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NEW TESTS FOR POSITIVE ITERATION SERIES

Abstract

We give new tests for series whose terms are defined by iteration of a positive map. Several known results are generalized and unified.

1 Introduction.

Let $f :]0,\infty[\to]0,\infty[$ be a map, such that $f(x) < x$ for every $x > 0$. For each $n \in \mathbb{N}$, let $f^{[n]}$ denote the nth iterate of f, that is,

$$f^{[0]} = \mathrm{id}_{]0,\infty[}, \quad f^{[n+1]} = f \circ f^{[n]} \text{ for every } n \in \mathbb{N}.$$

A standard question is: when the functions series $\sum_{n\geq 0} f^{[n]}$ converges pointwise? The ratio test decides on the matter whenever $\limsup_{x\searrow 0} \frac{f(x)}{x} < 1$, but the study is much more delicate if this upper limit equals the unit. Obviously, for every $x_0 > 0$ the sequence $(x_n)_{n\in\mathbb{N}} = (f^{[n]}(x_0))_{n\in\mathbb{N}}$ is strictly decreasing. A necessary condition for pointwise convergence of $\sum_{n\geq 0} f^{[n]}$ is

$$\lim_{n\to\infty} f^{[n]}(x) = 0 \text{ for every } x > 0. \tag{1}$$

Note that (1) holds whenever f is continuous to the right on $]0,\infty]$. Let us briefly recall several known related results, all involving the auxiliary map

$$\omega_f :]0,\infty[\to]0,\infty[, \quad \omega_f(x) = \frac{x}{x - f(x)}.$$

Assume one of the following conditions holds for some $a > 0$:

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1. ω_f is decreasing on $]0,a]$ (Altman[1]),
2. $]0,a] \ni x \mapsto \frac{f(x)}{x} \in]0,1[$ is decreasing (equivalent to Altman's condition) and $f|_{]0,a]}$ is differentiable, with $\inf_{x\in]0,a]} f'(x) > 0$ (Fort and Schuster[6]),
3. $f|_{]0,a]}$ is Lipschitzian and increasing, with $\lim_{x\searrow 0} \frac{f(x)}{x} = 1$ (Švarcman[7]),
4. $f|_{]0,a]}$ is Lipschitzian (Brauer[2, 3]).

If in addition

$$\int_0^a \omega_f(t)\mathrm{d}t < \infty, \tag{2}$$

then $\sum_{n\geq 0} f^{[n]}$ converges pointwise on $]0,a]$. Convergence and (2) are equivalent under the more restrictive assumptions of the Fort-Schuster and Švarcman tests. Nevertheless, none of these applies to the *pointwise convergent* series defined by

$$f(x) = x - \frac{x\sqrt{x}}{2 + x + \sin(\pi/x)} \tag{3}$$

(see Example 10), since for every $a > 0$ we have $\inf_{x\in]0,a]} f'(x) = -\infty$ and ω_f is not monotone on $]0,a]$. For such problems, local conditions on f (near 0) seem to be more natural than monotony or Lipschitz restrictions if (1) holds.

Our main result (Theorem 1) provides a test without monotony or Lipschitz conditions. Its corollaries from Section 3.1 generalize the above cited results. Section 3.2 presents a sequence of tests with local conditions (upper/lower limits) and of strictly increasing strength. Examples are discussed in the last section.

2 A General Principle.

Let $f:]0,\infty[\to]0,\infty[$ be a map, such that $f(x) < x$ for every $x > 0$. Let us also consider a map $\varphi \in L^1_{\mathrm{loc}}(]0,\infty[)$ (locally Lebesgue integrable), with $\varphi > 0$.

Theorem 1 (φ-test). (i) $\sum_{n\geq 0} f^{[n]}$ *converges pointwise if* (1) *holds and*

$$\int_0^1 t\varphi(t)\mathrm{d}t < \infty \quad \textit{and} \quad \liminf_{x\searrow 0} \int_{f(x)}^x \varphi(t)\mathrm{d}t > 0.$$

(ii) $\sum_{n\geq 0} f^{[n]}$ *diverges everywhere if*

$$\int_0^1 t\varphi(t)\mathrm{d}t = \infty \quad \textit{and} \quad \limsup_{x\searrow 0} \int_{f(x)}^x \varphi(t)\mathrm{d}t < \infty.$$

PROOF. Since $\varphi \in L^1_{\rm loc}(]0,\infty[)$, we can define the map

$$\Phi :]0,\infty[\to \mathbb{R}, \quad \Phi(t) = -\int_1^x \varphi(t)\mathrm{d}t.$$

