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DESCRIPTIVE PROPERTIES OF σ-POROUS SETS

Abstract

We show that there exists a closed set $H \subset \mathbb{R}$ such that the set $P(H)$ of all points $x \in H$ at which H is porous can be covered by no $F_{\sigma\delta}$ σ-porous set. This improves Tkadlec's result ([T]). We also show that there exists a perfect nowhere dense non-σ-porous set $L \subset \mathbb{R}$ such that the set $P(L)$ is G_δ. This answers a question posed by Zajíček.

1 Introduction.

The notions of porosity and σ-porosity were studied in many papers from different points of view. We refer the reader to [Z_1] and [Z_3] for motivations and applications of these notions. Let us recall their definitions. Let (P, ρ) be a metric space, $M \subset P$, $x \in P$, and $R > 0$. Then we define

$$\begin{aligned}\theta(x, R, M) = &\sup\{r > 0;\ \text{there exists an open ball } B(z, r) \\ &\text{such that } \rho(x, z) < R \text{ and } B(z, r) \cap M = \emptyset\}, \\ p(x, M) = &\limsup_{R \to 0+} \frac{\theta(x, R, M)}{R}.\end{aligned}$$

We say that $M \subset P$ is *porous* if $p(x, M) > 0$ whenever $x \in M$. A set $M \subset P$ is said to be *σ-porous* if it is a countable union of porous sets.

Let $M \subset P$. We say that $x \in P$ is a *point of porosity of* M if $p(x, M) > 0$. We denote

$$P(M) = \{x \in M;\ p(x, M) > 0\}.$$

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It is known that each σ-porous set can be covered by a $G_{\delta\sigma}$ σ-porous set (see [FH]). On the other hand Foran – Humke ([FH]) and Tkadlec ([T]) showed that there exists a porous subset of $\mathbb{R}$ which can be covered by no F_σ σ-porous set and by no G_δ σ-porous set respectively. The problem whether each σ-porous set can be covered by an $F_{\sigma\delta}$ σ-porous set is implicitly posed in [Z$_1$]. We show that this is not the case. Namely, we prove the following theorem in Section 3.

Theorem 1.1. *There exists a closed set $H \subset \mathbb{R}$ such that the set $P(H)$ can be covered by no $F_{\sigma\delta}$ σ-porous set.*

Tkadlec's porous set with no G_δ σ-porous envelope is of the form $P(H)$, where $H \subset \mathbb{R}$ is a suitable perfect nowhere dense non-σ-porous set. Zajíček asked a question whether $P(H)$ has no G_δ σ-porous envelope whenever $H \subset \mathbb{R}$ is a perfect nowhere dense non-σ-porous set. In Section 4 we prove that this is not the case as the next theorem says.

Theorem 1.2. *There exists a non-σ-porous perfect nowhere dense set $L \subset \mathbb{R}$ such that $P(L)$ is G_δ.*

2 Several Lemmas.

We use the technique of construction of non-σ-porous sets developed in [ZP] to prove Theorem 1.1.

Notation 2.1. The symbols $\mathbb{N}$ and $\mathbb{N}_0$ stand for the sets of positive integers and non-negative integers respectively.

Let $M \subset \mathbb{R}$. Then the complement of M in $\mathbb{R}$ is denoted by M^c. Any set of the form $M \cap G$, where G is an open subset of $\mathbb{R}$ intersecting M, is called a portion of M.

Open ball and closed ball in $\mathbb{R}$ with center x and radius $s > 0$ are denoted by $B(x,s)$ and $\overline{B}(x,s)$ respectively. Let $B \subset \mathbb{R}$ be an open ball and $\omega > 1$. Then $\omega \star B$ denotes the open ball with the same center and with ω times greater radius. The symbol $\omega \star B$ has an analogical meaning when B is a closed ball. The center of a ball B is denoted by $c(B)$.

