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TANGENTIAL BEHAVIOR OF FUNCTIONS AND CONICAL DENSITIES OF HAUSDORFF MEASURES

Abstract

We construct a C^1-function $f\colon [0,1] \to \mathbb{R}$ such that for almost all $x \in (0,1)$, there is $r > 0$ for which $f(y) > f(x) + f'(x)(y-x)$ when $y \in (x, x+r)$ and $f(y) < f(x) + f'(x)(y-x)$ when $y \in (x-r, x)$. The existence of such functions is related to a problem concerning conical density properties of Hausdorff measures on $\mathbb{R}^n$. We also discuss the tangential behavior of typical C^1-functions, using an improvement of Jarník's theorem on essential derived numbers.

1 Introduction and Notation.

Let us begin by introducing some notation. For $0 \le s \le n$, let $\mathcal{H}^s$ denote the s-dimensional Hausdorff measure on $\mathbb{R}^n$, and on the real line, let $\mathcal{L}$ stand for the Lebesgue measure. We use the common notation $B(x,r)$ for open balls and for the unit sphere on $\mathbb{R}^n$ the notation $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is used. If $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, then $d(x,A)$ stands for the Euclidean distance between x and A. The length of an interval $I \subset \mathbb{R}$ is denoted by $\ell(I)$ and the notion ∂A is used for the boundary of a given set $A \subset \mathbb{R}$. The (symmetrical) upper and lower densities of a measurable set $A \subset \mathbb{R}$ at $x \in \mathbb{R}$ are defined as the upper and lower limits, respectively, of the ratio $\mathcal{L}((x-r, x+r) \cap A)/(2r)$ when $r \downarrow 0$. If $f\colon [0,1] \to \mathbb{R}$ is differentiable at $x \in (0,1)$, the sets $A^+(f,x)$ and $A^-(f,x)$ are given by

$$A^+(f,x) = \{y \in (0,1) : f(y) > f(x) + f'(x)(y-x)\},$$
$$A^-(f,x) = \{y \in (0,1) : f(y) < f(x) + f'(x)(y-x)\}.$$

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If $x \in \mathbb{R}^n$, $\theta \in S^{n-1}$, and $0 \leq \eta < 1$, we define the cone $H(x,\theta,\eta)$ by setting $H(x,\theta,\eta) = \{y \in \mathbb{R}^n : (y-x) \cdot \theta > \eta|y-x|\}$. For half-spaces we use shorter notation, $H(x,\theta) = H(x,\theta,0)$. Finally, we let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ denote the set of extended real numbers.

Given a set $A \subset \mathbb{R}^n$, it is often of interest to know how it is distributed near a "generic" point. This paper was inspired by the following conical density theorem of Marstrand [5, pp. 293–297].

Theorem 1.1. *Let $0 \leq s < 2$ and $A \subset \mathbb{R}^2$ with $\mathcal{H}^s(A) < \infty$.*

1. *If $0 \leq s < 1$ and $\theta \in S^1$, then $\liminf_{r \downarrow 0} \mathcal{H}^s(B(x,r) \cap H(x,\theta) \cap A)/r^s = 0$ for $\mathcal{H}^s$-almost all $x \in A$.*
2. *If $1 < s < 2$, then for $\mathcal{H}^s$-almost all $x \in A$, there is $\theta \in S^1$ such that $\liminf_{r \downarrow 0} \mathcal{H}^s(B(x,r) \cap H(x,\theta) \cap A)/r^s = 0$.*

It seems that 1-sets (sets A with $0 < \mathcal{H}^1(A) < \infty$) play a special role in connection with the above theorem. Marstrand's proof yields that claim (2) is valid also for 1-sets if half-spaces $H(x,\theta)$ are replaced by cones $H(x,\theta,\eta)$ for any $\eta > 0$. On the other hand, Besicovitch [1, Theorem 13] had shown before that even (1) holds for purely unrectifiable 1-sets, that is, for 1-sets which intersect every rectifiable curve only in a set of zero $\mathcal{H}^1$-measure. A question arises whether (2) actually holds for all 1-sets.

However, the answer to the above question is negative: Consider a Cantor set $C \subset [0,1]$ with $\mathcal{L}(C) > 0$ and define $f(x) = \int_0^x \operatorname{dist}(t,C)\,dt$. Then the graph of f gives us a counterexample, see §3. In Section 2.1 we construct a C^1-function $f\colon [0,1] \to \mathbb{R}$ whose graph does not satisfy claim (2) of Theorem 1.1 anywhere except a set of zero length. In Section 2.2, inspired by our examples, we study the distribution of the sets $A^+(f,x)$ and $A^-(f,x)$ for functions $f \in C^k[0,1]$, $k \in \mathbb{N}$. We show among other things that for a typical $f \in C^1[0,1]$, both of the sets $A^+(f,x)$ and $A^-(f,x)$ have zero lower density and unit upper density at x for all $x \in (0,1)$ except a set of Hausdorff dimension zero. This is a corollary to an extension of Jarník's theorem on essential derived numbers, Theorem 2.5, which is essentially due to Zajíček [9].

