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ALGEBRAIC SUMS OF SETS IN MARCZEWSKI-BURSTIN ALGEBRAS

Abstract

Using almost-invariant sets, we show that a family of Marczewski–Burstin algebras over groups are not closed under algebraic sums. We also give an application of almost-invariant sets to the difference property in the sense of de Bruijn. In particular, we show that if G is a perfect Abelian Polish group then there exists a Marczewski null set $A \subseteq G$ such that $A + A$ is not Marczewski measurable, and we show that the family of Marczewski measurable real valued functions defined on G does not have the difference property.

1 Introduction.

The *algebraic sum* of two subsets A, B of a group G is the set $A+B = \{a+b : a \in A, b \in B\}$. If $\mathcal{A}$ is an algebra of subsets of the group G it is natural to ask whether $\mathcal{A}$ is closed under algebraic sums. It is a well-known result that the algebras of Lebesgue measurable sets and sets with the Baire property are not closed under algebraic sums over $\mathbb{R}$. In fact, there is a null (resp. meager) $A \subseteq \mathbb{R}$ such that $A + A$ is not Lebesgue measurable (resp. $A + A$ does not have the Baire property). For various proofs of these facts (and some generalizations) see [9], [15] and [10], for example.

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In this paper we show that certain of Marczewski–Burstin algebras, including Marczewski and Miller algebras on Abelian Polish groups, are not closed under algebraic sums. If $\mathcal{K}$ is a family of subsets of an infinite Abelian group G, we define

$$\mathcal{S}(\mathcal{K}) = \{A \subseteq G \colon (\forall K \in \mathcal{K})(\exists K' \in \mathcal{K}) K' \subseteq K \cap A \vee K' \subseteq K \setminus A\},$$
$$\mathcal{S}_0(\mathcal{K}) = \{A \subseteq G \colon (\forall K \in \mathcal{K})(\exists K' \in \mathcal{K}) K' \subseteq K \setminus A\}.$$

It is easy to see that $\mathcal{S}(\mathcal{K})$ is an algebra of subsets of G and $\mathcal{S}_0(\mathcal{K}) \subseteq \mathcal{S}(\mathcal{K})$ is an ideal. The set $\mathcal{S}(\mathcal{K})$ (resp. $\mathcal{S}_0(\mathcal{K})$) is the *Marczewski–Burstin algebra* (resp. *Marczewski–Burstin ideal*) associated with the family $\mathcal{K}$. (*cf.* [2] or [1].)

A set $B \subseteq G$ is *$\mathcal{K}$-Bernstein* if $K \cap B \neq \emptyset$ and $K \setminus B \neq \emptyset$ for all $K \in \mathcal{K}$. Obviously, $B \notin \mathcal{S}(\mathcal{K})$ when B is $\mathcal{K}$-Bernstein.

We also address the question of whether the family of $\mathcal{S}(\mathcal{K})$-measurable functions on G has the difference property. For any function $f : G \to \mathbb{R}$ and $y \in G$ we define the *difference function* $\Delta_y f \colon G \to \mathbb{R}$ by $\Delta_y f(x) = f(x + y) - f(x)$ for every $x \in G$. A family $\mathcal{F}$ of real valued functions defined on G is said to have the *difference property* (in the sense of de Bruijn) if every function $f \colon G \to \mathbb{R}$ such that $\Delta_y f \in \mathcal{F}$ for each $y \in G$ is of the form $f = g + h$, where $g \in \mathcal{F}$ and h is an algebraic homomorphism. The notion of the difference property was introduced by de Bruijn [4], see [12] for a recent survey.

The key to our approach is to relate these questions to the existence of appropriate almost-invariant sets. Let $\mathcal{J}$ be an arbitrary ideal on G. A set $A \subseteq G$ is *$\mathcal{J}$-almost-invariant* if the symmetric difference $(A + g)\Delta A \in \mathcal{J}$ for every $g \in G$. We simply say that A is *almost-invariant* if it is $[G]^{<|G|}$-almost-invariant.