Let us note that Φ is strictly decreasing and locally absolutely continuous, with $\Phi' = -\varphi$ almost everywhere. Its range $J := \Phi(]0,\infty[)$ is an open interval containing 0, and its inverse $\Phi^{-1} : J \to]0,\infty[$ is a decreasing homeomorphism. Fix $x_0 > 0$ and $(x_n)_{n\in\mathbb{N}} = (f^{[n]}(x_0))_{n\in\mathbb{N}}$. If $\lim_{n\to\infty} x_n = 0$, then applying the Stolz-Cesàro theorem to the sequences $(\Phi(x_n))_{n\geq 1}$ and $(n)_{n\geq 1}$ yields

$$\liminf_{n\to\infty} \frac{\Phi(x_n)}{n} \geq \liminf_{n\to\infty} \int_{f(x_n)}^{x_n} \varphi(t)\mathrm{d}t \geq \liminf_{x\searrow 0} \int_{f(x)}^{x} \varphi(t)\mathrm{d}t, \tag{4}$$

$$\limsup_{n\to\infty} \frac{\Phi(x_n)}{n} \leq \limsup_{n\to\infty} \int_{f(x_n)}^{x_n} \varphi(t)\mathrm{d}t \leq \limsup_{x\searrow 0} \int_{f(x)}^{x} \varphi(t)\mathrm{d}t. \tag{5}$$

(i). Assume that (1) holds, and that $\liminf_{x\searrow 0} \int_{f(x)}^x \varphi(t)\mathrm{d}t > 0$. By (4), there exists $\lambda \in]0,\infty[$, such that $\frac{\Phi(x_n)}{n} > \lambda$ for sufficiently large n (that is, for $n \geq n_0 \in \mathbb{N}^*$). For such n we have $\Phi(x_n) > \lambda n > 0 = \Phi(1)$ (hence $\lim_{x\searrow 0}\Phi(x) = \infty$), and consequently $\lambda n \in J$ and $x_n < \Phi^{-1}(\lambda n)$. The assertion follows if we prove that the series $\sum_{n\geq n_0} \Phi^{-1}(\lambda n)$ converges. According to the integral criterion, this is equivalent to $\int_0^\infty \Phi^{-1}(\lambda s)\mathrm{d}s < \infty$. An easy computation gives for this integral

$$\int_0^\infty \Phi^{-1}(\lambda s)\mathrm{d}s = \lim_{x\searrow 0} \int_0^{\frac{\Phi(x)}{\lambda}} \Phi^{-1}(\lambda s)\mathrm{d}s = \frac{1}{\lambda}\lim_{x\searrow 0}\int_x^1 t\varphi(t)\mathrm{d}t = \frac{1}{\lambda}\int_0^1 t\varphi(t)\mathrm{d}t. \tag{6}$$

The conclusion is now evident.

(ii). Assume that $\limsup_{x\searrow 0} \int_{f(x)}^x \varphi(t)\mathrm{d}t < \infty = \int_0^1 t\varphi(t)\mathrm{d}t$. We also can assume that $\lim_{n\to\infty} x_n = 0$, since otherwise $\sum_{n\geq 0} x_n$ diverges. We have $[0,\infty[\subset J$, since $\lim_{x\searrow 0}\Phi(x) = \int_0^1 \varphi(t)\mathrm{d}t \geq \int_0^1 t\varphi(t)\mathrm{d}t = \infty$. By (5), there exists $\lambda \in]0,\infty[$, such that $\frac{\Phi(x_n)}{n} < \lambda$ for sufficiently large n. For such n we have $\Phi(x_n) < \lambda n$, and consequently $x_n > \Phi^{-1}(\lambda n)$. Now using again the integral test together with (6) shows that the series $\sum_{n\geq 0} x_n$ diverges. $\square$

Remark 2. **(a)** In Theorem 1 we can replace $\int_0^1 t\varphi(t)\mathrm{d}t$ by $\int_0^a t\varphi(t)\mathrm{d}t$, for any $a > 0$. All our results also work for maps $f :]0,a] \to]0,a]$ and $\varphi \in L^1_{\rm loc}(]0,a])$ with similar properties.

(b) The above proof is based on a comparison of the series $\sum_{n\geq 0} f^{[n]}(x_0)$ with the integral $\int_0^1 t\varphi(t)\mathrm{d}t = \int_1^\infty \frac{\varphi(1/t)}{t^3}\mathrm{d}t$. The same integral characterizes the nature of the series $\sum_{n\geq 1} \frac{\varphi(1/n)}{n^3}$, if the latter integrand is monotone for large $t \geq 1$. In Section 3.2 this will allow for an interpretation of the p-test.

