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STABILITY OF THE CAUCHY FUNCTIONAL EQUATION OVER p-ADIC FIELDS

Dedicated to the memory of S. M. Ulam.

Abstract

This paper corrects some errors found in [2], which discusses an extension of Lorentz transformations over a non-Archimedean valued field; namely, the p-adic field $\mathbb{Q}_p$. The paper [2] is based on the results given by Hyers [7] which showed that for a continuous function f defined on $\mathbb{R}$, the Cauchy functional equation $f(x+y) = f(x) + f(y)$ is stable. By stable we mean that if there exists $\epsilon > 0$ such that $\|f(x+y) - f(x) - f(y)\| < \epsilon$, $\forall x, y$, then there exists a unique and continuous $\mathcal{L}$ such that $\|\mathcal{L}(x) - f(x)\| \leq \epsilon$, $\forall x$ and $\mathcal{L}(x+y) = \mathcal{L}(x) + \mathcal{L}(y)$. In this paper, we show this result is true on the p-adic field $\mathbb{Q}_p$.

1 Introduction.

In 1908, K. Hensel [6] introduced the concept of p-adic numbers as a tool for solving problems in algebra and number theory. Specifically, his idea was to extend the analogies between the ring of integers $\mathbb{Z}$ and the field of rational numbers $\mathbb{Q}$ to the field of rational functions and Laurent series. The way this was accomplished was by expressing any rational number $x \in \mathbb{Q}$ as the sum

$$x = \sum_{n \geq n_0}^{\infty} a_n p^n,$$

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where $n_0 \in \mathbb{Z}$, p is a prime number, and $a_n \in \mathbb{Z}$ where $a_n \leq p-1$. For a fixed value of p, we denote the complete field of p-adic numbers as $\mathbb{Q}_p$ [5] .

In 1941, Hyers [7] showed that if a continuous function was "nearly" linear; that is, there exists an $\epsilon > 0$ such that $\|f(x+y) - f(x) - f(y)\| < \epsilon$, $\forall x, y \in \mathbb{R}$, then there exists a unique and continuous $\mathcal{L}$ such that $\|\mathcal{L}(x) - f(x)\| \leq \epsilon$, and $\mathcal{L}(x+y) = \mathcal{L}(x) + \mathcal{L}(y)$. Interest in this problem arose from the question of the stability of the Cauchy functional equation $f(x+y) = f(x) + f(y)$. For a comprehensive survey of the origins and evolution of this problem see [4, 8, 11, 12].

Later, Everett and Ulam [3] presented results on generalizing Lorentz groups over p-adic fields. More recently, p-adic fields have become of considerable interest to physicists. A key property of p-adic fields is that they do not satisfy the Archimedean axiom; $\forall a, b > 0$, there exists an integer n, such that $a < nb$. This property has been found to be useful in theoretical physics. In quantum mechanics [10, 9] it has long been recognized that fundamental limitations on measuring conjugate quantities such as position-momentum or energy-time exist because of the Heisenberg uncertainty principle. For example, any attempt at taking gravitational measurements at sub-Planck domains, say of the order of $l = 10^{-35}$m, would change the underlying geometry and introduce distortions to l. Introducing a p-adic space-time could provide a means of quantifying the non-localization affects.

In this paper, we correct the mistakes in the proof given in [2]; namely, we prove the stability of the functional equation $f(x+y) = f(x) + f(y)$, where x, y are in the p-adic field $\mathbb{Q}_p$.

2 Basic Definitions.

In this section, we recall some definitions and results that will be needed later when discussing ϵ-linear transformations over p-adic fields.

Definition 1 (Non-Archimedean Valuation)**.** Let $\mathbb{K}$ denote a scalar field, and let $|\cdot|$ denote the usual absolute value (valuation) where $|\cdot| : \mathbb{K} \to \mathbb{R}$. A non-Archimedean valuation is a function $|\cdot|_p$ that satisfies the strong triangle inequality; namely,

$$|x+y|_p \leq \max\{|x|_p, |y|_p\} \leq |x|_p + |y|_p,$$

$\forall x, y \in \mathbb{K}$. The associated field $\mathbb{K}$ is referred to as a non-Archimedean field.

Lemma 1 (**p**-adic Valuation)**.** *Given any nonzero rational number $x \in \mathbb{Q}$, there exists a unique integer $n \in \mathbb{Z}$, such that $x = \frac{a}{b} p^n$, where a and b are integers not divisible by p [5]. The p-adic valuation is defined as $|x|_p := p^{-n}$.*

Definition 2 (**p**-adic Field)**.** For each prime p, define the p-adic field $\mathbb{Q}_p$ to be the set of all p-adic expansions

$$\mathbb{Q}_p := \Big\{x | x = \sum_{k \geq n_0}^{\infty} a_k p^k\Big\},$$

where $a_k \leq p-1$ are integers.

Definition 3 (ϵ-Linear Function)**.** The function $f : \mathbb{Q}_p \to \mathbb{R}$ is said to be ϵ-linear if there exists an $\epsilon \in \mathbb{R}^+$, such that

$$\|f(x+y) - f(x) - f(y)\| < \epsilon, \ \forall x, y \in \mathbb{Q}_p. \tag{1}$$

3 ϵ-Linear Functions.