Let $\mathcal{V}$ be a system of closed balls in $\mathbb{R}$. Then $c(\mathcal{V})$ denotes the set of all centers of balls from $\mathcal{V}$. Let $S \subset \mathbb{R}$. The set of all points of accumulation of S is denoted by S'.

The following definitions and lemmas can be found in [ZP]. In [ZP], they are introduced in nonempty complete metric spaces without isolated points. However, from now on we will work on $\mathbb{R}$ with the usual metric.

Definition 2.2. (cf. [ZP, Definitions 2.3 and 2.5])

(i) Let $\mathcal{V}$ be a system of closed balls in $\mathbb{R}$. Then the symbol $\operatorname{ap}(\mathcal{V})$ stands for the set of all points $x \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exist infinitely many $B \in \mathcal{V}$ with $B \cap B(x, \varepsilon) \neq \emptyset$.

(ii) Let $\mathcal{V}$ be a nonempty system of closed balls in $\mathbb{R}$ satisfying

(a) $\mathcal{V}$ is point finite, i.e., each $x \in \mathbb{R}$ is contained at most in finitely many balls from $\mathcal{V}$,

(b) $\operatorname{ap}(\mathcal{V}) \subset c(\mathcal{V})$.

Then we say that $\mathcal{V}$ is a *B-system*.

(iii) Let $M \subset \mathbb{R}$, $x \in \mathbb{R}$ and B_1, B_2 be two closed balls in $\mathbb{R}$ with $x \in B_2 \subset B_1$. Then we denote

$$\Gamma(x, B_1, B_2, M) = \sup\{r/\rho(x,z);\ z \in B_1 \setminus B_2,\ B(z,r) \subset B_1 \setminus M\}.$$

Lemma 2.3. ([ZP, Lemma 2.4(i)]) *Let $\mathcal{V}$ be a B-system and, for every $B \in \mathcal{V}$, let $\mathcal{V}(B)$ be a B-system such that $\bigcup \mathcal{V}(B) \subset B$ and $c(B) \in c(\mathcal{V}(B))$. Then $\mathcal{U} = \bigcup\{\mathcal{V}(B);\ B \in \mathcal{V}\}$ is a B-system.*

The next two notions are quite technical but we will use mainly their properties described in Lemmas 2.8 – 2.10.

Definition 2.4. ([ZP, Definition 2.6]) Let $B \subset \mathbb{R}$ be a closed ball, S be a closed nonempty subset of B, and $n \in \mathbb{N}$, $\delta, \kappa, \alpha \in (0,1)$. We say that S has the *$\mathcal{C}(0, \delta, \kappa, \alpha)$-property in B* if $S = \{c(B)\}$. We say that S has the *$\mathcal{C}(n, \delta, \kappa, \alpha)$-property in B* if

$(\mathrm{C1})_n$ $\forall x \in S:\ \operatorname{dist}(x, B^c) > \delta^n \operatorname{diam} B$,

$(\mathrm{C2})_n$ $\sup\{r/\rho(y,z);\ B(z,r) \subset B \setminus S,\ y \neq z\} \le \kappa$ whenever $y \in S'$,

$(\mathrm{C3})_n$ $\forall x \in S':\ p(x, S) < \alpha\kappa$,

$(\mathrm{C4})_n$ S' has the $\mathcal{C}(n-1, \delta, \kappa, \alpha)$-property in B.

Definition 2.5. ([ZP, Definition 2.7]) Let $B \subset \mathbb{R}$ be a closed ball, $\mathcal{V}$ be a B-system, $n \in \mathbb{N}$, $\delta, \beta, \varepsilon \in (0,1)$. We say that $\mathcal{V}$ has the *$\mathcal{P}(0, \delta, \beta, \varepsilon)$-property in B* if $\mathcal{V} = \{B_0\}$, $c(B_0) = c(B)$, and $B_0 \subset B$. We say that $\mathcal{V}$ has the *$\mathcal{P}(n, \delta, \beta, \varepsilon)$-property in B* if

$(\mathrm{P1})_n$ $\forall V \in \mathcal{V}:\ \operatorname{dist}(V, B^c) > \operatorname{diam} V$,

