Real Analysis Exchange
ISSN:0147-1937
Vol. 31(1), 2005/2006, pp. 239–252

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ON RIEMANN INTEGRAL QUASICONTINUITY

Abstract

A function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies condition $(Q_{r,i}(x))$ (resp. $(Q_{r,s}(x))$, $[Q_{r,o}(x)]$) at a point x if for each real $r > 0$ and for each set U containing x and belonging to Euclidean topology in $\mathbb{R}^n$ (resp. to the strong density topology [to the ordinary density topology]) there is a regular domain I such that $\text{int}(I) \cap U \neq \emptyset$, $f \restriction I$ is integrable in the sense of Riemann and $|\frac{1}{\mu(U \cap I)} \int_{U \cap I} f(t)\,dt - f(x)| < r$. These notions are particular cases of their analogues for the Lebesgue integral. In this article we compare these notions with the classical quasicontinuity and integral quasicontinuities.

Let $\mathbb{R}$ be the set of all reals and let $\mathbb{R}^n$ be the n-dimensional product space. For a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and positive reals $r_1, \dots, r_n$ put

$$I_i = (x_i - r_i, x_i + r_i) \text{ for } i = 1, 2, \dots, n,$$

and

$$P(x; r_1, \dots, r_n) = I_1 \times \dots \times I_n.$$

The symbol $Q(x, r)$ denotes the cube $P(x; r_1, \dots, r_n)$, where $r_1 = \dots = r_n = r$.

Let μ denote Lebesgue measure in $\mathbb{R}^n$. For a Lebesgue measurable set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ we define the lower strong density (compare [3] or [10], IV § 10) $D_l(A, x)$ of the set A at the point x as

$$\liminf_{h_1,\dots,h_n \to 0^+} \frac{\mu(A \cap P(x; h_1, \dots, h_n))}{\mu(P(x; h_1, \dots, h_n))}.$$

Key Words: Riemann integral quasicontinuity, density topology, uniform limit, functions of two variables
Mathematical Reviews subject classification: 26A05, 26A15
Received by the editors April 11, 2005
Communicated by: B. S. Thomson

Similarly, for a measurable set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ we define the lower ordinary density (compare [3] or [10], IV § 10) $d_l(A, x)$ of the set A at the point x as

$$\liminf_{h \to 0^+} \frac{\mu(A \cap Q(x,h))}{\mu(Q(x,h))}.$$

A point x is said to be a strong density point (an ordinary density point) of a measurable set A if $D_l(A, x) = 1$ (if $d_l(A, x) = 1$).

The family T_{sd} (T_{od}) of all Lebesgue measurable sets $A \subset \mathbb{R}^n$ for which the implication

$$x \in A \Longrightarrow x \text{ is a strong (resp. an ordinary) density point of } A$$

is true, is a topology called the strong (resp. ordinary) density topology (compare [1, 3] and for the case $n = 1$ compare [12]). If T_e denotes the Euclidean topology in $\mathbb{R}^n$, then evidently $T_e \subset T_{sd} \subset T_{od}$. The continuity of applications f from $(\mathbb{R}^n, T_{sd})$ (resp. from $(\mathbb{R}^n, T_{od})$) to $(\mathbb{R}, T_e)$ is called the strong (ordinary) approximate continuity ([1, 3]).

For an arbitrary function $f : \mathbb{R}^n \to \mathbb{R}$ denote by $C(f)$ the set of all continuity points of f. Moreover, let $D(f) = \mathbb{R}^n \setminus C(f)$.

In [7, 9] the following notion is investigated. A function $f : \mathbb{R}^n \to \mathbb{R}$ is quasicontinuous at a point x ($f \in Q(x)$) if for each positive real r and for each set $U \in T_e$ containing x there is a nonempty open set I such that $I \subset U$ and $|f(t) - f(x)| < r$ for all points $t \in I$. A function f is quasicontinuous, if $f \in Q(x)$ for every point $x \in \mathbb{R}^n$.

