*, lectures available at www.math.sunsyb.edu/∼bishop/classes/math639.S01/math639.html

[2] D. Broomhead, J. Montaldi and N. Sidorov, *Golden Gaskets: Variations on the Sierpiński sieve*, Nonlinearity, **17, no. 4** (2004), 1455–1480.

[3] Manev Das, Sze-Man Ngai, *Graph-directed iterated function systems with overlaps*, Indiana Univ. Math. J., **53, no. 1** (2004), 109–134.

[4] K. Falconer, *Fractal Geometry; Mathematical foundations and applications*, Wiley, Chichester, 1990.

[5] B. Hunt and V. Kaloshin, *How projections affect the dimension spectrum of fractal measures*, Nonlinearity, **10, no 5** (1997), 1031–1046.

[6] J. Hutchinson, *Fractals and self-similarity*, Indiana University Math. J., **30** (1981), 271–280.

[7] L. Jonker and J. J. P. Veerman, *Semi-continuity of dimension and measure of locally scaled fractals*, Fund. Math., **173, no. 2** (2002), 113–131.

[8] Steven P. Lalley, *$\beta$-expansions with deleted digits for Pisot numbers $\beta$*, Trans. Amer. Math. Soc., **349, no. 11** (1997), 4355–4365.

[9] J. M. Marstrand, *The dimension of Cartesian product sets*, Proc. Camb. Phil. Soc., **50** (1954), 198–202.

[10] Sze-Man Ngai, Yang Wang, *Hausdorff dimension of self-similar sets with overlaps*, J. London Math. Soc. (2), **63, no. 3** (2001), 655–672.

[11] Y. Peres and B. Solomyak, *Self-similar measures and the intersection of Cantor sets*, Trans. Amer. Math. Soc., **350** (1998), 4065–4087.

[12] M. Pollicott and K. Simon, *The Hausdorff dimension of $\lambda$-expansions with deleted digits*, Trans. Amer. Math. Soc., **347, no. 3** (1995), 967–983.

[13] K. Simon and B. Solomyak, *On the dimension of self-similar sets*, Fractals, **10** (2003), 59–65.

[14] K. Simon, B. Solomyak and M. Urbanski, *Invariant measures for parabolic IFS with overlaps and random continued fractions*, Trans. Amer. Math. Soc., **353** (2001), 5145–5164.

[15] B. Solomyak, *On the random series $\sum_{i=1}^{\infty} \pm\lambda^i$ (an Erdös problem)*, Annals of Maths, **142** (1995), 611–625.