3 Consequences.

3.1 Integral Tests.

In Theorem 1, the connection between the given map f and the integrand $\varphi \in L^1_{\mathrm{loc}}(]0,\infty[)$ is ensured by the limits (upper and lower) of $\int_{f(x)}^x \varphi(t)\mathrm{d}t$. The results of this section are based on the choice $\varphi(x) = \frac{1}{x-f(x)}$, which gives good chances for suitable limits. Other possible choices will be explored in Section 3.2. Our first corollary generalizes both Altman and Fort-Schuster tests.

Corollary 3 (monotone test). *Let $u :]0,\infty[\to]0,\infty[$ be a monotone map, with $\liminf_{x\searrow 0} \frac{u(x)}{u(2x)} > 0$. Assume the auxiliary map*

$$\Omega_{f,u} :]0,\infty[\to]0,\infty[, \quad \Omega_{f,u}(x) = u(x)(x - f(x))$$

to be increasing. Then

$$\int_0^1 \frac{t}{t - f(t)}\mathrm{d}t < \infty \Longrightarrow \sum_{n\geq 0} f^{[n]} \text{ converges pointwise.} \tag{7}$$

If $f|_{]0,a]}$ is differentiable for some $a > 0$ and if $\inf_{x\in]0,a]} f'(x) > 0$, then

$$\int_0^1 \frac{t}{t - f(t)}\mathrm{d}t = \infty \Longrightarrow \sum_{n\geq 0} f^{[n]} \text{ diverges everywhere.} \tag{8}$$

PROOF. If u is decreasing, then the map $]0,\infty[\ni x \mapsto \frac{\Omega_{f,u}(x)}{u(x)} = x - f(x) \in]0,\infty[$ is increasing, and the problem reduces to the case when u is constant. Thus we can assume u to be increasing. For abbreviation, we write Ω instead of $\Omega_{f,u}$. We first show that (1) holds. Let $x_0 > 0$ and the strictly decreasing sequence $(x_n)_{n\in\mathbb{N}} = (f^{[n]}(x_0))_{n\in\mathbb{N}}$. If $\xi := \lim_{n\to\infty} x_n > 0$, we get

$$0 < \Omega(\xi) \leq \lim_{n\to\infty} \Omega(x_n) = \lim_{n\to\infty} u(x_n)(x_n - x_{n+1}) \leq u(x_0) \cdot 0 = 0, \tag{9}$$

a contradiction. We conclude that $\xi = 0$, and consequently that (1) holds. Let us define the map $\varphi = \frac{u}{\Omega} :]0,\infty[\to]0,\infty[$. As Ω and u are increasing, we deduce that $\varphi \in L^1_{\mathrm{loc}}([0,\infty[)$. Let us note that $\int_0^1 t\varphi(t)\mathrm{d}t = \int_0^1 \frac{t}{t-f(t)}\mathrm{d}t$. For $x > 0$ and $\bar{x} := \max\{f(x), \frac{x}{2}\}$ we have $x - f(x) < 2(x - \bar{x})$, and so

$$\int_{f(x)}^{x} \varphi(t)\mathrm{d}t \geq \int_{\bar{x}}^{x} \frac{u(t)}{\Omega(t)}\mathrm{d}t \geq \frac{u(\bar{x})(x-\bar{x})}{\Omega(x)} \geq \frac{u(x/2)}{u(x)} \cdot \frac{x-\bar{x}}{x-f(x)} \geq \frac{u(x/2)}{2u(x)}.$$

Hence $\liminf_{x\searrow 0} \int_{f(x)}^{x} \varphi(t)\mathrm{d}t > 0$. Thus (7) follows by applying Theorem 1(i). Now assume $f|_{]0,a]}$ to be differentiable, with $\inf_{x\in\,]0,a]} f'(x) > \frac{1}{2^p}$ for some $a > 0$ and some $p \in \mathbb{N}$. For $x \in]0,a]$ we have $f(x) \geq \frac{x}{2^p}$, and so

$$\int_{f(x)}^{x} \varphi(t)\mathrm{d}t \leq \frac{u(x)}{\Omega(f(x))}(x - f(x)) \leq \frac{u(x)}{u(x/2^p)} \cdot \frac{x-f(x)}{f(x)-f(f(x))} \leq \frac{2^p u(x)}{u(x/2^p)}.$$

Hence $\limsup_{x\searrow 0} \int_{f(x)}^{x} \varphi(t)\mathrm{d}t \leq 2^p \left(\liminf_{x\searrow 0} \frac{u(x)}{u(2x)}\right)^{-p} < \infty$. Thus (8) follows by applying Theorem 1(ii). □

Altman's result follows from the above corollary by taking $u(x) = \frac{1}{x}$. Stronger tests are obtained for increasing u, since $\Omega_{f,u}$ is increasing whenever ω_f is decreasing (Altman's hypothesis). Our next corollary generalizes Brauer's test.

