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ANOTHER APPLICATION OF ROLLES'S THEOREM

Abstract

We find an analogue of Rolle's Theorem in (real variable) calculus for continuous complex valued functions defined on convex subsets of the complex plane.

In this note U is a convex subset of the complex plane $\mathbb{C}$ and f is a continuous complex valued function defined on U. If $u \in U$ and $v \in \mathbb{C}$, $|v| = 1$, and if f is defined on the set $\{u + tv : t \in (-\delta, \delta)\}$ for some positive number δ, then by the derivative of f at u in the direction v we mean the limit

$$\lim_{t \to 0,\, t \text{ real}} \frac{f(u + tv) - f(u)}{tv}.$$

This derivative is denoted $f'_v(u)$.

In this note we require that if $u_1 \in U$, $u_2 \in U$, $v = \dfrac{u_1 - u_2}{|u_1 - u_2|}$, then $f'_v(u)$ exists at all points $u \in U$ for which $u = u_2 + tv$ for some positive number t.

Continuous functions on $\mathbb{C}$ abound that are nowhere analytic, but nonetheless have derivatives in all directions. Witness for example

$$f_1(a + ib) = a - ib, \text{ and } f_2(a + ib) = 2a + 3ib \ (a, b \text{ real}).$$

It appears unlikely that Rolle's Theorem [2, p. 95] could be of much use for complex valued functions. Consider for example, the function $g(z) = e^z$ on the segment joining points 0 and $2\pi i$. Observe that $g(2\pi i) - g(0) = 0$, but $g'(z) = e^z$ vanishes nowhere. Nonetheless we have for f and U as given here:

Theorem 1. *Let $u_1, u_2 \in U$, $u_1 \neq u_2$, and $f(u_1) = f(u_2)$. Then*

$$\frac{\text{diameter } X}{\sqrt{2}} \geq \text{dist}(0, X),$$

where $X = \{f'_v(u) : u \in U \text{ and } v \text{ are such that } f'_v(u) \text{ is defined}\}$.

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Proof. Let u and v be complex numbers and let d be a positive number such that $|v| = 1$, $u \in U$, $u + dv \in U$ and $f(u) = f(u + dv)$. On the real interval $0 \leq t \leq d$ define the complex valued function

$$g(t) = \frac{f(u + tv)}{v},$$

and let g_1 and g_2 be the real valued functions such that $g = g_1 + ig_2$.

Now g, g_1 and g_2 are differentiable on $0 < t < d$ because $f(u + tv)$ has derivatives in the direction of v for $0 < t < d$. Likewise g, g_1 and g_2 are continuous on $0 \leq t \leq d$. Moreover,

$$g(0) = g(d), \; g_1(0) = g_1(d) \text{ and } g_2(0) = g_2(d)$$

because $f(u) = f(u + dv)$.

By Rolle's Theorem there are real numbers t_1 and t_2 such that $0 < t_1 < d$, $0 < t_2 < d$ and $g_1'(t_1) = g_2'(t_2) = 0$. Thus

$$g'(t_1) = ig_2'(t_1), \; g'(t_2) = g_1'(t_2)$$

and

$$\Big|g'(t_1) - g'(t_2)\Big| = \Big[g_1'(t_2)^2 + g_2'(t_1)^2\Big]^{\frac{1}{2}}. \tag{1}$$

But $g'(t) = f_v'(u + tv)$, so

$$\text{diameter}\, X \geq \big|g'(t_1) - g'(t_2)\big|. \tag{2}$$

Say $g_2'(t_1)^2 \geq g_1'(t_2)^2$ for definiteness. Then

$$\Big[g_2'(t_1)^2 + g_1'(t_2)^2\Big]^{\frac{1}{2}} \geq \big|g_1'(t_2)\big| \cdot \sqrt{2} = \big|g'(t_2)\big| \cdot \sqrt{2} = \big|f_v'(u + t_2 v)\big| \cdot \sqrt{2}$$

and

$$\Big[g_2'(t_1)^2 + g_1'(t_2)^2\Big]^{\frac{1}{2}} \geq \big|f_v'(u + t_2 v)\big| \cdot \sqrt{2}. \tag{3}$$

We combine (1), (2) and (3) to obtain

$$\text{diameter}\, X \geq \big|f_v'(u + t_2 v)\big| \cdot \sqrt{2}. \qquad \square$$

Clearly if F is analytic on an open set, then F has derivatives in all possible directions on this set. We can give another proof that F is locally one-to-one around any point w where $F'(w) \neq 0$ [1, Chapter II, Theorem 5.1]. Use the continuity of F' at w to find a disc V centered at w such that

$$\text{diameter}\, F'(V) < \frac{|F'(w)|}{3}.$$

It follows from Theorem 1 that F is one-to-one on V. Note that power series expansions of F were not needed here.

For functions F let $(\Delta F)(u_1, u_2)$ denote the difference quotient

$$\frac{F(u_1) - F(u_2)}{u_1 - u_2}.$$

Now if F is a differentiable real valued function on a convex subset V of the real line, then all values assumed by ΔF lie in the set $F'(V)$. This is clear from the Mean Value Theorem. But it need not to hold for f satisfying our hypotheses. We now see that at least the values assumed by Δf are not so "far" from the set X in Theorem 1.

Theorem 2. *Let f, U and X be as in Theorem 1. If*

$$\Delta f = \frac{f(u_1) - f(u_2)}{u_1 - u_2}$$

for u_1 and u_2 in U, then

$$\operatorname{dist}(\Delta f, X) \leq \frac{\text{diameter}\, X}{\sqrt{2}}.$$

PROOF. Let y be a complex number such that the distance from y to X exceeds $\frac{\text{diameter}\, X}{\sqrt{2}}$. For complex numbers s, put $g(s) = f(s) - sy$ and define

$$W = \Big\{ g'_v(u) : u \in U \text{ and } v \text{ are such that } g'_v(u) \text{ is defined} \Big\}.$$

Then $g'_v(u) = f'_v(u) - y$ and it follows that the distance from 0 to W exceeds

$$\frac{\text{diameter}\, W}{\sqrt{2}} = \frac{\text{diameter}\, X}{\sqrt{2}}.$$

By Theorem 1, g is a one-to-one function on U. Thus if $u_1 \in U$, $u_2 \in U$, $u_1 \neq u_2$, then

$$f(u_2) - u_2 y = g(u_2) \neq g(u_1) = f(u_1) - u_1 y$$

and therefore

$$\frac{f(u_1) - f(u_2)}{u_1 - u_2} \neq y.$$

It follows that y is not in the range of Δf, and therefore the distance from any value in the range of Δf to X cannot exceed $\frac{\text{diameter}\, X}{\sqrt{2}}$. $\square$

References

[1] S. Lang, *Complex Analysis*, 2nd ed., Springer-Verlag, New York, 1985.

[2] A. Wayne Roberts, *Introductory Calculus*, 2nd ed., Academic Press, New York, 1972.