$(\mathrm{P2})_n$ $\forall V \in \mathcal{V}:\ \operatorname{dist}(V, B^c) > \delta^n \operatorname{diam} B$,

$(\mathrm{P3})_n$ $\forall V \in \mathcal{V}:\ \operatorname{diam} V \le \frac{1}{2} \operatorname{diam} B$,

$(\mathrm{P4})_n$ there exists a B-system $\mathcal{R} \subset \mathcal{V}$ with the $\mathcal{P}(n-1, \delta, \beta, \varepsilon)$-property in B such that, for an arbitrary set J intersecting each ball from $\mathcal{V}$, we have

$$\forall R \in \mathcal{R}\ \forall x \in R:\ \operatorname{dist}(x, R^c) > \beta \operatorname{diam} R \Rightarrow \Gamma(x, B, R, J) < \varepsilon.$$

We will need the following easy observation later.

Observation 2.6. ([ZP, Observation 2.8]) *If $B \subset \mathbb{R}$ is a closed ball and $S \subset \mathbb{R}$ is a set with the $\mathcal{C}(n, \delta, \kappa, \alpha)$-property in B for some $n \in \mathbb{N}_0$, $\delta, \kappa, \alpha \in (0,1)$, then S is countable.*

Definition 2.7. (cf. [ZP, Definition 3.2]) Let $\omega > 1, r > 0, n \in \mathbb{N}$, and $A \subset \mathbb{R}$. Then we define

$$D_{\omega,r}(A) = A \setminus \bigcup \{B(x, \omega s);\ B(x,s) \cap A = \emptyset \text{ and } s \le r\},$$
$$D^n_{\omega,r}(A) = \underbrace{D_{\omega,r} \circ \cdots \circ D_{\omega,r}}_{n\text{-times}}(A).$$

Using [ZP, Lemma 2.12] we easily get the following lemma.

Lemma 2.8. *Let $x \in \mathbb{R}$, $r > 0$, $m \in \mathbb{N}_0$, $\delta, \kappa, \alpha \in (0,1)$, $\omega > 1$, $40\delta < \kappa$, $1/\omega < \alpha\kappa/10$, and $P_0 \subset P_1 \subset \cdots \subset P_m$ be subsets of $\mathbb{R}$ such that $x \in P_0$ and $P_j \subset D_{\omega,r}(P_{j+1})$, $j = 0, \dots, m-1$. Then there exists a set $S \subset P_m$ with the $\mathcal{C}(m, \delta, \kappa, \alpha)$-property in $\overline{B}(x, r)$.*

Lemma 2.9. ([ZP, Lemma 2.13]) *Let $B \subset \mathbb{R}$ be a closed ball, $m \in \mathbb{N}$, $\delta, \kappa, \alpha, \varepsilon \in (0,1)$, $10\kappa < \varepsilon$, and $S_m \subset B$ be a set with the $\mathcal{C}(m, \delta, \kappa, \alpha)$-property in B. Then there exists a function $s : S_m \to (0, +\infty)$ such that, for every function $r : S_m \to (0, +\infty)$ with $r \le s$, we have that $\mathcal{V}_m = \{\overline{B}(x, r(x));\ x \in S_m\}$ forms a B-system with the $\mathcal{P}(m, \delta, \alpha, \varepsilon)$-property in B.*

Lemma 2.10. ([ZP, Lemma 2.22]) *Let $\varepsilon \in (0, 1/8)$, $\alpha_n, \delta_n \in (0,1)$ for every $n \in \mathbb{N}$, $B \subset \mathbb{R}$ be a closed ball, and let $(\mathcal{U}_n)_{n=0}^{\infty}$ be a sequence of B-systems such that*

(i) $\mathcal{U}_0 = \{B\}$,

(ii) $\mathcal{U}_{n+1} = \bigcup \{\mathcal{U}_{n+1}(C);\ C \in \mathcal{U}_n\}$, *where $\mathcal{U}_{n+1}(C)$ has the $\mathcal{P}(n+1, \delta_{n+1}, \alpha_{n+1}, \varepsilon)$-property in C, $n \in \mathbb{N}_0$,*

(iii) *for every* $n \in \mathbb{N}$ *we have* $\alpha_n < (\delta_{n+1})^{n+1}$.

