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A NEW CHARACTERIZATION OF BUCZOLICH'S UPPER SEMICONTINUOUSLY INTERGRABLE FUNCTIONS

Abstract

It is shown that if f is Henstock-Kurzweil integrable on a compact interval E in $\mathbb{R}^m$, then f is upper semicontinuously integrable on E if and only if there exists an increasing sequence $\{X_n\}$ of closed sets whose union is E, and $f|_{X_n}$ is bounded for each positive integer n.

1 Introduction.

In [2] Buczolich proved that if f is Henstock-Kurzweil integrable on a compact interval E in $\mathbb{R}^m$, then f is upper semicontinuously integrable on E if and only if there exists a Baire 1 function g on E such that $|f(x)| \leq g(x)$ for all $x \in E$. However, his proof involves transfinite induction. In this paper, we shall prove that f is upper semicontinuously integrable on E if and only if there exists an increasing sequence $\{X_n\}$ of closed sets whose union is E, and $f|_{X_n}$ is bounded for each positive integer n. As a result, we obtain a simpler proof of [2, Theorem 1].

2 Main Results.

For the rest of this paper, we shall follow the notations and terminologies used in [3]. The following results are crucial for this paper.

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Theorem 2.1. [3, Theorem 3.9] *Let X be a measurable subset of E. If F is the indefinite $\mathcal{HK}$-integral of a function f on E, then $f \in \mathcal{L}(X)$ if and only if $V_{\mathcal{HK}}F(X)$ is finite. Moreover,*

$$(L)\int_X |f| = V_{\mathcal{HK}}F(X)$$

even if one of the sides is equal to ∞.

Theorem 2.2. [4, Theorem 3.1] *Let F be an additive interval function on $\mathcal{I}$ such that $V_{\mathcal{HK}}F$ is absolutely continuous. If the following conditions are satisfied:*

(i) *$f : E \longrightarrow \mathbb{R}$ is a function such that $f = F'$ almost everywhere on E;*

(ii) *f is bounded on a nonempty closed set $X \subseteq E$ and $V_{\mathcal{HK}}F(X)$ is finite,*

then for $\varepsilon > 0$ there exists an upper semicontinuous gauge δ on X such that

$$\sum_{i=1}^{p} |f(\xi_i)\,|I_i| - F(I_i)| < \varepsilon$$

for each δ-fine partition $\{(I_i, \xi_i)\}_{i=1}^{p}$ anchored in X.

Our main result, which is equivalent to [2, Theorem 1], is

Theorem 2.3. *If F is the indefinite $\mathcal{HK}$-integral of a function f on E, then the following conditions are equivalent:*

(i) *There exists an increasing sequence $\{X_n\}$ of closed sets whose union is E, and $f|_{X_n}$ is bounded for each positive integer n.*

(ii) *For $\varepsilon > 0$ there exists an upper semicontinuous gauge δ on E such that*

$$\sum_{i=1}^{p} |f(x_i)\,|I_i| - F(I_i)| < \varepsilon$$

for each δ-fine partition $\{(I_1, x_1), \ldots, (I_p, x_p)\}$ in E.

(iii) *There exists a Baire 1 function $g : E \longrightarrow \mathbb{R}$ such that $|f(x)| \leq g(x)$ for all $x \in E$.*

PROOF. (i) $\implies$ (ii). An application of Theorem 2.1 yields

$$V_{\mathcal{HK}}F(X_k) = (L)\int_{X_k} |f|$$

for each positive integer k. Since $F' = f$ almost everywhere on E, and $f|_{X_n}$ is bounded for each positive integer n, an application of Theorem 2.2 shows that there exists an upper semicontinuous gauge Δ_k on X_k such that

$$\sum_{i=1}^{q} |f(\xi_i)\,|J_i| - F(J_i)| < \frac{\varepsilon}{2^{k+1}}$$

for each Δ_k-fine partition $\{(J_1, \xi_1), \dots, (J_q, \xi_q)\}$ anchored in X_k.

Define a gauge δ on E by

$$\delta(x) = \begin{cases} \Delta_1(x) & \text{if } x \in X_1 \\ \min\{\Delta_k, \operatorname{dist}(x, X_{k-1}\} & \text{if } x \in X_k \setminus X_{k-1} \text{ for some } k \in \{2, 3, \dots\}. \end{cases}$$

It is now clear that δ is upper semicontinuous on E, and assertion (ii) holds.

(ii) $\implies$ (iii). For $\varepsilon = 1$ there exists an upper semicontinuous gauge δ on E such that

$$\sum_{i=1}^{p} |f(x_i)\,|I_i| - F(I_i)| < 1 \tag{1}$$

for each δ-fine partition $\{(I_1, x_1), \dots, (I_p, x_p)\}$ in E.

For each positive integer n, we set $X_n := \{x \in E : \delta(x) \geq \frac{1}{n}\}$. The upper semicontinuity of the gauge δ on E implies that $\{X_n\}$ is an increasing sequence of closed sets whose union is E. Moreover, we may also assume that each X_n is nonempty. For each $x \in X_n$, we fix a subinterval J_n of E such that $x \in J_n$, $\operatorname{reg}(J_n) = 1$ and $|J_n| = \frac{1}{(2n)^m}$. Now, our choice of X_n implies that $\{(J_n, x)\}$ is a δ-fine partition anchored in $\{x\}$. By (1), we have

$$|f(x)| \leq \frac{1 + |F(J_n)|}{|J_n|} \leq \Big[1 + \omega(F, E)\Big](2n)^m,$$

observing that the finiteness of $\omega(F, E)$ follows from the continuity of F. Set $X_0 = \emptyset$. An application of [1, Theorem 10.12] shows that the function

$x \mapsto \left[1 + \omega(F, E)\right] \sum_{k=1}^{\infty} (2k)^m \chi_{X_k \setminus X_{k-1}}(x)$ gives the desired function g.

(iii) $\Longrightarrow$ (i). This follows immediately from [1, Theorem 10.12]. The proof is complete. □

Although the proof of Theorem 2.3 seems to be short and easy, the theorems used in the proof contain most of the difficulties of the proof of Theorem 2.3. As a simple application of our main result, we give a somewhat surprising corollary.

Corollary 2.4. *If f is upper semicontinuously integrable on E, then the indefinite $\mathcal{HK}$-integral of f is strongly derivable (see [5] for definition) almost everywhere on a portion of E.*

PROOF. This follows from Theorem 2.3, the Baire Category Theorem and Ward's Theorem for the Strong derivative [5]. □

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