Finally, in Section 3, we discuss the above conical density problem in higher dimensions.

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2 Tangential Behavior of Functions.

2.1 An Example.

Our goal in this section is to prove that there are functions $f \in C^1[0,1]$ so that both $A^+(f,x)$ and $A^-(f,x)$ have positive lower density at x for almost every $x \in (0,1)$. This is an easy consequence of the following result. The proof given here is due to one of the referees. Though resulting to a somewhat similar function, it is much easier than the authors original construction.

Theorem 2.1. *There is a continuous function $g\colon [0,1] \to \mathbb{R}$ which is non-constant on any non-degenerate interval $I \subset [0,1]$ but has a local minimum almost everywhere.*

Proof. Let $C, D \subset [0,1]$ be Cantor sets with $\mathcal{L}(C) = 0$ and $\mathcal{L}(D) > 0$. Moreover, let $\varphi\colon [0,1] \to [0,1]$ be a Cantor function associated to the set C, that is, φ is continuous, nondecreasing, and it is constant on any component of $[0,1] \setminus C$. We also assume that $\varphi(t) = 0$ if and only if $t = 0$. For any non-degenerate interval $I = (a,b) \subset [0,1]$, put $\varphi_I(x) = \ell(I)\varphi\big(2\,\mathrm{dist}(x, \{a,b\})/\ell(I)\big)$ for $x \in I$ and $\varphi_I(x) = 0$ if $x \notin I$. Moreover, for any open interval $J = (c,d)$ put $D_J = \{c\} + (d-c)D$ and let

$$J \setminus D_J = \bigcup_{i=1}^{\infty} I_i \tag{2.1}$$

where intervals $I_1, I_2, \dots$ are open and disjoint. Define $\Phi_J = \sum_{i=1}^{\infty} \varphi_{I_i}$.

We can now define g by an inductive process as follows: Let $g_1 = \Phi_{(0,1)}$. If g_k has been defined for $k \in \mathbb{N}$, let $J_i, i \in \mathbb{N}$ denote its intervals of constancy and put $g_{k+1} = g_k + \sum_{i=1}^{\infty} \Phi_{J_i}$. Since $||g_{k+1} - g_k|| \le \ell^{k+1}$ where ℓ is the length of the longest complementary interval of D, the sequence g_k is uniformly convergent and thus $g = \lim_{k\to\infty} g_k \in C[0,1]$. It is easy to see that g is non-constant on any non-degenerate interval, and has local minimum almost everywhere. $\square$

Corollary 2.2. *There is a continuously differentiable function $f\colon [0,1] \to \mathbb{R}$ such that for almost all $x \in (0,1)$, there is $r > 0$ for which*

$$(x, x+r) \subset A^+(f,x), \tag{2.2}$$

$$(x-r, x) \subset A^-(f,x). \tag{2.3}$$

Proof. Let $g\colon [0,1] \to \mathbb{R}$ be the function of Theorem 2.1. Defining $f(x) = \int_0^x g(t)\,dt$ for $x \in [0,1]$ gives what we want. $\square$

Remarks. *1.* It is easy to construct functions $f \in C^\infty[0,1]$ so that (2.2) and (2.3) are valid in a set of positive measure: Define $g \in C^\infty[0,1]$ so that $g = 0$ on some Cantor set C with $\mathcal{L}(C) > 0$, and $g > 0$ outside this set. If f is given by $f(x) = \int_0^x g(t)dt$, then (2.2) and (2.3) hold for f in the set C. However, for any C^2-function, these conditions can not hold almost everywhere, see Proposition 2.3 (1).
2. It is not very hard, though it becomes more technical, to prove that g in Theorem 2.1 may have local minimum everywhere except a set of small Hausdorff dimension. In fact, given any nondecreasing $h\colon (0,\infty) \to (0,\infty)$ with $\lim_{r\downarrow 0} h(r) = 0$ we can construct g which is nowhere constant and has local minimum everywhere except a set of zero h-Hausdorff measure. The idea is as follows: Given g_k as in the proof of Theorem 2.1, let J_i, $i \in \mathbb{N}$ denote its intervals of constancy. Choose Cantor sets $D_i \subset J_i$ so that $\sum_{i,j} h(\ell_{ij}) < 2^{-k}$, where $\{\ell_{ij}\}_{j\in\mathbb{N}}$ denote the lengths of the complementary intervals of D_i. For any such complementary interval I, define a Cantor bump, φ_I, such that it does all its increasing and decreasing in a set of h-measure zero. Now proceed as in the proof of Theorem 2.1.