In [5] the following properties are investigated. A function $f : \mathbb{R}^n \to \mathbb{R}$ is integrally quasicontinuous at a point x ($f \in Q_i(x)$) if for each positive real r and for each set $U \in T_e$ containing x there is a nonempty bounded open set I such that $I \subset U$, the restricted function $f \restriction I$ is Lebesgue integrable and

$$\left| \frac{\int_I f(t)\,dt}{\mu(I)} - f(x) \right| < r.$$

A function f is integrally quasicontinuous ($f \in Q_i$), if $f \in Q_i(x)$ for every point $x \in \mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to $Q_s(x)$ (resp. $f \in Q_o(x)$), if for each positive real η and for each set $U \in T_{sd}$ (resp. $U \in T_{od}$) containing x there is an open set I such that $I \cap U \neq \emptyset$, the function f is Lebesgue integrable on $I \cap U$ and

$$\left| \frac{1}{\mu(I \cap U)} \int_{I \cap U} f(t)\,dt - f(x) \right| < \eta.$$

If $f \in Q_s(x)$ (resp. $f \in Q_o(x)$) for every point $x \in \mathbb{R}^n$, then we will write that $f \in Q_s$ (resp. $f \in Q_o$).

In this article I investigate some analogues of these properties defined by the application of the integral of Riemann.

We will say that a nonempty set $I \subset \mathbb{R}^n$ is a regular domain if it is a bounded Jordan measurable set. If, for a regular domain I, the interior $\text{int}(I) \neq \emptyset$, then we will say that I is a nondegenerate regular domain.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is R-integrally quasicontinuous at a point x ($f \in Q_{r,i}(x)$) if, for each positive real r and for each set $U \in T_e$ containing x, there is a nondegenerate regular domain I such that $I \subset U$, the restricted function $f \restriction I$ is integrable in the sense of Riemann and

$$\left| \frac{\int_I f(t)\,dt}{\mu(I)} - f(x) \right| < r.$$

A function f is R-integrally quasicontinuous ($f \in Q_{r,i}$), if $f \in Q_{r,i}(x)$ for every point $x \in \mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to $Q_{r,s}(x)$ (resp. $f \in Q_{r,o}(x)$), if, for each positive real η and for each set $U \in T_{sd}$ (resp. $U \in T_{od}$) containing x, there is a nondegenerate regular domain I such that $\text{int}(I) \cap U \neq \emptyset$, the function $f \restriction I$ is integrable in the sense of Riemann and

$$\left| \frac{1}{\mu(I \cap U)} \int_{I \cap U} f(t)\,dt - f(x) \right| < \eta,$$

where the integral on $I \cap U$ in the last inequality is the integral of Lebesgue. If $f \in Q_{r,s}(x)$ (resp. $f \in Q_{r,o}(x)$) for every point $x \in \mathbb{R}^n$, then we will write that $f \in Q_{r,s}$ (resp. $f \in Q_{r,o}$).

In [5] it is observed that, if a function $f : \mathbb{R}^n \to \mathbb{R}$ is integrally quasicontinuous, then the set $Z(f)$ of all points $x \in \mathbb{R}^n$ at which f is locally Lebesgue integrable is open and dense in $\mathbb{R}^n$. Analogously we can observe the following.

Remark 1. *If a function $f : \mathbb{R}^n \to \mathbb{R}$ is R-integrally quasicontinuous, then it is integrally quasicontinuous and there is a dense open set $U \subset \mathbb{R}^n$ such that $\mu(U \setminus C(f)) = 0$.*

Proof. Evidently, if f is R-integrally quasicontinuous, then it is also integrally quasicontinuous. If W is a nonempty open set, then there is a nondegenerate regular domain $I \subset W$ such that the restricted function $f \restriction I$ is integrable in the sense of Riemann on I. Consequently, $\mu(\text{int}(I) \setminus C(f)) = 0$. So for each open set $W \neq \emptyset$ there is an open cube $J \subset W$ whose vertexes have rational coordinates such that $\mu(\text{int}(J) \setminus C(f)) = 0$. Let U be the union of all open cubes $J \subset W$ whose vertexes have rational coordinates and such that $\mu(\text{int}(J) \setminus C(f)) = 0$. Then the open set U satisfies all requirements. □

Example 1. Let $A \subset (0,1)$ be a nonempty F_σ-set such that $D_l(A,x) = 1$ for each point $x \in A$ and the closure $\mathrm{cl}(A)$ is a nowhere dense set. There is ([13] and [2], p. 28, Th. 6.5) an approximately continuous function $f : \mathbb{R} \to [0,1]$ such that $f(A) = (0,1]$, $f(x) = 0$ for $x \in \mathbb{R} \setminus A$ and $C(f) = \mathbb{R} \setminus A$. Then f is integrally quasicontinuous and $\mathrm{int}(C(f)) = \mathbb{R} \setminus \mathrm{cl}(A)$ is open and dense, but f is not R-integrally quasicontinuous at any point $x \in A$.