The relationship between algebraic sums and almost-invariant sets is provided by the following theorem.

Theorem 1 (Ciesielski–Fejzić–Freiling [3]). *Let G be an infinite Abelian group of size κ and let $\mathcal{K}$ be a family of subsets of G such that $|\mathcal{K}| = \kappa$ and $|K| = \kappa$ for every $K \in \mathcal{K}$. If there is a set $A \subseteq G$ such that $|(A + g) \cap (-A)| = \kappa$ for every $g \in G$ then there is $B \subseteq A$ such that $B + B$ is a $\mathcal{K}$-Bernstein set.*

Although Ciesielski, Fejzić and Freiling only consider the above theorem for $G = \mathbb{R}$, the reader will have no problem adapting their proof to our more general context. Observe that if A is *symmetric* (*i.e.* $A = -A$) almost-invariant and $|A| = \kappa$ then the condition $|(A + g) \cap (-A)| = \kappa$ follows immediately.

To relate the difference property to symmetric almost-invariant sets, we use a theorem of the second author.

Theorem 2 (Filipów [5])**.** *Let G be an infinite Abelian group, $\mathcal{A}$ a σ-algebra of subsets of G and $\mathcal{I} \subseteq \mathcal{A}$ an ideal. If $\mathcal{A}$ is closed under reflections; i.e., $A \in \mathcal{A}$ implies $-A \in \mathcal{A}$, and there is a symmetric $\mathcal{I}$-almost-invariant set $S \notin \mathcal{A}$, then the family of $\mathcal{A}$-measurable functions does not have the difference property.*

2 Construction of Almost-Invariant Sets.

Throughout this section G will stand for an uncountable Abelian group of size κ, $G = \{g_\alpha : \alpha < \kappa\}$ is a fixed enumeration of G and G_α denotes the subgroup of G generated by $\{g_\beta : \beta < \alpha\}$. Note that $|G_\alpha| \leq |\alpha|\omega < \kappa$ since κ is uncountable. Our results also apply for countable groups G provided that we may write $G = \bigcup_{n<\omega} G_n$ where G_n is an increasing sequence of finite subgroups of G.

Sierpiński formulated a construction of almost-invariant sets. Most constructions use his method which is summarized in the following proposition.

Proposition 3 (Sierpiński [16])**.** *For any sequence $\{x_\alpha : \alpha < \kappa\} \subseteq G$, the set $A = \bigcup_{\alpha<\kappa}(G_\alpha + x_\alpha)$ is almost-invariant.*

It is easy to use this to construct $\mathcal{K}$-Bernstein almost-invariant sets.

Theorem 4. *If $\mathcal{K} \subseteq [G]^\kappa$ is a family of size at most κ, then there is a symmetric almost-invariant set that is $\mathcal{K}$-Bernstein.*

PROOF. Write $\mathcal{K} = \{K_\alpha : \alpha < \kappa\}$. We will construct two sequences $\{x_\alpha : \alpha < \kappa\}$ and $\{y_\alpha : \alpha < \kappa\}$ as follows. Take

$$x_\alpha \in K_\alpha \setminus (G_\alpha + (\{\pm x_\beta : \beta < \alpha\} \cup \{\pm y_\beta : \beta < \alpha\}))$$

and

$$y_\alpha \in K_\alpha \setminus (G_\alpha + (\{\pm x_\beta : \beta \leq \alpha\} \cup \{\pm y_\beta : \beta < \alpha\})).$$

Now we put $S = \bigcup_{\alpha<\kappa}(G_\alpha \pm x_\alpha)$. It follows from Proposition 3 that S is almost-invariant. It is easy to see that S is symmetric since each $G_\alpha \pm x_\alpha$ is.