Corollary 4 (Lipschitz test). *Let $u, v :]0,\infty[\to]0,\infty[$ be continuous maps, such that u and the auxiliary map*

$$\Omega_{f,u,v} :]0,\infty[\to\mathbb{R}, \quad \Omega_{f,u,v}(x) = u(x)(v(x) - f(x))$$

are both increasing. Assume that on some $]0,a]$ the following two conditions hold:

(u) *u is convex or concave, with $\liminf_{x\searrow 0} \frac{u(x)}{u(2x)} > 0$, or alternatively, u is Lipschitzian, with $\liminf_{x\searrow 0} \frac{u(x)}{x} > 0$,*

(v) *v is Lipschitzian, with $\lim_{x\searrow 0} v(x) = 0$.*

If $\int_0^a \frac{t}{t-f(t)}\mathrm{d}t < \infty$, then $\sum_{n\geq 0} f^{[n]}$ converges pointwise.

Proof. For abbreviation, we write Ω instead of $\Omega_{f,u,v}$. Since Ω is increasing and $\lim_{x\searrow 0} \Omega(x) = 0$, we have $\Omega \geq 0$. As in the proof of Corollary 3, with the argument from (9) replaced by

$$\Omega(\xi) \leq \lim_{n\to\infty} \Omega(x_n) = \lim_{n\to\infty} u(x_n)(v(x_n) - x_{n+1}) = u(\xi)(v(\xi) - \xi) < \Omega(\xi),$$

we deduce that (1) holds. Let us define the map

$$\varphi :]0,\infty[\to]0,\infty[, \quad \varphi(x) = \begin{cases} \frac{1}{x-f(x)}, & \text{if } x \le a, \\ \frac{1}{a-f(a)}, & \text{if } x \ge a. \end{cases}$$

On $]0,a]$ we have $\varphi(x) = \frac{u(x)}{\Omega(x)-u(x)w(x)}$, where $w(x) := v(x) - x$. As Ω is increasing, we deduce that φ is measurable. That $\varphi \in L^1_{\mathrm{loc}}([0,\infty[)$ follows from $\int_0^a t\varphi(t)\mathrm{d}t < \infty$. Fix $x \in]0,a]$ and $\bar{x} := \max\{f(x), \frac{x}{2}\}$. For every $t \in [\bar{x}, x[$, we have

$$\frac{1}{\varphi(t)} = \frac{\Omega(t)}{u(t)} - w(t) \le \frac{\Omega(x)}{u(\bar{x})} - w(t) = \frac{u(x)(x-f(x)) + u(x)w(x) - u(\bar{x})w(t)}{u(\bar{x})}.$$

Let $\lambda > 0$ be a Lipschitz constant for $w|_{]0,a]}$. Thus

$$\begin{aligned} u(x)w(x) - u(\bar{x})w(t) &= u(x)(w(x) - w(t)) + w(t)(u(x) - u(\bar{x})) \\ &\le \lambda(x-t)u(x) + \lambda t(u(x) - u(\bar{x})) \le \lambda(x-\bar{x})(u(x) + x\delta_u(x,\bar{x})), \end{aligned}$$

where $\delta_u(y,z) := \frac{u(y)-u(z)}{y-z}$ for $y \ne z$. Since $x - f(x) < 2(x - \bar{x})$, it follows that

$$\frac{1}{\varphi(t)} \le \frac{(x-\bar{x})[(2+\lambda)u(x) + \lambda x\delta_u(x,\bar{x})]}{u(x/2)} \quad \text{for every } t \in [\bar{x}, x[,$$

$$\int_{f(x)}^{x} \varphi(t)\mathrm{d}t \ge \int_{\bar{x}}^{x} \varphi(t)\mathrm{d}t \ge \frac{u(x/2)}{u(x)} \cdot \frac{1}{2+\lambda+\lambda\frac{x\delta_u(x,\bar{x})}{u(x)}} =: E(x).$$

As $\int_0^a t\varphi(t)\mathrm{d}t < \infty$, the conclusion will follow by Theorem 1(i) if we prove that $\liminf_{x\searrow 0} E(x) > 0$. Set $\widetilde{u}(0) := \lim_{x\searrow 0} u(x) \ge 0$. According to the properties of $u|_{]0,a]}$, we shall analyze three cases.