Theorem 1. *If a function $f : \mathbb{R}^n \to \mathbb{R}$ is integrally quasicontinuous and locally integrable in the sense of Riemann at a point x, then f is R-integrally quasicontinuous at x.*

PROOF. Fix an open set $U \ni x$ and a real $\eta > 0$. Since f is locally integrable in the sense of Riemann, there is a regular domain $I \subset U$ such that $x \in \mathrm{int}(I)$ and f is integrable on I in the sense of Riemann. From the integral quasicontinuity of f at x it follows that there is a bounded open set $V \subset \mathrm{int}(I)$ such that

$$\left| \frac{\int_V f}{\mu(V)} - f(x) \right| < \eta.$$

There is a regular domain $J \subset V$ such that

$$\left| \frac{\int_J f}{\mu(J)} - f(x) \right| < \eta.$$

Since f is integrable on J in the sense of Riemann. □

The next example shows that an R-integrally quasicontinuous function may be nonmeasurable.

Example 2. Let $A \subset (0,1)$ be a nowhere dense, perfect set of positive measure and let $A = B \cup C$, where B and C are nonmeasurable and disjoint. In each component (a,b) of the complement $\mathbb{R} \setminus A$ find a nondegenerate closed interval $I(a,b) = [c(a,b), d(a,b)]$ and put

$$f_{(a,b)}(x) = \begin{cases} 1 & \text{for } x \in I(a,b) \\ 0 & \text{for } x \in (a,b) \setminus I(a,b). \end{cases}$$

If

$$f(x) = \begin{cases} f_{(a,b)}(x) & \text{for } x \in (a,b), \text{ where } (a,b) \text{ is an component of } \mathbb{R} \setminus A \\ 0 & \text{for } x \in B \\ 1 & \text{for } x \in C, \end{cases}$$

then f is evidently R-integrally quasicontinuous and nonmeasurable.

In [5] an example of a quasicontinuous bounded function $f:\mathbb{R}\to\mathbb{R}$ such that $Z(f)=\emptyset$ is shown and the following theorem is proved.

Theorem 2. *If $f:\mathbb{R}^n\to\mathbb{R}$ is a quasicontinuous function and if there is a dense open set $G\subset\mathbb{R}^n$ such that the restricted function $f\restriction G$ is measurable, then f is integrally quasicontinuous.*

In this article I prove the following assertion.

Theorem 3. *If $f:\mathbb{R}^n\to\mathbb{R}$ is a quasicontinuous function and if there is a dense open set $G\subset\mathbb{R}^n$ such that $\mu(G\setminus C(f))=0$, then f is R-integrally quasicontinuous.*

Proof. Fix a point x, a real $\eta>0$ and an open set $W\ni x$. Since f is quasicontinuous, the set $C(f)$ of all continuity points of f is dense and there is a nonempty open set $V\subset W$ such that $f(V)\subset(f(x)-\eta,f(x)+\eta)$. There is a point $u\in V\cap G\cap C(f)$. Let $h_1>0$ be a real such that $\operatorname{cl}(Q(u,h_1))\subset V\cap G$. Since $\operatorname{cl}(Q(u,h_1))\subset V$ and $\mu(\operatorname{cl}(Q(u,h_1))\setminus C(f))=0$, the function f is integrable on $\operatorname{cl}(Q(u,h_1))$ in the sense of Riemann. From the continuity of f at u it follows that

$$\lim_{h\to 0^+}\frac{\int_{\operatorname{cl}(Q(u,h))}f(t)\,dt}{\mu(Q(u,h))}=f(u).$$

Since $f(u)\in(f(x)-\eta,f(x)+\eta)$, there is a real $h>0$ such that $h<h_1$ and

$$\frac{\int_{\operatorname{cl}(Q(u,h))}f(t)\,dt}{\mu(Q(u,h))}\in(f(x)-\eta,f(x)+\eta).$$ □

The next example (considered also in [5]) shows that there is a uniform limit of a sequence of R-integrally quasicontinuous functions which is not R-integrally quasicontinuous.

Example 3. If $A\subset\mathbb{R}$ is a bounded nowhere dense closed set of positive measure, then we find a nonmeasurable (in the sense of Lebesgue) set $B\subset A\setminus\{\inf A,\sup A\}$ such that the interior measures $\mu_i(B)$ and $\mu_i(A\setminus B)$ are 0 and we put

$$f_A(x)=\begin{cases}1 & \text{for } x\in B\\ 0 & \text{for } x\in(A\setminus B)\cup(-\infty,\inf A]\cup[\sup A,\infty),\end{cases}$$

and if (a,b) is a component of the set $(\inf A,\sup A)\setminus A$, then for $x\in(a,b)$ we put

$$f_A(x)=\sin\Big(\frac{1}{\min(x-a,b-x)}\Big).$$

Evidently, the function f_A is quasicontinuous,

$$f_A(\mathbb{R}) = [-1,1], \quad C(f_A) = \mathbb{R} \setminus A$$

and the restricted function $f_A \restriction A$ is not measurable (in the Lebesgue sense).