Finally, it remains to show that S is $\mathcal{K}$-Bernstein. We see that $S \cap K \neq \varnothing$ for every set $K \in \mathcal{K}$ (since $x_\alpha \in S$ for all $\alpha < \kappa$). On the other hand, we show that $y_\alpha \notin S$ for all $\alpha < \kappa$. Suppose instead that there is $\alpha < \kappa$ such that $y_\alpha \notin G \setminus S$. Then there is β such that $y_\alpha \in G_\beta \pm x_\beta$. If $\alpha \geq \beta$, then we get a contradiction with the definition of points y_α. So $\beta > \alpha$, but in that case $x_\beta \in G_\beta \pm y_\alpha$ which is a contradiction. □

As a consequence of this we obtain our main result regarding the difference property for $\mathcal{S}(\mathcal{K})$-measurable functions.

Theorem 5. *Suppose that $\mathcal{K} \subseteq [G]^\kappa$ is a family of size at most κ that satisfies the following property*

($*$) *For every set $K \in \mathcal{K}$ and $Z \in [G]^{<\kappa}$, there is a set $K' \in \mathcal{K}$ with $K' \subseteq K \setminus Z$.*

If moreover $\mathcal{S}(\mathcal{K})$ is a σ-algebra that is closed under reflections, then the family of $\mathcal{S}(\mathcal{K})$-measurable functions does not have the difference property.

Proof. Note that property ($*$) is necessary and sufficient for $[G]^{<\kappa} \subseteq \mathcal{S}_0(\mathcal{K})$. So every almost-invariant set is also $\mathcal{S}_0(\mathcal{K})$-almost-invariant. The result then follows immediately from Theorem 2. □

We can also use Sierpiński's method to construct almost-invariant sets in $\mathcal{S}_0(\mathcal{K})$ for many families $\mathcal{K}$.

Theorem 6. *Suppose that $\mathcal{K} \subseteq [G]^\kappa$ is a family of size at most κ with property ($*$). If $\mathcal{K}$ is invariant under translations and no collection of fewer than κ sets from $\mathcal{K}$ cover G, then there is an almost-invariant set $T \in \mathcal{S}_0(\mathcal{K})$ with size κ.*

Proof. Write $\mathcal{K} = \{K_\alpha : \alpha < \kappa\}$. We will construct two sequences, $\{Q_\alpha : \alpha < \kappa\}$ and $\{x_\alpha : \alpha < \kappa\}$, which satisfy the following induction hypotheses:

1. $x_\alpha \neq x_\beta$ for $\alpha \neq \beta$,
2. $Q_\alpha \in \mathcal{K}$,
3. $Q_\alpha \subseteq K_\alpha$ for every $\alpha < \kappa$,
4. $(\bigcup_{\beta<\alpha} G_\beta + x_\beta) \cap \bigcup_{\beta<\alpha} Q_\beta = \varnothing$.

Let $\alpha < \kappa$ and suppose that we have already constructed Q_β and x_β for $\beta < \alpha$. First we show that we can find $x_\alpha \in G$ with

$$(G_\alpha + x_\alpha) \cap \Big(\bigcup_{\beta<\alpha} Q_\beta \cup K_\alpha \cup \{x_\beta : \beta < \alpha\}\Big) = \varnothing.$$

For the sake of contradiction, suppose that for every $x \in G$ we have

$$(G_\alpha + x) \cap \Big(\bigcup_{\beta<\alpha} Q_\beta \cup K_\alpha \cup \{x_\beta : \beta < \alpha\}\Big) \neq \varnothing.$$