Case 1. Assume $u|_{]0,a]}$ to be μ-Lipschitzian ($\mu > 0$), with $\liminf_{x\searrow 0} \frac{u(x)}{x} > 0$. Then

$$\limsup_{x\searrow 0} \frac{x\delta_u(x,\bar{x})}{u(x)} \le \limsup_{x\searrow 0} \frac{\mu x}{u(x)} = \frac{\mu}{\liminf_{x\searrow 0} \frac{u(x)}{x}} < \infty.$$

We also have $\liminf_{x\searrow 0} \frac{u(x)}{u(2x)} > 0$. Indeed, if $\widetilde{u}(0) = 0$ (otherwise the claimed property is obvious), then

$$\liminf_{x\searrow 0} \frac{u(x)}{u(2x)} = \liminf_{x\searrow 0} \frac{u(x)}{u(2x) - \widetilde{u}(0)} \ge \liminf_{x\searrow 0} \frac{u(x)}{\mu \cdot 2x} = \frac{1}{2\mu} \liminf_{x\searrow 0} \frac{u(x)}{x} > 0.$$

We conclude that $\liminf_{x\searrow 0} E(x) > 0$.
Case 2. Assume $u|_{]0,a]}$ to be convex, with $\liminf_{x\searrow 0} \frac{u(x)}{u(2x)} > 0$. Convexity yields $\delta_u(x,\bar{x}) \le \delta_u(2x,x) = \frac{u(2x)-u(x)}{x}$ for $x \in]0,\frac{a}{2}]$, which leads to

$$\limsup_{x\searrow 0} \frac{x\delta_u(x,\bar{x})}{u(x)} \le \limsup_{x\searrow 0} \frac{x\delta_u(2x,x)}{u(x)} = \frac{1}{\liminf_{x\searrow 0} \frac{u(x)}{u(2x)}} - 1 < \infty.$$

We conclude that $\liminf_{x\searrow 0} E(x) > 0$.
Case 3. Assume $u|_{]0,a]}$ to be concave. Then $2u(x) \ge u(2x) + \widetilde{u}(0) \ge u(2x)$ for $x \in]0,\frac{a}{2}]$, and consequently $\liminf_{x\searrow 0} \frac{u(x)}{u(2x)} \ge \frac{1}{2} > 0$. Also by concavity we have

$$\delta_u(x,\bar{x}) \le \frac{u(x)-\widetilde{u}(0)}{x} \le \frac{u(x)}{x}, \quad \limsup_{x\searrow 0} \frac{x\delta_u(x,\bar{x})}{u(x)} \le 1 < \infty.$$

We conclude that $\liminf_{x\searrow 0} E(x) > 0$. The proof is complete. □

Brauer's result follows from the above corollary by taking $u \equiv 1$ and $v = f$.

3.2 Sequence of Tests with Limits. Iterative Condensation.

The results of this section are based on other possible choices for the integrand φ. For all $p \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, let us consider the map $\varphi_{p,\alpha} :]0, \frac{1}{\exp^{[p]}(1)}] \to]0,\infty[$, defined by[1]

$$\begin{aligned}\varphi_{p,\alpha}(x) &= \frac{1}{x^2\left(\prod_{k=1}^{p} \ln^{[k]}\frac{1}{x}\right)\left(\ln^{[p]}\frac{1}{x}\right)^{\alpha}} \\ &= \frac{1}{x^2\left(\ln\frac{1}{x}\right)\left(\ln^{[2]}\frac{1}{x}\right)\cdots\left(\ln^{[p]}\frac{1}{x}\right)\left(\ln^{[p]}\frac{1}{x}\right)^{\alpha}}\end{aligned}$$

(where $\exp(x) = \mathrm{e}^x$). Here $\ln^{[k]}$ denotes the kth iteration, defined on the interval $]\exp^{[k-1]}(0),\infty[$, of the logarithm map. We have

$$\varphi_{0,\alpha}(x) = \frac{1}{x^{2-\alpha}}, \quad \varphi_{1,\alpha}(x) = \frac{1}{x^2(-\ln x)^{1+\alpha}},$$
$$\varphi_{2,\alpha}(x) = \frac{1}{x^2(-\ln x)[\ln(-\ln x)]^{1+\alpha}}, \ \ldots$$

Our next result provides a test for each $p \in \mathbb{N}$. The proof will show that its strength is increasing with p.

[1] We adhere to the convention that an empty product of real numbers equals 1.

Corollary 5 (p-test). *Let $p \in \mathbb{N}$. Then*

(i) $\sum_{n\ge 0} f^{[n]}$ *converges pointwise if* (1) *holds and*

$$\liminf_{x\searrow 0}[\varphi_{p,\alpha}(x)(x-f(x))] > 0 \text{ for some } \alpha > 0. \tag{10}$$

(ii) $\sum_{n\ge 0} f^{[n]}$ *diverges everywhere if*

$$\limsup_{x\searrow 0}[\varphi_{p,\alpha}(x)(x-f(x))] < \infty \text{ for some } \alpha \le 0. \tag{11}$$

PROOF. Let us first observe that

$$\lim_{x\searrow 0}\frac{\varphi_{p+1,\alpha}(x)}{\varphi_{p,\alpha}(x)} = \lim_{y\to\infty}\frac{y^\alpha}{(\ln y)^{\alpha+1}} = \begin{cases}\infty & \text{if } \alpha > 0,\\ 0 & \text{if } \alpha \le 0.\end{cases}$$