Now let $E \subset \mathbb{R}$ be a dense G_δ-set of measure zero and let $H = \mathbb{R} \setminus E$. Since H is an F_σ-set of the first category, by Sierpiński's theorem from [11] there are pairwise disjoint bounded closed sets F_n such that $H = \bigcup_n F_n$. Without loss of generality we can suppose that $\mu(F_n) > 0$ for $n \geq 1$. Let

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n} f_{F_n}. \tag{$*$}$$

If $x \in E$, then for each integer $n \geq 1$ the point x belongs to $\mathbb{R} \setminus F_n = C(f_{F_n})$. Consequently, by the uniform convergence of the series in $(*)$, the function f is continuous at x. So, $f \in Q(x)$.

Now let $x \in H$. There is a unique integer k with $x \in F_k$. For $n \neq k$ the functions f_{F_n} are continuous at x, so the sum $\sum_{n \neq k} \frac{1}{2^n} f_{F_n}$ is also continuous at x. Since the function f_{F_k} is quasicontinuous at x, by Theorem 1 from [4] the sum

$$\sum_{n=1}^{\infty} \frac{1}{2^n} f_{F_n} = \sum_{n \neq k} \frac{1}{2^n} f_{F_n} + \frac{1}{2^k} f_{F_k}$$

is also quasicontinuous at x. So the function f is quasicontinuous.

In the same way we can prove that the partial sums

$$f_k = \sum_{n=1}^{k} \frac{1}{2^n} f_{F_n} \text{ for } k \geq 1,$$

are also quasicontinuous at each point x. Since the sets

$$C(f_n) = \mathbb{R} \setminus \bigcup_{i=1}^{n} F_i$$

are open and dense, the functions f_n are R-integrally quasicontinuous for $n \geq 1$.

Now let $K \subset \mathbb{R}$ be a measurable set of positive measure. Then there is an integer $j \geq 1$ with $\mu(K \cap F_j) > 0$. Since the sum $\sum_{n \neq j} \frac{1}{2^n} f_{F_n}$ is continuous on $K \cap F_j$ and the restricted function $f_{F_j} \restriction K$ is not measurable, the restricted function $f \restriction K$ is not measurable. Consequently, $Z(f) = \emptyset$ and f is not integrally quasicontinuous at any point. It follows that it is not R-integrally quasicontinuous.

Theorem 4. *Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be R-integrally quasicontinuous functions such that $\mu(\mathbb{R}^n \setminus C(f_k)) = 0$ for $k \geq 1$. If the sequence (f_k) uniformly converges to a function f, then f is R-integrally quasicontinuous and $\mu(\mathbb{R}^n \setminus C(f)) = 0$.*

Proof. Since uniform convergence preserves continuity,

$$C(f) \supset \bigcap_{k=1}^{\infty} C(f_k) \text{ and } \mu(\mathbb{R}^n \setminus C(f)) \leq \sum_{k=1}^{\infty} \mu(\mathbb{R}^n \setminus C(f_k)) = 0.$$

So f is almost everywhere continuous. For the proof that f is R-integrally quasicontinuous fix a point x, a positive real η and an open set $U \ni x$. From the uniform convergence of (f_k) it follows that there is a positive integer k_1 such that

$$|f_{k_1}(y) - f(y)| < \frac{\eta}{3} \text{ for all } y \in \mathbb{R}^n.$$

Since f_{k_1} is R-integrally quasicontinuous at x, there is a regular domain $I \subset U$ such that

$$\left| \frac{\int_I f_{k_1}}{\mu(I)} - f_{k_1}(x) \right| < \frac{\eta}{3}.$$

The function f is almost everywhere continuous and bounded on I, so it is integrable on I in the sense of Riemann. Moreover,

$$\left| \frac{\int_I f}{\mu(I)} - f(x) \right| \leq \left| \frac{\int_I f}{\mu(I)} - \frac{\int_I f_{k_1}}{\mu(I)} \right| + \left| \frac{\int_I f_{k_1}}{\mu(I)} - f_{k_1}(x) \right| + |f_{k_1}(x) - f(x)|$$

$$< \frac{\int_I |f - f_{k_1}|}{\mu(I)} + \frac{\eta}{3} + \frac{\eta}{3} \leq \frac{\eta}{3} + \frac{2\eta}{3} = \eta,$$

so f is R-integrally quasicontinuous at x. □

Theorem 5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an R-integrally quasicontinuous function and let $A \subset \mathbb{R}^n$ be a dense set. Then for each point $x \in \mathbb{R}^n$ the inequalities*

$$\lim_{r\to 0^+} (\inf\{f(t); t \in A,\ |t-x| < r\}) \leq f(x) \leq \lim_{r\to 0^+} (\sup\{f(t); t \in A,\ |t-x| < r\})$$

are true.