However small, the set of local minima for $g \in C[0,1]$ which is nowhere constant is always a first category set. This follows from Proposition 2.3 (5) by considering $f = \int g$.
3. It might be an interesting question whether there are functions $f \in C^1[0,1]$ so that both of the one-sided lower densities,

$$\liminf_{r\downarrow 0} \mathcal{L}((x, x+r) \cap A^+(f,x))/r,$$
$$\liminf_{r\downarrow 0} \mathcal{L}((x, x+r) \cap A^-(f,x))/r,$$

are strictly positive in a set of positive measure. Similar questions may be posed when intervals $(x, x+r)$ are replaced by some other sets.

2.2 Typical Behavior.

Given a property for functions, it is natural to ask if this property is typical on a function class in question. The theme of this section is to study the typicality of some properties related to the above examples. For $k \in \mathbb{N} \cup \{0, \infty\}$, the space $C^k[0,1]$ is given the norm $||f|| = \sup\{|f^j(x)| : x \in (0,1), j = 0, \ldots, k\}$ where $f^0 = f$. When we say that some property holds for a typical $f \in C^k[0,1]$ we mean that this property is valid on a residual set of functions on $C^k[0,1]$.

Some very basic facts are listed in the proposition below. To help discussion, we write for $f \in C^1[0,1]$,

$$A_f = \{x \in (0,1) : x \in \overline{A^+(f,x)} \bigcap \overline{A^-(f,x)}\},$$
$$B_f^+ = \{x \in (0,1) : \text{ conditions (2.2) and (2.3) hold for some } r > 0\}, \text{ and}$$
$$B_f^- = B_{-f}^+.$$

Proposition 2.3. *1. For any $f \in C^2[0,1]$, the set A_f is nowhere dense.*

2. If $2 \leq k \in \mathbb{N} \cup \{\infty\}$, then $\mathcal{L}(A_f) = 0$ for a typical $f \in C^k[0,1]$.

3. The set $(0,1) \setminus A_f$ is dense for any $f \in C^1[0,1]$.

4. The sets B_f^+ and B_f^- are dense for a typical $f \in C^1[0,1]$.

5. For any $f \in C^1[0,1]$, the sets B_f^+ and B_f^- are first category sets.

PROOF. All five claims may be proved in an elementary manner and thus we give only the main lines. If $f \in C^2[0,1]$ and $x \in (0,1)$ is such that both of the sets $(x-r, x+r) \cap A^+(f,x)$ and $(x-r, x+r) \cap A^-(f,x)$ are nonempty for any $r > 0$, then $f''(x) = 0$. This implies (1). Also (2) follows since $\mathcal{L}(\{x : f''(x) = 0\}) = 0$ for a typical $f \in C^k[0,1]$ when $k \geq 2$. To see this consider $j \in \mathbb{N}$ and the set

$$E_j = \{f \in C^k[0,1] : \mathcal{L}(\{x \in (0,1) : f''(x) = 0\}) > 1/j\}.$$

Fix $f \in C^k[0,1]$, and $r > 0$. Then we may find $c \in (-r/2, r/2)$ and $0 < \delta < r$ such that

$$\mathcal{L}(\{x \in (0,1) : f''(x) \in (c-\delta, c+\delta)\}) \leq 1/j.$$

If $g(x) = f(x) - cx^2/2$, then $B(g, \delta/2) \subset B(f,r) \setminus E_j$. Thus E_j is nowhere dense and consequently $\bigcup_j E_j$ is a first category set.

Claim (3) is geometrically obvious: Given $0 \leq a < b \leq 1$, choose $c \in (a,b)$ so that the point $(c, f(c))$ maximizes the distance to the line segment joining $(a, f(a))$ and $(b, f(b))$ among all the points on the graph of $f|_{(a,b)}$. Then $c \notin A_f$.

Claim (4) follows using a similar argument as in Corollary 2.2 since the derivative of a typical C^1-function has a dense set of minima and maxima, see [2, Theorem 10.20] for example.