Then the set $\bigcap_{n=0}^{\infty} \bigcup \mathcal{U}_n$ *is a closed non-*σ*-porous set.*

Definition 2.11. Let $A \subset \mathbb{R}$. Then we define $W(A)$ by

$$x \in W(A) \overset{\text{def}}{\iff} \forall \omega \in \mathbb{R}, \omega > 1\, \forall k \in \mathbb{N}\, \exists r \in \mathbb{R}, r > 0 : \ x \in D^k_{\omega,r}(A).$$

Lemma 2.12. *Let* $A \subset \mathbb{R}$ *be a closed set such that each portion of* A *is non-*σ*-porous. Then* $W(A)$ *is dense in* A.

PROOF. It is easy to see that it is sufficient to prove that $W(A) \neq \emptyset$. We may and do assume that A is compact. Observe that if $F \subset \mathbb{R}$ is closed then $D_{\omega,r}(F)$ is closed as well. Observe also that if $\omega > 1$ and $F \subset \mathbb{R}$ is non-σ-porous, then there exists $r > 0$ such that $D_{\omega,r}(A)$ is non-σ-porous. Indeed, the set $F \setminus \bigcup_{n=1}^{\infty} D_{\omega,1/n}(F)$ is porous, hence there exists $n_0 \in \mathbb{N}$ such that $D_{\omega,1/n_0}(F)$ is non-σ-porous. These observations enable us to find a sequence $\{r_n\}_{n=1}^{\infty}$ of positive real numbers such that

$$\{D_{n+1,r_n} \circ \cdots \circ D_{2,r_1}(A)\}_{n=1}^{\infty}$$

is a decreasing sequence of compact non-σ-porous sets. Thus there exists $x \in \mathbb{R}$ such that

$$x \in D_{n+1,r_n} \circ \cdots \circ D_{2,r_1}(A)$$

for every $n \in \mathbb{N}$. To show that $x \in W(A)$ take $\omega > 1$ and $k \in \mathbb{N}$. Choosing $n > \omega + k$ we have

$$x \in D_{n+1,r_n} \circ \cdots \circ D_{2,r_1}(A) \subset D_{\omega,r_n} \circ \cdots \circ D_{\omega,r_{n-k+1}}(A).$$

Setting $r := \min\{r_n, \ldots, r_{n-k+1}\}$ we obtain

$$x \in D_{\omega,r_n} \circ \cdots \circ D_{\omega,r_{n-k+1}}(A) \subset \underbrace{D_{\omega,r} \circ \cdots \circ D_{\omega,r}}_{k\text{-times}}(A) = D^k_{\omega,r}(A).$$

□

3 Proof of Theorem 1.1.

Set $a_k = 2^k + 1$ for $k \in \mathbb{N}$. Let $I = [a, b]$ be a nondegenerate closed bounded interval. The system $\mathcal{H}_k(I)$ of closed intervals is defined by

$$\mathcal{H}_k(I) = \left\{ \left[a + (j-1) \cdot \frac{b-a}{a_k}, a + j \cdot \frac{b-a}{a_k} \right];\ j = 1, \ldots, a_k \right\}.$$

The interval $J \in \mathcal{H}_k(I)$ containing the center of I is denoted by $C(I,k)$. We define further systems of intervals by $\mathcal{H}_0 = \mathcal{J}_0 = \{[n,n+1];\ n \in \mathbb{Z}\}$, $\mathcal{H}_k = \bigcup\{\mathcal{H}_k(I);\ I \in \mathcal{H}_{k-1}\}$, and $\mathcal{J}_k = \bigcup\{\mathcal{H}_k(I) \setminus \{C(I,k)\};\ I \in \mathcal{J}_{k-1}\}$, $k \in \mathbb{N}$.