Proof. Fix a point $x \in \mathbb{R}^n$ and positive reals η and r. Since the function f is R-integrally quasicontinuous at x, there is a nondegenerate regular domain $I \subset K(x, r) = \{t; |t - x| < r\}$ such that f is integrable on I in the sense of Riemann and

$$\left| \frac{\int_I f}{\mu(I)} - f(x) \right| < \eta.$$

From the Riemann integrability of f on I it follows that $\mu(I \setminus C(f)) = 0$ and that f is bounded on I. There are points

$$u, v \in I \cap C(f) \text{ with } f(u) > f(x) - \eta \text{ and } f(v) < f(x) + \eta.$$

But the set A is dense, so there are points $u_1, v_1 \in I \cap A$ such that

$$f(u_1) > f(x) - \eta \text{ and } f(v_1) < f(x) + \eta.$$

So the inequalities from the statement of the theorem are true. □

Corollary 1. *Let $f : (\mathbb{R}^n \times \mathbb{R}^m) \to \mathbb{R}$ be a function such that the horizontal sections $f^y(x) = f(x, y)$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, are R-integrally quasicontinuous and almost everywhere continuous, and the vertical sections $f_x(y) = f(x, y)$ are Lebesgue measurable. Then f is Lebesgue measurable.*

PROOF. By the previous theorem our corollary follows immediately from Theorem 2 from [8]. □

It is well known (see for example, [6]) that there are functions $f : \mathbb{R}^2 \to \mathbb{R}$ with continuous sections f_x and f^y, $x, y \in \mathbb{R}$, such that $\mu(I \cap (\mathbb{R}^2 \setminus C(f))) > 0$ for each nondegenerate regular domain I. Evidently, such functions are not R-integrally quasicontinuous. By Kempisty's theorem from [7] they are quasicontinuous. Since they are also Lebesgue measurable, by Theorem 2 they are integrally quasicontinuous.

Theorem 6. *Let $f : (\mathbb{R}^n \times \mathbb{R}^m) \to \mathbb{R}$ be a function locally integrable in the sense of Riemann such that for each point $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ there is an open set $A(x, y) \subset \mathbb{R}^n$ containing x for which the sections f_u, $u \in A(x, y)$, are R-integrally equiquasicontinuous at y; i.e., for each real $\eta > 0$ and for each open set $U \ni y$ contained in $\mathbb{R}^m$ there is a nondegenerate regular domain $I \subset U$ such that f_u, $u \in A(x, y)$, are integrable in the sense of Riemann on I and*

$$\left| \frac{\int_I f_u}{\mu(I)} - f(u, y) \right| < \eta \text{ for } u \in A(x, y).$$

If the sections f^v, $v \in \mathbb{R}^m$, are R-integrally quasicontinuous, then f is also R-integrally quasicontinuous.

PROOF. Fix a point $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, an open set $U \ni (x, y)$ and a positive real η. Since f is locally integrable in the sense of Riemann, there is a regular domain $I \subset U$ such that f is integrable on I in the sense of Riemann and $(x, y) \in \text{int}(I)$. The section f^y is R-integrally quasicontinuous at x, so there is a nondegenerate regular domain

$$J \subset A(x, y) \cap (\text{int}(I))^y = A(x, y) \cap \{u \in \mathbb{R}^n; (u, y) \in \text{int}(I)\}$$

such that f^y is integrable on J in the sense of Riemann and

$$\left|\frac{\int_J f^y}{\mu(J)} - f(x,y)\right| < \frac{\eta}{2}.$$

From the inclusion $J \times \{y\} \subset \operatorname{int}(I)$ it follows that for each point $u \in J$ there is an open regular domain $X(u) \times Y(u) \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $u \in X(u)$, $y \in Y(u)$ and $X(u) \times Y(u) \subset U$. Since the set $J \times \{y\}$ is compact and