Then
$$G = \bigcup_{g \in G_\alpha} \left(\left(\bigcup_{\beta<\alpha} Q_\beta \cup K_\alpha \cup \{x_\beta : \beta < \alpha\} \right) - g \right) = \bigcup \mathcal{F}$$
where
$$\mathcal{F} = \{P + x_\beta - g : \beta < \alpha, g \in G_\alpha\} \cup \{Q_\beta - g : \beta < \alpha, g \in G_\alpha\} \cup \{K_\alpha - g : g \in G_\alpha\}$$
and P is any element of $\mathcal{K}$ with $0 \in P$ (so $x \in P + x$ for every $x \in G$). Since $\mathcal{K}$ is invariant under translation, we have $\mathcal{F} \subseteq \mathcal{K}$ and $|\mathcal{F}| \leq (2|\alpha| + 1)|G_\alpha| < \kappa$ since $|G_\alpha| < \kappa$ by convention. This contradicts the fact that no collection of fewer than κ sets from $\mathcal{K}$ cover G so there must be an $x_\alpha \in G$ as claimed above.

Now it follows immediately from $(*)$ that there is a $Q_\alpha \in \mathcal{K}$ such that $Q_\alpha \subseteq K_\alpha$ and
$$Q_\alpha \cap \bigcup_{\beta \leq \alpha} (G_\beta + x_\beta) = \varnothing.$$
It is easy to see that this choice of Q_α, x_α satisfies our four induction hypotheses. Now let
$$T = \bigcup_{\alpha<\kappa} (G_\alpha + x_\alpha).$$
We will show that the set T is as required; i.e., $T \in \mathcal{S}_0(\mathcal{K})$ is an almost-invariant set of size κ.

Proposition 3 implies that T is almost-invariant and since x_α are distinct and $x_\alpha \in G_\alpha + x_\alpha$ we have $|T| = \kappa$.

To see that $T \in \mathcal{S}_0(\mathcal{K})$, fix any $K \in \mathcal{K}$ and let $\alpha < \kappa$ be such that $K = K_\alpha$. We show that $Q_\alpha \cap T = \varnothing$. Take any $\beta < \kappa$ and let $\delta = \max\{\alpha, \beta\} + 1$. By condition 4 we have
$$\left(\bigcup_{\gamma<\delta} (G_\gamma + x_\gamma) \right) \cap \left(\bigcup_{\gamma<\delta} Q_\gamma \right) = \varnothing$$
so $(G_\beta + x_\beta) \cap Q_\alpha = \varnothing$ as well. But the latter holds for every $\beta < \kappa$, hence $Q_\alpha \cap T = \varnothing$ as required. This shows that for every $K \in \mathcal{K}$ there is a $Q \in \mathcal{K}$ with $Q \subseteq K$ and $Q \cap T = \varnothing$ and hence $T \in \mathcal{S}_0(\mathcal{K})$ as required. □

As a corollary we get our main result regarding algebraic sums of sets in Marczewski–Burstin algebras.

Theorem 7. *Suppose that $\mathcal{K} \subseteq [G]^\kappa$ is a family of size at most κ with property $(*)$. If $\mathcal{K}$ is invariant under translations and reflections and no collection of fewer than κ sets from $\mathcal{K}$ cover G, then there is a set $A \in \mathcal{S}_0(\mathcal{K})$ such that $A + A$ is $\mathcal{K}$-Bernstein and hence not in $\mathcal{S}(\mathcal{K})$.*

Proof. Since $\mathcal{K}$ satisfies all the hypotheses of Theorem 6, let T be as in the conclusion of that theorem. Then the symmetric set $S = T \cup (-T) \in \mathcal{S}_0(\mathcal{K})$ is also almost-invariant since $\mathcal{K}$, and hence $\mathcal{S}_0(\mathcal{K})$, is invariant under reflections. The sets $(S+g) \cap S = (S+g) \cap (-S)$ for $g \in G$ necessarily have size κ for every $g \in G$ since $|(S+g)\Delta S| < \kappa$ and $|(S+g) \cup S| = \kappa$. By Theorem 1, there is a set $A \subseteq S$ (hence $A \in \mathcal{S}_0(\mathcal{K})$) such that $A + A$ is $\mathcal{K}$-Bernstein. □

3 Applications.

In this section, we apply our two main results about algebraic sums and the difference property to Marczewski and Miller measurable sets.