Notation 3.1. (i) The symbol $\mathfrak{S}$ stands for the set of all sequences $\mathcal{D} = \{\mathcal{D}_n\}_{n=0}^\infty$ of systems of intervals, such that for every $n \in \mathbb{N}_0$ we have

- $\emptyset \neq \mathcal{D}_n \subset \mathcal{J}_n$,
- $\forall I \in \mathcal{D}_n\ \exists J \in \mathcal{D}_{n+1} :\ J \subset I$,
- $\forall I \in \mathcal{D}_{n+1}\ \exists J \in \mathcal{D}_n :\ I \subset J$.

(ii) If $\mathcal{D}^1 = \{\mathcal{D}_n^1\}_{n=0}^\infty \in \mathfrak{S}$, $\mathcal{D}^2 = \{\mathcal{D}_n^2\}_{n=0}^\infty \in \mathfrak{S}$, then the symbol $\mathcal{D}^1 \prec \mathcal{D}^2$ means that $\mathcal{D}_n^1 \subset \mathcal{D}_n^2$ for every $n \in \mathbb{N}_0$.

(iii) Let $\mathcal{S}$ be a system of intervals and I be an interval. Then cardinality of the set $\{J \in \mathcal{S};\ J \subset I\}$ is denoted by $\alpha(\mathcal{S}, I)$.

(iv) Let $\mathcal{D} = \{\mathcal{D}_n\}_{n=0}^\infty \in \mathfrak{S}$ and $m \in \mathbb{N}$. Then we denote $q(\mathcal{D}, m) = \min\{\alpha(\mathcal{D}_m, J);\ J \in \mathcal{D}_{m-1}\}$.

(v) Let $\mathcal{D} = \{\mathcal{D}_n\}_{n=0}^\infty \in \mathfrak{S}$. Then we denote $\mathbf{F}(\mathcal{D}) = \bigcap_{n=0}^\infty \bigcup \mathcal{D}_n$.

The next observation is obvious.

Observation 3.2. *Let $\mathcal{D} = \{\mathcal{D}_n\}_{n=0}^\infty \in \mathfrak{S}$ and $j \in \mathbb{N}$. If an interval I contains an element of $\mathcal{D}_j$, then $\mathbf{F}(\mathcal{D}) \cap I \neq \emptyset$.*

The desired set H is defined by $H = \bigcap_{n=0}^\infty \bigcup \mathcal{J}_n$, i.e., $H = \mathbf{F}(\mathcal{J})$, where $\mathcal{J} = \{\mathcal{J}_n\}_{n=0}^\infty$. It is obvious that H is a nonempty perfect nowhere dense subset of $\mathbb{R}$. To prove that $P(H)$ can be covered by no $F_{\sigma\delta}$ σ-porous set we need the following auxiliary notions.

Definition 3.3. We say that $\mathcal{D} \in \mathfrak{S}$ is *good with constant* $c \in \mathbb{R}$ if there exists $n_0 \in \mathbb{N}$ such that $q(\mathcal{D}, m) \geq a_m - c$ for every $m \geq n_0$. We say that $\mathcal{D} \in \mathfrak{S}$ is *good* if there exists $c \in \mathbb{R}$ such that $\mathcal{D}$ is good with the constant c.

Lemma 3.4. *Let $\mathcal{D} \in \mathfrak{S}$ be good. Then the set $\mathbf{F}(\mathcal{D}) \cap P(H)$ is residual in $\mathbf{F}(\mathcal{D})$.*