$$J \times \{y\} \subset \bigcup_{u \in J} (X(u) \times Y(u)),$$

there is a finite subset $\{u_1, u_2, \dots, u_k\} \subset J$ with

$$J \times \{y\} \subset \bigcup_{i=1}^{k} (X(u_i) \times Y(u_i)).$$

Let $Y = \bigcap_{i=1}^{k} Y(u_i)$. Then Y is an open regular domain containing y such that $J \times \{y\} \subset J \times Y$. Since the sections f_u, $u \in J$, are R-integrally equiquasicontinuous at y, there is a nondegenerate regular domain $K \subset Y$ such that f_u, $u \in J$, are integrable on K in the sense of Riemann and

$$\left|\frac{\int_K f_u}{\mu(K)} - f(u,y)\right| < \frac{\eta}{2} \text{ for } u \in J.$$

Let $W = J \times K$. Then $W \subset I$ is a nondegenerate regular domain, f is integrable on W in the sense of Riemann and

$$\begin{aligned}
\Big|\frac{\int_W f}{\mu(W)} - f(x,y)\Big| &\le \Big|\frac{\int_J (\int_K f(u,v)\,dv)\,du}{\mu(W)} - \frac{\int_J f(u,y)\mu(K)\,du}{\mu(J)\mu(K)}\Big| \\
&\quad + \Big|\frac{\int_J f(u,y)\mu(K)\,du}{\mu(J)\mu(K)} - f(x,y)\Big| \\
&\le \frac{\int_J \Big|\frac{\int_K f(u,v)\,dv}{\mu(K)} - f(u,y)\Big|\,du}{\mu(J)} + \Big|\frac{\int_J f(u,y)\,du}{\mu(J)} - f(x,y)\Big| \\
&< \frac{\eta \int_J du}{2\mu(J)} + \frac{\eta}{2} = \eta
\end{aligned}$$

□

Theorem 7. *There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that the sections f_x and f^y, $x, y \in \mathbb{R}$, are continuous and R-integrally equiquasicontinuous and which is not locally integrable in the sense of Riemann.*

PROOF. Let $A \subset [0,1]$ be a nowhere dense perfect set of positive measure such that $0, 1 \in A$ and let $((a_k, b_k))_k$ be an enumeration of all components of the set $[0,1] \setminus A$ such that $(a_k, b_k) \cap (a_i, b_i) = \emptyset$ for $k \neq i$. For each $k \geq 1$ there is a strictly decreasing sequence $(c_{k,i})$ with $\lim_{i\to\infty} c_{k,i} = a_k$ and $c_{k,i} \in (a_k, b_k)$ for $i \geq 1$. The set N of all positive integers is the union of an infinite family of pairwise disjoint infinite subsets $N_{k,s}$, where $k, s \geq 1$. Evidently, for all integers $k, m, i \geq 1$ the sequence $(c_{k,j})_{j\in N_{m,i}}$ is strictly decreasing and converges to a_k. For each point $c_{k,i}$ put

$$r(c_{k,i}) = \frac{\inf(\inf\{|c_{k,i} - c_{k,j}|; j \neq i \text{ and } j \geq 1\}, |c_{k,i} - b_k|)}{3}.$$

Moreover, for all pairs $(c_{k,i}, c_{s,t})$ let

$$I_{c_{k,i},c_{s,t}} = [c_{k,i} - r(c_{k,i}), c_{k,i} + r(c_{k,i})] \times [c_{s,t} - r(c_{s,t}), c_{s,t} + r(c_{s,t})].$$

Now we will define a function f. On the rectangles $I_{c_{k,i},c_{s,i}}$, where $i \in N_{k,s}$, $k, s \geq 1$, we define f in such a way that f is continuous on $I_{c_{k,i},c_{s,i}}$,

$$f(u,v) = 0 \text{ on } I_{c_{k,i},c_{s,i}} \setminus \operatorname{int}(I_{c_{k,i},c_{s,i}})$$

and $f(I_{c_{k,i},c_{s,i}}) = [0,1]$. Moreover, on the set $\mathbb{R}^2 \setminus \bigcup_{k,s=1}^{\infty} \bigcup_{i\in N_{k,s}} I_{c_{k,i},c_{s,i}}$ we put $f(u,v) = 0$. For each point $x \in \mathbb{R}$ the set of all pairs $(c_{k,i}, c_{s,i})$ ($k, s \geq 1$, $i \in N_{k,s}$) giving a nonempty intersection

$$\{(x,v); v \in \mathbb{R}\} \cap I_{c_{k,i},c_{s,i}} \neq \emptyset,$$

is empty or contains only one pair. Similarly, for each point $y \in \mathbb{R}$ the set of all pairs $(c_{k,i}, c_{s,i})$ giving a nonempty intersection

$$\{(u,y); u \in \mathbb{R}\} \cap I_{c_{k,i},c_{s,i}} \neq \emptyset,$$

is empty or contains only one pair.