3.1 Marczewski Measurable Sets.

Let X be a Polish space. By a *perfect set* in X we mean a nonempty, closed subset of X without isolated points. The algebra of *Marczewski measurable* subsets of X is defined by $(s^X) = \mathcal{S}(\mathrm{Perf}_X)$ where Perf_X is the family of perfect subsets of X. The ideal of *Marczewski null* subsets of X is similarly defined by $(s_0^X) = \mathcal{S}_0(\mathrm{Perf}_X)$.

It is well known (*cf.* [13]) that (s^X) is a σ-algebra and that $(s_0^X) \subseteq (s^X)$ is a σ-ideal. This is a proper σ-ideal if and only if X is not σ-discrete; i.e., X is not a countable union of discrete subspaces. Moreover, we always have $[X]^{<\mathfrak{c}} \subseteq (s_0^X)$ since a perfect set can always be split into $\mathfrak{c}$ many disjoint perfect subsets.

If G is a perfect Abelian Polish group, then (s^G) and (s_0^G) are invariant under translations and reflections since these transformations are homeomorphisms.

Theorem 8. *If G is a perfect Abelian Polish group, then there is a Marczewski null set $A \subseteq G$ such that $A + A$ is not Marczewski measurable.*

Remark. Theorem 8 was proved later by Kysiak [11] using different methods.

The following easy lemma is key to the proofs of Theorem 8 and, later, for Theorem 11.

Lemma 9. *Every perfect Abelian Polish group G has a proper σ-compact subgroup H with $|H| = |G/H| = \mathfrak{c}$.*

Proof. A well-known theorem of Mycielski [14] says that we can always find a nonempty independent perfect set $P \subseteq G$. Choose a compact perfect set

$P_0 \subseteq P$ with $P_1 = P \setminus P_0$ of size $\mathfrak{c}$. The subgroup H generated by P_0 is σ-compact and $|H| = \mathfrak{c}$ since P_0 is perfect. Since P is independent, the elements of P_1 belong to different cosets in G/H and so $|G/H| = \mathfrak{c}$ also. □

PROOF OF THEOREM 8. Let H be as in Lemma 9 and let $\mathcal{K}$ be the family of all perfect sets $P \subseteq G$ such that either

- $P \subseteq H + g$ for some $g \in G$, or else
- $|P \cap (H + g)| \leq 1$ for all $g \in G$.

Clearly, the family $\mathcal{K}$ is invariant under translations and reflections, and $|\mathcal{K}| = \mathfrak{c}$. Therefore it suffices to verify that no collection of fewer than $\mathfrak{c}$ many sets from $\mathcal{K}$ can cover G and the result will follow from Theorem 7. Given $\mathcal{F} \in [\mathcal{K}]^{<\mathfrak{c}}$ we can always find a $g \in G$ such that $|P \cap (H + g)| \leq 1$ for all $P \in \mathcal{F}$. But then $|(H + g) \cap \bigcup \mathcal{F}| \leq |\mathcal{F}| < \mathfrak{c} = |H + g|$ and so $\bigcup \mathcal{F} \neq G$.

Finally we show that $\mathcal{K}$ is cofinal in Perf_G, from which it follows that $(s^G) = \mathcal{S}(\mathcal{K})$ and $(s_0^G) = \mathcal{S}_0(\mathcal{K})$. But first we recall a well-known result of Galvin [7] (or [8], Theorem 19.7), which says that if Q is a perfect Polish space and $c : [Q]^2 \to \{0, 1\}$ is Borel, then there is a perfect set $P \subseteq Q$ such that c is constant on $[P]^2$.