The results of this section will show that given a continuous, ϵ-linear function $f : \mathbb{Q}_p \to \mathbb{R}$, there exists a unique, additive and continuous function $\mathcal{L} : \mathbb{Q}_p \to \mathbb{R}$ such that

$$\|\mathcal{L}(x) - f(x)\| \leq \epsilon, \ \forall x \in \mathbb{Q}_p.$$

Lemma 2. *Every ϵ-linear function f satisfies the inequality*

$$\|f(mx) - mf(x)\| \leq (m-1)\epsilon, \ \forall x \in \mathbb{Q}_p, \forall m \in \mathbb{Z}. \tag{2}$$

Furthermore,

$$\|f(p^{-n}x) - p^{-n}f(x)\| \leq (1 - p^{-n})\epsilon, \ \forall x \in \mathbb{Q}_p, \forall n \in \mathbb{Z}, \tag{3}$$

where p prime.

PROOF. Since the proof for $m \in \mathbb{Z}^-$ is the same as for $m \in \mathbb{N}$, we only need to show the latter case. Substituting $y = x$ into inequality (1) gives the result for $m = 2$.

$$\|f(2x) - f(x) - f(x)\| = \|f(2x) - 2f(x)\| < (2-1)\epsilon.$$

Using induction, assume (2) is true for some m. For the $m+1$ case we have

$$\begin{aligned}\|f((m+1)x) - (m+1)f(x)\| =& \|f((m+1)x) - f(mx) \\ &+ f(mx) - mf(x) - f(x)\| \\ \leq& \|f((m+1)x) - f(mx) - f(x)\| \\ &+ \|f(mx) - mf(x)\| \\ \leq& \epsilon + (m-1)\epsilon = m\epsilon.\end{aligned}$$

To show that f satisfies inequality (3), substitute $m = p^n$ and replace x with $p^{-n}x$ in inequality (2) and divide by p^n to obtain the stated result. $\square$

Lemma 3. *The sequence* $\{q_k(x)\}_{k=1}^{\infty}$, *where* $q_k(x) := p^{-k} f(p^k x)$, $x \in \mathbb{Q}_p$ *is a Cauchy sequence.*

Proof. Let $l \geq k$, where $l, k \in \mathbb{N}$. Consider the difference

$$\begin{aligned} \|q_k(x) - q_l(x)\| =& \|p^{-k} f(p^k x) - p^{-l} f(p^l x)\| \\ =& p^{-k} \|f(p^{k-l} p^l x) - p^{k-l} f(p^l x)\|. \end{aligned}$$

Using inequality (3), this difference can be made arbitrarily small because

$$\|q_k(x) - q_l(x)\| = p^{-k} \|f(p^{k-l} p^l x) - p^{k-l} f(p^l x)\| \leq p^{-k}(1 - p^{k-l})\epsilon.$$

Hence $\{q_k(x)\}$ is a Cauchy sequence. $\square$

Lemma 4. *Let*

$$\mathcal{L}(x) := \lim_{k \to \infty} q_k(x). \tag{4}$$

Then there exists an $\epsilon > 0$, *such that* $\forall x \in \mathbb{Q}_p$, $\|\mathcal{L}(x) - f(x)\| \leq \epsilon$.

Proof. In inequality (3) make the replacement $x \leftrightarrow p^n x$ to get

$$\|f(x) - p^{-n} f(p^n x)\| \leq (1 - p^{-n})\epsilon.$$

Taking the limit as $n \to \infty$ gives the desired result. $\square$

We now show that the function $\mathcal{L}$ is unique, additive, and assuming f continuous, $\mathcal{L}$ is also continuous.

Lemma 5. *For* $n \in \mathbb{Z}$, $\mathcal{L}(nx) = n\mathcal{L}(x)$.

Proof.

$$\begin{aligned} \|p^{-k} f(p^k n x) - n p^{-k} f(p^k x)\| =& p^{-k} \|f(n p^k x) - n f(p^k x)\| \\ \leq& p^{-k}(n-1)\epsilon, \end{aligned}$$

in which case

$$\begin{aligned}\|\mathcal{L}(nx) - n\mathcal{L}(x)\| &= \| \lim_{k\to\infty} p^{-k} f(p^k nx) - \lim_{k\to\infty} np^{-k} f(p^k x)\| \\ &= \lim_{k\to\infty} p^{-k} \| f(p^k nx) - np^{-k} f(p^k x)\| \\ &\leq \lim_{k\to\infty} p^{-k}(n-1)\epsilon = 0,\end{aligned}$$

and therefore $\mathcal{L}(nx) = n\mathcal{L}(x)$. □

This result will be needed in order to show that the function $\mathcal{L}$ is unique. The next theorem will prove this fact by contradiction.