So the sections f_x and f^y are continuous. Since

$$\{(x,y); f(x,y) \neq 0\} \subset \bigcup_{k,s=1}^{\infty} \bigcup_{i\in N_{k,s}} I_{c_{k,i},c_{s,i}},$$

by the definitions of $I_{c_{k,i},c_{s,t}}$ we obtain that the sections f_x, $x \in \mathbb{R}$, and f^y, $y \in \mathbb{R}$, are integrally equiquasicontinuous at all points. Moreover,

$$\mathbb{R}^2 \setminus C(f) = A \times A$$

and consequently f is not locally R-integrable at some points of $A \times A$. □

Observe that the function f from the proof of the last theorem is R-integrally quasicontinuous.

Problem 1. Suppose that the sections f_x, $x \in \mathbb{R}^n$, of a function $f : (\mathbb{R}^n \times \mathbb{R}^m) \to \mathbb{R}$ are R-integrally equiquasicontinuous and the sections f^y, $y \in \mathbb{R}^m$, are R-integrally quasicontinuous. Is the function f R-integrally quasicontinuous?

Finishing, we will prove a natural characterization of the classes $Q_{r,s}$ and $Q_{r,o}$.

Theorem 8. *A function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to $Q_{r,s}$ (resp. to $Q_{r,o}$) if and only if $\mu(D(f)) = 0$ and $f \in Q_s$ (resp. $f \in Q_o$).*

Proof. Let $f \in Q_{r,o} \subset Q_{r,s}$. Assume, to a contradiction, that $\mu(\mathbb{R}^n \setminus C(f)) > 0$. Since Lebesgue's density theorem is true for the topologies T_{sd} and T_{od} (see [1] or [10], IV § 10, Th. (10.1)), there is a nonempty set $U \in T_{sd}$ contained in $\mathbb{R}^n \setminus C(f)$. Since for each point $x \in U$ and for each regular domain I with $U \cap \operatorname{int}(I) \neq \emptyset$ the restricted function $f \restriction I$ is not integrable in the sense of Riemann, we obtain a contradiction. So, $\mu(\mathbb{R}^n \setminus C(f)) = 0$. Immediately from the definition it follows that, if $f \in Q_{r,s}$ (resp. $f \in Q_{r,o}$), then $f \in Q_s$ (resp. $f \in Q_o$).

Now we will prove that, if $\mu(D(f)) = 0$ and $f \in Q_s$ (resp. $f \in Q_o$), then $f \in Q_{r,s}$ (resp. $f \in Q_{r,o}$). For this fix a function $f \in Q_s$, a point x, a set $U \in T_{sd}$ containing x and a real $\eta > 0$. Since $f \in Q_s$, there is an open set W such that $W \cap U \neq \emptyset$ and

$$\left| \frac{\int_{U \cap W} f}{\mu(W \cap U)} - f(x) \right| < \eta.$$

For each point $u \in W \cap C(f)$ there is a nondegenerate closed box $I(u) \subset W$ whose vertexes have rational coordinates such that $u \in \operatorname{int}(I(u))$ and f is integrable on $I(u)$ in the sense of Riemann. Since $\mu(D(f)) = 0$, there is a regular domain $I \subset W$ being the finite union of some family $(I(u_i))_{i \le k}$, where $u_i \in W \cap C(f)$, such that $\operatorname{int}(I) \cap U \neq \emptyset$ and

$$\left| \frac{\int_{U \cap I} f}{\mu(U \cap I)} - f(x) \right| < \eta.$$

Evidently, $f \restriction I$ is integrable in the sense of Riemann and the proof in this case is completed. For the case $f \in Q_o$ the proof is analogous. □

In the definitions of the classes $Q_{r,s}$ and $Q_{r,o}$ we use the integral of Lebesgue. For locally bounded functions we have a characterization in which only the integral of Riemann is used.

The present form of Theorem 9 and its proof is an idea of the referee.