For (5) we give some details: Suppose on the contrary that there is for example an $f \in C^1[0,1]$ so that B_f^+ is a second category set. Since $B_f^+ = \bigcup_{k=1}^{\infty} B_k$, where $B_k = \{x \in (0,1) : (x, x+1/k) \subset A^+(f,x) \text{ and } (x-1/k, x) \subset A^-(f,x)\}$ it follows that there is $k \in \mathbb{N}$ and a nonempty open interval $I \subset (0,1)$ with $\ell(I) < 1/k$ so that B_k is dense on I. It follows that f is both convex and concave on I which forces f to be linear on I, a contradiction. $\square$

A natural question arising from our examples is the following: Is it true that for a typical $f \in C^1[0,1]$, both of the sets $A^+(f,x)$ and $A^-(f,x)$ have strictly positive lower density at x for almost every $x \in (0,1)$. This is, however, not the case; as we shall see, both of these sets have typically unit upper density and thus also zero lower density for almost every x. We shall prove this by using a generalization of a well known theorem of Jarník [3]. We start by proving a simple special case and then discuss a more general result using Zajíček's notion of $[g]$-porosity. We say that $c \in \overline{\mathbb{R}}$ is a symmetrical essential derived number of $f\colon [0,1] \to \mathbb{R}$ at $x \in (0,1)$, denote $c \in \mathrm{SEDN}(f,x)$, if there is a set $E = E(x,c) \subset \mathbb{R}$ such that E has unit upper density at x and

$$\lim_{\substack{y \in E \\ y \to x}} \frac{f(y)-f(x)}{y-x} = c. \tag{2.4}$$

To avoid confusion, we note that in this context the term "symmetrical" does not refer to symmetric differentiation but to symmetrical upper density.

Theorem 2.4. *For a typical $f \in C[0,1]$, we have $\mathrm{SEDN}(f,x) = \overline{\mathbb{R}}$ for almost every $x \in (0,1)$.*

Remarks. A number $c \in \overline{R}$ is called a right essential derived number of f at x if there is $E \subset \mathbb{R}$ satisfying (2.4) with $\limsup_{r \downarrow 0} \mathcal{L}((x,x+r) \cap E)/r = 1$. Left essential derived numbers are defined in an analogous way. A point $x \in (0,1)$ is an essential knot point of f if every $c \in \overline{R}$ is simultaneously left and right essential derived number of f at x. Jarník [3] proved that almost all points are essential knot points for a typical function $f \in C[0,1]$. The above theorem is stronger than Jarník's result since it allows us to choose E such that it is simultaneously big at both sides of x for some small scales, and not only big at left for some scales and big at right for some (possibly different) scales.

If $w \in \mathbb{R}^2$, $c \in \mathbb{R}$, and $\alpha > 0$, we denote by $\ell_{w,c}$ the line through w with slope c and put $X(w,c,\alpha) = \{v \in \mathbb{R}^2 : d(v-w, \ell_{w,c}) \le \alpha |v-w|\}$. These cones are useful since $c \in \mathrm{SEDN}(f,x)$ if and only if

$$\begin{aligned} x \in \bigcap_{\varepsilon,\alpha,r_0} \{ & z \in (0,1) : \exists\, 0 < r < r_0 \text{ such that} \\ & \mathcal{L}(\{y \in (z-r,z+r) : (y,f(y)) \in X((z,f(z)),c,\alpha)\}) > (2-\varepsilon)r\} \end{aligned} \tag{2.5}$$

where the intersection is taken over all positive rationals ε, α and r_0, see also [10, Lemma 1].

PROOF OF THEOREM 2.4. For $f \in C[0,1]$, let

$$F = F(f) = \{x \in (0,1) : \mathrm{SEDN}(f,x) \ne \overline{\mathbb{R}}\}.$$

It follows from (2.5) that

$$F = \bigcup_{c,\varepsilon,\alpha,r_0} \{x \in (0,1) : \mathcal{L}(\{y \in (x-r, x+r) : (y, f(y)) \notin X((x, f(x)), c, \alpha)\}) > \varepsilon r \text{ for all } 0 < r < r_0\}$$

where the union is taken over all $c \in \mathbb{Q}$, and $0 < \varepsilon, \alpha, r_0 \in \mathbb{Q}$. Thus

$$\{f \in C[0,1] : \mathcal{L}(F(f)) > 0\} = \bigcup_{\delta,c,\varepsilon,\alpha,r_0} \mathcal{A}(\delta, c, \varepsilon, \alpha, r_0)$$

where $c \in \mathbb{Q}$, $0 < \delta, \varepsilon, \alpha, r_0 \in \mathbb{Q}$, and $\mathcal{A}(\delta, c, \varepsilon, \alpha, r_0) \subset C[0,1]$ is given by

$$\begin{aligned}\mathcal{A}(\delta, c, \varepsilon, \alpha, r_0) =& \{f \in C[0,1] : \mathcal{L}(\{x \in (0,1) : \mathcal{L}(\{y \in (x-r, x+r) : \\ & (y, f(y)) \notin X((x, f(x)), c, \alpha)\}) > \varepsilon r \text{ for all } 0 < r < r_0\}) > \delta\}.\end{aligned}$$