For a perfect set $Q \subseteq G$, let $c : [Q]^2 \to \{0, 1\}$ be given by $c\{x, y\} = 1$ iff $x - y \in H$. This is a Borel map since H is σ-compact, so by Galvin's Theorem there is a perfect set P such that c is constant on $[P]^2$. But c has constant value 1 iff $P \subseteq H + g$ for some $g \in G$, and c has constant value 0 iff $|P \cap (H + g)| \leq 1$ for all $g \in G$. So $P \in \mathcal{K}$ is a subset of Q as required. □

Since (s^G) is a σ-algebra, we obtain a strengthening of a result of Recław and the second author [6] as an immediate consequence of Theorem 5.

Theorem 10. *If G is a perfect Abelian Polish group, then the family of Marczewski measurable functions on G does not have the difference property.*

3.2 Miller Measurable Sets.

Miller measurability is defined in a similar way to Marczewski measurability. By a *superperfect set* we mean a nonempty, closed subset of X in which compact sets are nowhere dense; i.e., have empty interior. The algebra of *Miller measurable* subsets of X is defined by $(m^X) = \mathcal{S}(\text{Super}_X)$ where Super_X is the family of superperfect subsets of X. The ideal of *Miller null* subsets of X is similarly defined by $(m_0^X) = \mathcal{S}_0(\text{Super}_X)$.

Again, it is well known that (m^X) is a σ-algebra and that $(m_0^X) \subseteq (m^X)$ is a σ-ideal. This is a proper σ-ideal if and only if X is not σ-compact. Moreover, we always have $[X]^{<\mathfrak{c}} \subseteq (m_0^X)$ since a superperfect set can always be split into $\mathfrak{c}$ many disjoint superperfect subsets.

If G is a superperfect Abelian Polish group, then (m^G) and (m_0^G) are invariant under translations and reflections since these transformations are homeomorphisms.

Theorem 11. *If G is a superperfect Abelian Polish group, then there is a Miller null set $A \subseteq G$ such that $A + A$ is not Miller measurable.*

Proof. Let H be as in Lemma 9 and let $\mathcal{K}$ be the family of all superperfect sets $S \subseteq G$ such that $|S \cap (H + g)| \leq 1$ for all $g \in G$. Clearly, this family is invariant under translations, $|\mathcal{K}| = \mathfrak{c}$, and no collection of fewer than $\mathfrak{c}$ many elements of $\mathcal{K}$ can cover G (or even H). Therefore the family $\mathcal{K}$ satisfies the assumptions of Theorem 7.

To finish we show that the family $\mathcal{K}$ is cofinal in Super_G, from which it follows that $(s^G) = \mathcal{S}(\mathcal{K})$ and $(s_0^G) = \mathcal{S}_0(\mathcal{K})$. To do this we appeal to a recent result of Spinas [17], which is a generalization to superperfect sets of the result of Galvin that we used in the proof of Theorem 8: if T is a superperfect Polish space and $c : [T]^2 \to \{0, 1\}$ is Borel, then there is a superperfect set $S \subseteq T$ such that c is constant on $[S]^2$.

For a superperfect set $T \subseteq G$, let $c : [T]^2 \to \{0, 1\}$ be given by $c\{x, y\} = 1$ iff $x - y \in H$. This is a Borel map since H is σ-compact, so by Spinas' Theorem there is a superperfect set $S \subseteq T$ such that c is constant on $[S]^2$. Now c cannot have constant value 1 on $[S]^2$ for then we would have $S \subseteq H + g$ for some $g \in G$, which is impossible since $H + g$ is σ-compact by definition. So c must have constant value 0 on $[S]^2$, which means that $|S \cap (H + g)| \leq 1$ for all $g \in G$. Hence $S \in \mathcal{K}$ is a subset of T as required. $\square$

Also, since (m^G) is a σ-algebra, the following result follows immediately from Theorem 5.

Theorem 12. *If G is a superperfect Abelian Polish group, then the family of Miller measurable functions on G does not have the difference property.*

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