Theorem 9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally bounded function. The function f belongs to $Q_{r,s}$ (resp. to $Q_{r,o}$) if and only if it satisfies the following condition*

(a) for each point $x \in \mathbb{R}^n$, for each set $U \ni x$ belonging to T_{sd} (resp. to T_{od}), for each open set $Z \ni x$ and for each real $\eta > 0$ there is a nondegenerate regular domain $I \subset Z$ such that $f \restriction I$ is integrable in the sense of Riemann, $\operatorname{int}(I) \cap U \neq \emptyset$, $\mu(I \setminus U) < \eta\mu(I \cap U)$ and

$$\left| \frac{\int_I f}{\mu(I)} - f(x) \right| < \eta.$$

PROOF. Fix a point x, a set $U \ni x$ belonging to T_{sd}, an open set $Z \ni x$ and a positive real η. Since f is locally bounded, there are an open set $V \ni x$ contained in Z and a real $M > 0$ with $|f(t)| < M$ for $t \in V$. Let $U_1 = V \cap U$. If $f \in Q_{r,s}$, then there is a nondegenerate regular domain I such that $\operatorname{int}(I) \cap U_1 \neq \emptyset$ and

$$\left| \frac{\int_{I \cap U_1} f}{\mu(I \cap U_1)} - f(x) \right| < \eta.$$

f is measurable by Theorem 8 and bounded on V. Hence we find an open set W, $\operatorname{int}(I) \cap U_1) \subset W \subset \operatorname{int}(I) \cap V$, that approximates $\operatorname{int}(I) \cap U_1$ from outside such that

$$\left| \frac{\int_W f}{\mu(W)} - f(x) \right| < \eta$$

and $\mu(W \setminus U_1) < \eta\mu(W \cap U_1)$. The last estimate gives $\mu(W \setminus U) < \eta\mu(W \cap U)$, because $W \cap U_1 = W \cap (V \cap U) = W \cap U$ and similarly $W \setminus U_1 = W \setminus U$, and in particular $W \cap U \neq \emptyset$.

Since $\mu(D(f)) = 0$ by Theorem 8, we obtain, as in the previous proof, a nondegenerate regular domain $J \subset W$ with $\operatorname{int}(J) \cap U \neq \emptyset$, $\mu(J \setminus U) < \eta\mu(J \cap U)$, and

$$\left| \frac{\int_J f}{\mu(J)} - f(x) \right| < \eta.$$

This proves (a).

On the other hand, if f satisfies condition (a), then there is a nondegenerate regular domain $I \subset V$ such that $\operatorname{int}(I) \cap U \neq \emptyset$, $\mu(I \setminus U) < \frac{\eta}{4M}\mu(I \cap U)$, $f \restriction I$ is integrable in the sense of Riemann, and

$$\left| \frac{\int_I f}{\mu(I)} - f(x) \right| < \frac{\eta}{2}.$$

Then

$$\begin{aligned}\Big|\frac{\int_{I\cap U} f}{\mu(I\cap U)} - f(x)\Big| &\le \Big|\frac{\int_{I\cap U} f}{\mu(I\cap U)} - \frac{\int_I f}{\mu(I)}\Big| + \Big|\frac{\int_I f}{\mu(I)} - f(x)\Big| \\ &< \Big|\int_{I\cap U}\Big(\frac{f}{\mu(I\cap U)} - \frac{f}{\mu(I)}\Big) - \int_{I\setminus U}\frac{f}{\mu(I)}\Big| + \frac{\eta}{2} \\ &\le \Big|\Big(\frac{1}{\mu(I\cap U)} - \frac{1}{\mu(I)}\Big)\int_{I\cap U} f\Big| + \Big|\frac{1}{\mu(I)}\int_{I\setminus U} f\Big| + \frac{\eta}{2} \\ &\le \Big(\frac{1}{\mu(I\cap U)} - \frac{1}{\mu(I)}\Big)M\mu(I\cap U) + \frac{1}{\mu(I)}M\mu(I\setminus U) + \frac{\eta}{2} \\ &\le \Big(\frac{\mu(I)}{\mu(I)} - \frac{\mu(I\cap U)}{\mu(I)} + \frac{\mu(I\setminus U)}{\mu(I)}\Big)M + \frac{\eta}{2} \\ &= 2\frac{\mu(I\setminus U)}{\mu(I)}M + \frac{\eta}{2} \le 2\frac{\mu(I\setminus U)}{\mu(I\cap U)}M + \frac{\eta}{2} \\ &< 2\frac{\eta}{4M}M + \frac{\eta}{2} = \eta.\end{aligned}$$

This yields $f \in Q_{r,s}$.

The equivalence of the inclusion $f \in Q_{r,o}$ with the respective version of (a) can be proved in the same way.

Acknowledgement. I would like to thank the referee for his valuable remarks which allowed me to correct many mistakes in the first version of the paper. I am especially grateful to him for the correction of condition (a) in Theorem 9.

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