Fix numbers $c \in \mathbb{R}$, and $\delta, \varepsilon, \alpha, r_0 > 0$. It suffices to prove that the set $\mathcal{A}(\delta, c, \varepsilon, \alpha, r_0)$ is nowhere dense on $C[0,1]$. Take $f \in C[0,1]$ and let $0 < r < r_0$. Let $g \in C[0,1]$ be piecewise linear with $||f-g|| < r/2$ so that (see [9, Lemma 1]) there are disjoint intervals $I_1, \dots, I_k \subset (0,1)$ with $\sum_{i=1}^k \ell(I_i) > 1 - \delta/2$ such that g has slope c on each interval I_i. Let $0 < \ell < \min_{i=1,\dots,k} \ell(I_i)$, $0 < t < \min\{\delta/4, r_0\}$, and $0 < s < \min\{\alpha\varepsilon t/4, r/2\}$. Take $h \in B(g, s\ell)$. It is easy to see that if $x \in I_i$ with $d(x, \partial I_i) > t\ell$, then $(y, h(y)) \in X((x, h(x)), c, \alpha)$ for all $y \in (x - t\ell, x + t\ell) \setminus (x - 2s\ell/\alpha, x + 2s\ell/\alpha)$. It follows that for such x, we have

$$\mathcal{L}(\{y \in (x - t\ell, x + t\ell) : (y, h(y)) \notin X((x, h(x)), c, \alpha)\}) < 4s\ell/\alpha < t\ell\varepsilon. \quad (2.6)$$

Since (2.6) holds in a measurable set whose measure is greater than $1 - \delta/2 - k2t\ell > 1 - \delta$, we conclude that $h \notin \mathcal{A}(\delta, c, \varepsilon, \alpha, r_0)$. Thus $B(g, s\ell) \subset B(f, r) \setminus \mathcal{A}(\delta, c, \varepsilon, \alpha, r_0)$ and the claim follows. $\square$

Zajíček [9] strengthened Jarník's result using porosity notions. His result is also in a sense one-sided and does not seem to imply Theorem 2.4. However, only a minor change in his method gives even stronger theorem. The following notation is from [9]. If $A \subset \mathbb{R}$ and $I \subset \mathbb{R}$ is an interval, the number $p(A, I)$ denotes the length of the largest subinterval $I' \subset I \setminus A$. We denote by $\mathcal{G}$ the collection of all strictly increasing continuous functions g on $(0, \infty)$ for which $g(x) > x$ for all $0 < x < \infty$. If $g \in \mathcal{G}$, we say that $E \subset \mathbb{R}$ is $[g]$-porous from the right (left) at $x \in \mathbb{R}$ if there is a sequence $r_i \downarrow 0$ such that $g(p(E, (x, x + r_i))) > r_i$ $(g(p(E, (x - r_i, x))) > r_i)$ for all $i \in \mathbb{N}$. A number $c \in \overline{\mathbb{R}}$ is a right (left) $[g]$-derived number of f at x if there is a set $E \subset \mathbb{R}$ for which $\mathbb{R} \setminus E$ is $[g]$-porous from the right (left) at x such that (2.4) holds for E. A point x is a $[g]$-knot point of f if every $c \in \overline{\mathbb{R}}$ is both left and right $[g]$-derived number of f at x. Zajíček [9, Theorem 2] showed that for a typical $f \in C[0,1]$, the set of points from $(0,1)$ which are not $[g]$-knot points of f is σ-$[g]$-totally porous. A set $A \subset \mathbb{R}$ is called totally $[g]$-porous if for any $\varepsilon > 0$, we can find a point $a \in \mathbb{R}$ and a number $0 < \delta < \varepsilon$ so that $g(p(A, [a + n\delta, a + (n+1)\delta])) > \delta$ for all $n \in \mathbb{Z}$. A set A is σ-$[g]$-totally porous if it is a countable union of $[g]$-totally porous sets. To compare $[g]$-porosity with other notions of porosity, see [8] and [9].