Therefore, if one of (10) and (11) holds for $\varphi_{p,\alpha}$, then it holds for $\varphi_{p+1,\alpha}$ too. An easy computation shows that

$$\int_0^{\frac{1}{\exp^{[p]}(1)}} t\varphi_{p,\alpha}(t)\mathrm{d}t = \begin{cases}\frac{1}{\alpha} & \text{if } \alpha > 0,\\ \infty & \text{if } \alpha \le 0.\end{cases}$$

We thus can assume that $p \in \mathbb{N}^*$, for if not, we replace p by $p+1$. Consequently, $\varphi_{p,\alpha}$ is decreasing on a sufficiently small interval $]0,\varepsilon]$. For $x \in]0,\varepsilon]$ we have

$$\varphi_{p,\alpha}(x)(x-f(x)) \le \int_{f(x)}^x \varphi_{p,\alpha}(t)\mathrm{d}t \le \varphi_{p,\alpha}(f(x))(x-f(x)). \tag{12}$$

(i). Assume (10) to hold. The conclusion follows by Theorem 1(i), since (12) yields

$$\liminf_{x\searrow 0}\int_{f(x)}^x \varphi_{p,\alpha}(t)\mathrm{d}t \ge \liminf_{x\searrow 0}[\varphi_{p,\alpha}(x)(x-f(x))] > 0.$$

(ii). Assume (11) to hold. As $\lim_{x\searrow 0}(x\varphi_{p,\alpha}(x)) = \infty$, we have

$$\limsup_{x\searrow 0}\left|1-\frac{f(x)}{x}\right| = \limsup_{x\searrow 0}\left(\varphi_{p,\alpha}(x)(x-f(x))\cdot\frac{1}{x\varphi_{p,\alpha}(x)}\right) = 0,$$

and so $\lim_{x\searrow 0}\frac{f(x)}{x} = 1$. It follows that $\lim_{x\searrow 0}\frac{\varphi_{p,\alpha}(f(x))}{\varphi_{p,\alpha}(x)} = 1$. The conclusion follows by Theorem 1(ii), since (12) yields

$$\limsup_{x\searrow 0}\int_{f(x)}^x \varphi_{p,\alpha}(t)\mathrm{d}t \le \limsup_{x\searrow 0}\left(\varphi_{p,\alpha}(x)(x-f(x))\cdot\frac{\varphi_{p,\alpha}(f(x))}{\varphi_{p,\alpha}(x)}\right) < \infty. \quad \square$$

Some comments are needed for a better understanding of p-tests. It is easily seen that the map $[\exp^{[p]}(1), \infty[\ni x \mapsto \frac{\varphi_{p,\alpha}(1/x)}{x^3} =: g_{p,\alpha}(x) \in]0, \infty[$ is monotone for large x. If a p-test succeeds, the answer (pointwise convergence or divergence everywhere) is given by the nature of the series

$$\sigma_p := \sum_{n \geq \exp^{[p]}(1)} g_{p,\alpha}(n) = \sum_{n \geq \exp^{[p]}(1)} \frac{1}{n(\prod_{k=1}^{p} \ln^{[k]} n)(\ln^{[p]} n)^{\alpha}},$$

according to Remark 2(b). But $g_{p,\alpha}(n) = \mathrm{e}^n g_{p+1,\alpha}(\mathrm{e}^n)$ for every $n \geq \exp^{[p]}(1)$, and so the series σ_p can be obtained from σ_{p+1} by Cauchy condensation[2]. σ_p and σ_{p+1} have the same nature, but the latter converges or diverges slower, thus giving more chances for a successful comparison. Our interpretation is that *the difference between successive p-tests is near to Cauchy condensation.*

The main difficulty in applying the p-test is that of finding a suitable α. This problem is solved by the following equivalent test.

Corollary 6 (equivalent p-test). *Let $p \in \mathbb{N}$. Then*

(i) $\sum_{n\geq 0} f^{[n]}$ *converges pointwise if* (1) *holds and*

$$\liminf_{x \searrow 0} \frac{\ln(x - f(x)) + \ln(\varphi_{p,0}(x))}{\ln^{[p+1]} \frac{1}{x}} > 0. \tag{13}$$

(ii) $\sum_{n\geq 0} f^{[n]}$ *diverges everywhere if*

$$\limsup_{x \searrow 0} [\varphi_{p,0}(x)(x - f(x))] < \infty. \tag{14}$$

Proof. For all $\alpha \in \mathbb{R}$ and $x \in]0, \frac{1}{\exp^{[p]}(1)}[$, set

$$u_\alpha(x) = \varphi_{p,\alpha}(x)(x - f(x)), \quad v(x) = \frac{\ln(u_0(x))}{\ln^{[p+1]} \frac{1}{x}} = \frac{\ln(x - f(x)) + \ln(\varphi_{p,0}(x))}{\ln^{[p+1]} \frac{1}{x}}.$$