Theorem 1. *The limiting function $\mathcal{L}$ is unique.*

PROOF. Seeking a contradiction, suppose that the limiting function $\mathcal{L}$ is not unique; i.e., there exists an $\hat{\mathcal{L}}$, such that

$$\|\hat{\mathcal{L}}(x) - f(x)\| \leq \epsilon \; \forall x \in \mathbb{Q}_p;$$

however, for some $a \in \mathbb{Q}_p$, $\hat{\mathcal{L}}(a) \neq \mathcal{L}(a)$. Since $\|\hat{\mathcal{L}}(a) - \mathcal{L}(a)\| \neq 0$ and ϵ specified, choose the smallest $n \in \mathbb{N}$ such that

$$n > \frac{3\epsilon}{\|\hat{\mathcal{L}}(a) - \mathcal{L}(a)\|},$$

in which case $\|\hat{\mathcal{L}}(na) - \mathcal{L}(na)\| > 3\epsilon$. Furthermore,

$$\|\hat{\mathcal{L}}(na) - \mathcal{L}(na)\| \leq \|\hat{\mathcal{L}}(na) - f(na)\| + \|\mathcal{L}(na) - f(na)\| \leq 2\epsilon,$$

in which case $3\epsilon < \|\hat{\mathcal{L}}(na) - \mathcal{L}(na)\| \leq 2\epsilon$, which is clearly a contradiction. Therefore the limiting function $\mathcal{L}$ is unique. □

Lemma 6. *The limiting function $\mathcal{L}$ is additive; that is,*

$$\mathcal{L}(x+y) = \mathcal{L}(x) + \mathcal{L}(y), \; \forall x, y \in \mathbb{Q}_p.$$

PROOF. Since f is ϵ-linear, $\forall x, y \in \mathbb{Q}_p$ and $k \in \mathbb{N}$,

$$\|f(p^k x + p^k y) - f(p^k x) - f(p^k y)\| < \epsilon.$$

Multiplying by p^{-k} and taking the limit as $k \to \infty$ gives

$$\begin{aligned}\|\mathcal{L}(x+y)-\mathcal{L}(x)-\mathcal{L}(y)\| &= \lim_{k\to\infty} \|p^{-k}f(p^kx+p^ky) \\ &\quad - p^{-k}f(p^kx)-p^{-k}f(p^ky)\| \\ &\le \lim_{k\to\infty} p^{-k}\epsilon = 0.\end{aligned}$$

□

Theorem 2. *If f is continuous on $\mathbb{Q}_p$, then $\mathcal{L}$ is continuous on $\mathbb{Q}_p$.*

Proof. Seeking a contradiction, suppose that $\mathcal{L}$ is not continuous at some point $a \in \mathbb{Q}_p$. Specifically, assume there exists a sequence $\{x_n\}_{n=0}^{\infty}$, such that $x_n \to a$ but $\lim\limits_{n\to\infty} \mathcal{L}(x_n) \neq \mathcal{L}(a)$. By the additivity property,

$$\mathcal{L}(x_n - a) = \mathcal{L}(x_n) - \mathcal{L}(a),$$

in which case

$$\lim_{n\to\infty} \mathcal{L}(x_n - a) = \lim_{n\to\infty} \left(\mathcal{L}(x_n) - \mathcal{L}(a)\right).$$

Using the limit definition of $\mathcal{L}$ given in (4) we obtain

$$\begin{aligned}\lim_{n\to\infty} \mathcal{L}(x_n - a) &= \lim_{n\to\infty}\lim_{k\to\infty} p^{-k} f\left(p^k(x_n-a)\right) \\ &= \lim_{k\to\infty} p^{-k} \lim_{n\to\infty} f\left(p^k(x_n-a)\right).\end{aligned}$$

Since f is everywhere continuous on $\mathbb{Q}_p$ and $x_n \to a$,

$$\lim_{n\to\infty} f\left(p^k(x_n-a)\right) = f(0),$$

in which case

$$\lim_{n\to\infty} \mathcal{L}(x_n - a) = \lim_{n\to\infty} \left(\mathcal{L}(x_n) - \mathcal{L}(a)\right) = \mathcal{L}(0).$$

Using the additivity property $\mathcal{L}(0+0) = \mathcal{L}(0) + \mathcal{L}(0)$, which implies that $\mathcal{L}(0) = 0$. This produces the contradiction

$$\lim_{n\to\infty} \mathcal{L}(x_n) - \mathcal{L}(a) = 0,$$

and therefore $\mathcal{L}$ is continuous on $\mathbb{Q}_p$. □

4 Summary and Future Directions.

We have shown that if $f : \mathbb{Q}_p \to \mathbb{R}$ is ϵ-linear, then there exists a unique, continuous and additive function $\mathcal{L}$, such that

$$\|\mathcal{L}(x) - f(x)\| \leq \epsilon, \ \forall x \in \mathbb{Q}_p,$$

where additivity means $\mathcal{L}(x+y) = \mathcal{L}(x) + \mathcal{L}(y)$. Future work will be to determine the stability of the logarithmic functional equation $f(xy) = f(x) + f(y)$, and the exponential functional equation $f(x+y) = f(x)f(y)$ over the p-adic field.

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