Modifying the above notation, we say that $A \subset \mathbb{R}$ is symmetrically $[g]$-porous at x if there is a sequence $r_i \downarrow 0$ so that

$$\min\{g(p(A, (x - r_i, x))), g(p(A, (x, x + r_i)))\} > r_i$$

for each $i \in \mathbb{N}$. A number $c \in \overline{\mathbb{R}}$ is a symmetrical $[g]$-derived number of f at x if there is a set $E \subset \mathbb{R}$ such that $\mathbb{R} \setminus E$ is symmetrically $[g]$-porous at x and (2.4) holds for E. A point x is a symmetrical $[g]$-knot point of f if each $c \in \overline{\mathbb{R}}$ is a symmetrical $[g]$-derived number of f at x.

Theorem 2.5. *Let $g \in \mathcal{G}$. Then for a typical $f \in C[0,1]$, the set of points $x \in (0,1)$ which are not symmetrical $[g]$-knot points of f is σ-$[g]$-totally porous.*

This theorem can be proved by modifying Zajíček's method only slightly and thus we shall not repeat the argument. For the convenience of an interested reader we comment that the main point is that in [9, Lemma 2(b)] one may assert: For any $h \in U(a, b, s, \delta)$ and $x \in \cup_{k=0}^{n-1}[k/n + 2v, (k+1)/n - 2v]$ the inequalities $g(p(\{y : (h(y) - h(x))/(y - x) \notin [a, b]\}, [x, x + v])) > v$ and $g(p(\{y : (h(y) - h(x))/(y - x) \notin [a, b]\}, [x - v, x])) > v$ hold.

We now turn our attention back to our original question related to the distribution of the sets $A^+(f, x)$ and $A^-(f, x)$ for typical functions $f \in C^1[0,1]$.

Theorem 2.6. *Let $g \in \mathcal{G}$. For a typical $f \in C^1[0,1]$, both of the sets $\mathbb{R} \setminus A^+(f, x)$ and $\mathbb{R} \setminus A^-(f, x)$ are symmetrically $[g]$-porous at x for all $x \in (0,1) \setminus A$ where A is a σ-$[g]$-totally porous set (depending on f).*

PROOF. Making g smaller if necessary, we may assume that $\lim_{r \downarrow 0} g(r) = 0$. By symmetry, it suffices to prove that for a typical $f \in C^1[0,1]$, the set $\mathbb{R} \setminus A^+(f, x)$ is symmetrically $[g]$-porous at x for a set of points $x \in (0,1)$ whose complement is σ-$[g]$-totally porous. Take $\widetilde{g} \in \mathcal{G}$ so that

$$\lim_{r \downarrow 0} \frac{\left(r - g^{-1}(r)\right)^2}{r - \widetilde{g}^{-1}(r)} = \infty. \tag{2.7}$$

One may choose, for example, $\widetilde{g}$ for which $\widetilde{g}^{-1}(r) > r\left(1-(r-g^{-1}(r))^2\right)$ for all (small) $r > 0$. Since (2.7) implies that $g(r) \geq \widetilde{g}(r)$ for small r, using Theorem 2.5, we see that for typical $f \in C^1[0,1]$, there is a set $A \subset [0,1]$ so that $[0,1] \setminus A$ is σ-$[g]$-totally porous and all points $x \in A$ are $[\widetilde{g}]$-knot points of f'. To see this we use the fact that if $B \subset C[0,1]$ is residual on $C[0,1]$, then the set $\widetilde{B} = \{f \in C^1[0,1] : f' \in B\}$ is residual on $C^1[0,1]$. Fix such a function f and $x \in A = A(f)$. Now there is $E \subset (0,1)$ whose complement is symmetrically $[\widetilde{g}]$-porous at x and for which

$$\lim_{\substack{t \in E \\ t \to x}} \frac{f'(t)-f'(x)}{t-x} = 2. \tag{2.8}$$

Replacing E by its closure if necessary, we may assume that it is measurable.

Choose $0 < M < \infty$ so that $|f'(x)| < M$ for all $x \in [0,1]$. Using (2.8) and (2.7), we may choose $0 < r_0 < 1$ so that

$$(f'(t)-f'(x))/(t-x) > 1 \text{ for all } t \in (x-r_0, x+r_0) \cap E \setminus \{x\} \text{ and} \tag{2.9}$$

$$(r-g^{-1}(r))^2 > (36M+9)(r-\widetilde{g}^{-1}(r)) \text{ for all } 0 < r < r_0. \tag{2.10}$$