We have the identities $u_\alpha(x) = \left(\ln^{[p]} \frac{1}{x}\right)^{v(x)-\alpha}$ and $v(x) = \alpha + \frac{\ln(u_\alpha(x))}{\ln^{[p+1]} \frac{1}{x}}$.
(i). It suffices to prove the equivalence (10) $\Leftrightarrow$(13).
(10) $\Rightarrow$(13). Assume (10) to hold. Thus there is $\lambda > 0$, such that $u_\alpha(x) > \lambda$, and so $v(x) > \alpha + \frac{\ln \lambda}{\ln^{[p+1]} \frac{1}{x}}$, for sufficiently small $x > 0$. Thus $\liminf_{x \searrow 0} v(x) \geq \alpha > 0$.
(10) $\Leftarrow$(13). Assume (13) to hold. Thus there exists $\alpha > 0$, such that $v(x) > \alpha$,

[2]This is possible with the exponential e^n, since the map $g_{p+1,\alpha}$ is defined on an interval.

and so $u_\alpha(x) > 1$, for sufficiently small $x > 0$. Therefore $\liminf_{x\searrow 0} u_\alpha(x) \geq 1 > 0$.
(ii). For all $\alpha \leq 0$, we have the inequality $\limsup_{x\searrow 0} u_0(x) \leq \limsup_{x\searrow 0} u_\alpha(x)$. This proves the equivalence (11) $\Leftrightarrow$(14). □

The beginning of the proof of Corollary 5 and the following example show that each p-test is strictly weaker than the $(p+1)$-test.

Example 7. Let $p \in \mathbb{N}$. Choose $a > 0$, such that $x\varphi_{p+1,0}(x) > 1$ on $]0, a]$. Thus the map

$$f :]0, a] \to]0, a], \quad f(x) = x - \frac{1}{\varphi_{p+1,0}(x)},$$

is well defined, continuous, and $f(x) < x$ on $]0, a]$. For the series $\sum_{n\geq 0} f^{[n]}$ the p-test fails, but the $(p+1)$-test proves pointwise convergence, since

$$\lim_{x\searrow 0}[\varphi_{p+1,0}(x)(x - f(x))] = 1, \quad \lim_{x\searrow 0}[\varphi_{p,\alpha}(x)(x - f(x))] = \begin{cases} 0 & \text{if } \alpha > 0, \\ \infty & \text{if } \alpha \leq 0. \end{cases}$$

Even the 0-test ($p = 0$) suffices in many “decent” cases (see the examples from Section 4). Therefore we find it useful to write down both of its equivalent forms:

Corollary 8 (0-test). (i) *$\sum_{n\geq 0} f^{[n]}$ converges pointwise if* (1) *holds and*

$$\liminf_{x\searrow 0} \frac{x - f(x)}{x^\beta} > 0 \ \text{ for some } \beta < 2.$$

(ii) *$\sum_{n\geq 0} f^{[n]}$ diverges everywhere if*

$$\limsup_{x\searrow 0} \frac{x - f(x)}{x^\beta} < \infty \ \text{ for some } \beta \geq 2.$$

In [4] it is shown that if f is concave and $\beta > 1$, then the limits from the above corollary are both in $]0, \infty[$ if and only if

$$0 < \liminf_{n\to\infty}(n^{\frac{1}{\beta-1}} f^{[n]}(x)) \leq \limsup_{n\to\infty}(n^{\frac{1}{\beta-1}} f^{[n]}(x)) < \infty \ \text{ for every } x > 0.$$

In this particular case, the series $\sum_{n\geq 0} f^{[n]}(x_0)$ and $\sum_{n\geq 1} \frac{1}{n^{1/(\beta-1)}}$ have the same nature.

Corollary 9 (equivalent 0-test). **(i)** $\sum_{n\geq 0} f^{[n]}$ *converges pointwise if* (1) *holds and*

$$\limsup_{x\searrow 0} \frac{\ln(x-f(x))}{\ln x} < 2.$$

(ii) $\sum_{n\geq 0} f^{[n]}$ *diverges everywhere if*

$$\limsup_{x\searrow 0} \frac{x-f(x)}{x^2} < \infty.$$

4 Examples.

Example 10. Let f be the map defined in (3). Then $\sum_{n\geq 0} f^{[n]}$ converges pointwise. *The tests of Altman, Fort-Schuster, Švarcman, and Brauer cannot be applied.*