Now we may find arbitrary small radii $0 < r < r_0$ such that $\widetilde{g}(p(\mathbb{R} \setminus E, (x, x+r))) > r$ and $\widetilde{g}(p(\mathbb{R} \setminus E, (x-r, x))) > r$. For such a radius r we get $(x+(r-\widetilde{g}^{-1}(r)), x+r-(r-\widetilde{g}^{-1}(r))) \subset E$ and since $x+(r-\widetilde{g}^{-1}(r)) < x+(r-g^{-1}(r))/3 < x+r-(r-g^{-1}(r))/3 < x+r-(r-\widetilde{g}^{-1}(r))$ by (2.10), for any $y \in (x+(r-g^{-1}(r))/3, x+r-(r-g^{-1}(r))/3)$, we get $(x+r-\widetilde{g}^{-1}(r), y) \subset E$ and, using (2.9) and (2.10), we estimate

$$\begin{aligned}
\int_x^y f' d\mathcal{L} &= \int_{x+r-\widetilde{g}^{-1}(r)}^{y} f' d\mathcal{L} + \int_x^{x+r-\widetilde{g}^{-1}(r)} f' d\mathcal{L} \\
&> \int_{x+r-\widetilde{g}^{-1}(r)}^{y} (f'(x)+(t-x))\,dt - M(r-\widetilde{g}^{-1}(r)) \\
&> f'(x)(y-x) + \tfrac{1}{2}(y-x)^2 - \tfrac{1}{2}(r-\widetilde{g}^{-1}(r))^2 - 2M(r-\widetilde{g}^{-1}(r)) \\
&> f'(x)(y-x) + \tfrac{1}{18}\left((r-g^{-1}(r)\right)^2 - (2M+\tfrac{1}{2})(r-\widetilde{g}^{-1}(r)) \\
&> f'(x)(y-x).
\end{aligned}$$

Thus $f(y) = f(x) + \int_x^y f' d\mathcal{L} > f(x) + f'(x)(y-x)$ giving $y \in A^+(f,x)$. In a similar manner, we see that $y \in A^+(f,x)$ also if $y \in (x-r+(r-g^{-1}(r))/3, x-(r-g^{-1}(r))/3)$. It follows that

$$g(p(\mathbb{R} \setminus A^+(f,x), (x, x+r))) \geq g(r - \tfrac{2}{3}(r-g^{-1}(r))) > g(g^{-1}(r)) = r$$

and similarly $g(p(\mathbb{R}\setminus A^+(f,x),(x-r,x)))>r$. We conclude that $\mathbb{R}\setminus A^+(f,x)$ is symmetrically $[g]$-porous at x for all $x\in A$. □

In terms of densities and dimensions, we get the following corollary.

Corollary 2.7. *For a typical $f\in C^1[0,1]$, both of the sets $A^+(f,x)$ and $A^-(f,x)$ have unit upper density and zero lower density at x for all $x\in(0,1)$ except a set of Hausdorff dimension zero.*

Proof. This follows easily from Theorem 2.6 choosing, for example, $g(r)=r+e^{-1/r}$. □

Remarks. One can not use Theorem 2.4 to deduce information about the densities $\limsup_{r\downarrow 0}\mathcal{L}(A^+(f,x)\cap(x-r,x+r))/(2r)$, etc. If we argue as in the proof of Theorem 2.6 and try to prove that

$$\limsup_{r\downarrow 0}\mathcal{L}(A^+(f,x)\cap(x-r,x+r))/(2r)=1$$

assuming $\limsup_{r\downarrow 0}\mathcal{L}(E\cap(x-r,x+r))/(2r)=1$ where

$$\lim_{y\in E,y\to x}(f'(y)-f'(x))/(y-x)=2,$$

we end up with problems. We need to know that the set E can be chosen so that $\liminf_{r\downarrow 0}\mathcal{L}((x-r,x+r)\setminus E)/r^2=0$. We could prove this by modifying the proof of Theorem 2.4, but as we can see, using Theorem 2.5 gives a much stronger result with the same effort.

3 Conical Densities.

In this section we discuss briefly what do our examples tell about possible generalizations of Theorem 1.1 and what is still unknown.

Let f be as in Corollary 2.2 and let $C\subset(0,1)$ denote the set where conditions (2.2) and (2.3) hold. Define $\mathcal{G}=\{(t,f(t)):t\in(0,1)\}$ and $\mathcal{G}_C=\{(t,f(t)):t\in C\}$.

Fix $x=(t,f(t))\in\mathcal{G}_C$. It is clear that

$$\liminf_{r\downarrow 0}\mathcal{H}^1(B(x,r)\cap\mathcal{G}\cap H(x,\theta))/r>0 \tag{3.1}$$

whenever $\theta\in S^1$ is not perpendicular to the tangent of $\mathcal{G}$ at x. Therefore we consider only directions of the form $\theta=\pm(-f'(t),1)/(1+f'(t)^2)^{1/2}$. If $r>0$ is small, then $(s,f(s))\in B(x,r)$ for $s\in(t-cr,t+cr)$ and $c=(1+2f'(t)^2)^{-1/2}$.

It follows from (2.2)–(2.3) that $\mathcal{H}^1(B(x,r)\cap\mathcal{G}\cap H(x,\theta))\geq cr$. We conclude that (3.1) holds for all $x\in\mathcal{G}_C$ and $\theta\in S^1$. In particular, it holds $\mathcal{H}^1$-almost everywhere on the curve $\mathcal{G}$.

By modifying the above example, one can easily construct m-rectifiable surfaces S on $\mathbb{R}^n$, for any integer $1<m<n$, so that

$$\liminf_{r\downarrow 0} r^{-m}\mathcal{H}^m(B(x,r)\cap H(x,\theta)\cap S)>0 \tag{3.2}$$

for any $\theta\in S^{n-1}$ and for $\mathcal{H}^m$-almost all $x\in S$. One can take, for example, S to be the graph of the function

$$g\colon (0,1)\times\mathbb{R}^{m-1}\to\mathbb{R}^{n-m}\colon (x_1,\ldots,x_m)\mapsto (f(x_1),\ldots,f(x_1)),$$

where f is as above.

Marstrand's argument from [5, pp. 293–297] can be generalized to prove that in $\mathbb{R}^n$ claim (1) of Theorem 1.1 holds for $0<s<1$, and claim (2) for $n-1<s<n$, see also [6, Theorem 11.11]. For general $0<s<n$, the following is known, see Lorent [4] and Suomala [7]. If $m\in\mathbb{N}$, then a set $A\subset\mathbb{R}^n$ is called m-rectifiable if $\mathcal{H}^m$-almost all of it can be covered with a countable union of C^1-images of $\mathbb{R}^m$. A set A is called purely m-unrectifiable, if it intersects every C^1-image of $\mathbb{R}^m$ only in a set of $\mathcal{H}^m$ measure zero.

Theorem 3.1. *Let $A\subset\mathbb{R}^n$ with $\mathcal{H}^s(A)<\infty$ and let V be an m-dimensional linear subspace of $\mathbb{R}^n$. If either $0<s<m$ or if $s=m$ and A is purely m-unrectifiable, then for $\mathcal{H}^s$ almost every $x\in A$, there is $\theta\in V\cap S^{n-1}$ such that*

$$\liminf_{r\downarrow 0} r^{-s}\mathcal{H}^s(B(x,r)\cap H(x,\theta,\eta)\cap A)=0 \tag{3.3}$$

for any $\eta>0$.

The examples discussed above show that one cannot always take $\eta=0$ in (3.3) when $s\in[1,m)\cap\mathbb{N}$ and A is s-rectifiable. On the other hand, Besicovitch's argument from [1, pp. 317–320] can be modified to prove that even claim (1) of Theorem 1.1 holds for any purely 1-unrectifiable set $A\subset\mathbb{R}^n$ with $\mathcal{H}^1(A)<\infty$. It remains unknown if Theorem 3.1 holds with $\eta=0$ when either $s\in(1,n-1)$ is non-integral, or $s\in[2,n-1]\cap\mathbb{N}$ and A is purely s-unrectifiable.

References

[1] A. S. Besicovitch, *On the fundamental geometrical properties of linearly measurable plane sets of points II*, Math. Ann., **115** (1938), 296–329.

[2] A. M. Bruckner, J. B. Bruckner, and B. S. Thomson, *Real Analysis*, Prentice-Hall, Upper Saddle River, 1997.

[3] V. Jarník, *Sur les nombres dérivés approximatifs*, Fund. Math., **22** (1934), 4–16.

[4] A. Lorent, *A generalised conical density theorem for unrectifiable sets*, Ann. Acad. Sci. Fenn. Math., **28(2)** (2003), 415–431.

[5] J. M. Marstrand, *Some fundamental geometrical properties of plane sets of fractional dimensions*, Proc. London Math. Soc. (3), **4** (1954), 257–301.

[6] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and rectifiability*, Cambridge University Press, Cambridge, 1995.

[7] V. Suomala, *On the conical density properties of measures on* $\mathbb{R}^n$, Math. Proc. Cambridge Philos. Soc., **138(3)** (2005), 493–512.

[8] L. Zajíček, *Porosity and* σ*-porosity*, Real Anal. Exchange, **13(2)** (1987/88), 314–350.

[9] L. Zajíček, *Porosity, derived numbers and knot points of typical continuous functions*, Czechoslovak Math. J., **39(114)(1)** (1989), 45–52.

[10] L. Zajíček, *On essential derived numbers of typical continuous functions*, Tatra Mt. Math. Publ., **2** (1993), 123–125.