Proof. f is continuous and $f(x) < x$ for $x > 0$. As $\frac{\ln(x-f(x))}{\ln x} = \frac{3}{2} - \frac{\ln(2+x+\sin(\pi/x))}{\ln x}$ yields $\lim_{x\searrow 0} \frac{\ln(x-f(x))}{\ln x} = \frac{3}{2} < 2$, the conclusion follows by Corollary 9(i). $\square$

Example 11. Let $p, q > 0$. Define the sequence $(x_n)_{n\in\mathbb{N}}$ by

$$x_0 \in]0,1], \quad x_{n+1} = (\sin(x_n^p))^q \text{ for every } n \in \mathbb{N}.$$

Then $\sum_{n\geq 0} x_n$ converges if and only if

$$pq > 1 \text{ or } 2 < q = \frac{1}{p}.$$

Proof. Define $f :]0,1] \to]0,1]$, $f(x) = (\sin(x^p))^q$. Thus f is a continuous map and $x_n = f^{[n]}(x_0)$ for $n \in \mathbb{N}^*$. We have

$$\lim_{x\searrow 0} \frac{f(x)}{x} = \lim_{x\searrow 0} \left(\frac{\sin(x^p)}{x^p}\right)^q x^{pq-1} = \begin{cases} 0 & \text{if } pq > 1, \\ 1 & \text{if } pq = 1, \\ \infty & \text{if } pq < 1. \end{cases}$$

Therefore $\sum_{n\geq 0} x_n$ cannot converge if $pq < 1$ (even if $\lim_{n\to\infty} x_n = 0$).
Now assume that $pq \geq 1$. Hence $f(x) < x^{pq} \leq x$ for every $x \in]0,1]$. By the ratio test, we deduce that $\sum_{n\geq 0} x_n$ converges if $pq > 1$. Let us finally consider the case $pq = 1$. For every $\beta \in \mathbb{R}$, we have

$$\frac{x-f(x)}{x^\beta} = \frac{1-\left(\frac{\sin(x^p)}{x^p}\right)^q}{1-\frac{\sin(x^p)}{x^p}} \cdot \frac{x^p - \sin(x^p)}{(x^p)^3} \cdot x^{2p+1-\beta}.$$

Taking here $\beta = 2p + 1$ leads to $\lim_{x \searrow 0} \frac{x - f(x)}{x^\beta} = \frac{q}{6} \in]0, \infty[$. The proof is completed by using Corollary 8. $\square$

Example 12. Let $p > 0$. Define the sequence $(x_n)_{n \in \mathbb{N}}$ by

$$x_0 \in]0, \infty[, \quad x_{n+1} = (\arctan(x_n^p))^{1/p} \text{ for every } n \in \mathbb{N}.$$

Then $\sum_{n \geq 0} x_n$ converges if and only if $p < \frac{1}{2}$.

PROOF. Define $f :]0, \infty[\to]0, \infty[$, $f(x) = (\arctan(x^p))^{1/p}$. Thus f is continuous, $f(x) < x$ for $x > 0$, and $x_n = f^{[n]}(x_0)$ for $n \in \mathbb{N}^*$. For every $\beta \in \mathbb{R}$, we have

$$\frac{x - f(x)}{x^\beta} = \frac{1 - \left(\frac{\arctan(x^p)}{x^p}\right)^{1/p}}{1 - \frac{\arctan(x^p)}{x^p}} \cdot \frac{x^p - \arctan(x^p)}{(x^p)^3} \cdot x^{2p+1-\beta}.$$

Taking here $\beta = 2p + 1$ leads to $\lim_{x \searrow 0} \frac{x - f(x)}{x^\beta} = \frac{1}{3p} \in]0, \infty[$. The proof is completed by using Corollary 8. $\square$

Example 13. Let $p > 0$. Define the sequence $(x_n)_{n \in \mathbb{N}}$ by

$$x_0 \in]0, \infty[, \quad x_{n+1} = (\ln(1 + x_n^p))^{1/p} \text{ for every } n \in \mathbb{N}.$$

Then $\sum_{n \geq 0} x_n$ converges if and only if $p < 1$.

PROOF. The reasoning is similar to that from Example 12, and uses the equality

$$\frac{x - (\ln(1 + x^p))^{1/p}}{x^\beta} = \frac{1 - \left(\frac{\ln(1+x^p)}{x^p}\right)^{1/p}}{1 - \frac{\ln(1+x^p)}{x^p}} \cdot \frac{x^p - \ln(1 + x^p)}{(x^p)^2} \cdot x^{p+1-\beta}$$

for $x > 0$ and $\beta \in \mathbb{R}$, as well as the limit $\lim_{y \to 0} \frac{y - \ln(1+y)}{y^2} = \frac{1}{2}$. $\square$

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