PROOF. We employ the symbol $\operatorname{Int} X$ to denote the interior of $X \subset \mathbb{R}$. Since $\mathcal{D}$ is good we can clearly find $c \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $q(\mathcal{D}, m) \geq a_m - c > 3$ for every $m \geq n_0$. The set

$$A_k := \bigcup\{(c+3) \star \operatorname{Int} C(I,j);\ I \in \mathcal{H}_{j-1}, j \geq k\} \cap \mathbf{F}(\mathcal{D}), \quad k \in \mathbb{N},$$

is clearly open in $\mathbf{F}(\mathcal{D})$. Let $k \in \mathbb{N}$. Take $j \geq \max\{k, n_0\}$ and $I \in \mathcal{D}_j$. Since $(c+3) \star C(I, j+1) \subset I$ and $a_{j+1} \geq \alpha(\mathcal{D}_{j+1}, I) \geq q(\mathcal{D}, j+1) \geq a_{j+1} - c$, the interval $(c+3) \star \operatorname{Int} C(I, j+1)$ necessarily contains at least one interval from $\mathcal{D}_{j+1}$. Thus A_k intersects $\mathbf{F}(\mathcal{D}) \cap I$ by Observation 3.2. It implies that A_k is dense in $\mathbf{F}(\mathcal{D})$. Thus $A := \bigcap_{n=1}^{\infty} A_n$ is residual in $\mathbf{F}(\mathcal{D})$. We have $\operatorname{Int} C(I, j) \cap H = \emptyset$ for every $j \in \mathbb{N}$ and $I \in \mathcal{H}_{j-1}$. Thus $p(x, H) \geq 1/(c+3)$ whenever $x \in A$. Hence $A \subset P(H)$ and we get the conclusion. □

The next lemma relates the operation $D_{\omega,r}$ to good systems.

Lemma 3.5. *Let $c \geq 1$, $\omega > 1$, $r > 0$, and $\psi \geq 9c\omega$. Let $\mathcal{D} \in \mathfrak{S}$ be good with the constant c. Then there exist $r^* \in (0, r)$ and $\mathcal{Y} \in \mathfrak{S}$ such that $\mathcal{Y} \prec \mathcal{D}$, $\mathcal{Y}$ is good with the constant $2c\psi$, and $D_{\psi,r}(\mathbf{F}(\mathcal{D})) \subset \mathbf{F}(\mathcal{Y}) \subset D_{\omega,r^*}(\mathbf{F}(\mathcal{D}))$.*

PROOF. The intervals from $\mathcal{H}_j$, $j \in \mathbb{N}_0$, are of the same length, which we denote by b_j. Let $n_0 \in \mathbb{N}$ be such that $q(\mathcal{D}, m) \geq a_m - c > 0$ for every $m \geq n_0$. Choose $k \in \mathbb{N}$, $k \geq n_0$, such that $a_k - 2c\psi > 0$ and $b_k < r$. We define $\mathcal{Y} = \{\mathcal{Y}_j\}_{j=0}^{\infty} \in \mathfrak{S}$ by

$$\begin{aligned} &\mathcal{Y}_j = \mathcal{D}_j,\ j = 0, \ldots, k-1, \\ &\mathcal{D}_j^* = \{I \in \mathcal{D}_j;\ \exists J \in \mathcal{Y}_{j-1} :\ I \subset J\},\ j \geq k, \\ &\mathcal{Y}_j = \mathcal{D}_j^* \setminus \{I \in \mathcal{D}_j^*;\ \exists K \in \mathcal{H}_j \setminus \mathcal{D}_j :\ I \subset \psi \star K\},\ j \geq k. \end{aligned}$$

We have $\mathcal{Y}_j \neq \emptyset$, $j = 0, \ldots, k-1$. Suppose that $\mathcal{Y}_{m-1} \neq \emptyset$ and $m \geq k$. Let $J \in \mathcal{Y}_{m-1}$. If $K \in \mathcal{H}_m$, then at most $[\psi]$ intervals from $\mathcal{H}_m$ are contained in $\psi \star K$. (The symbol $[x]$ stands for the integer part of x.) So at most $[\psi][c]$ intervals from $\mathcal{H}_m(J)$ are covered by an interval of the form $\psi \star I$, where $I \in \mathcal{H}_m \setminus \mathcal{D}_m$, $I \subset J$. At most $[\psi] - 1$ intervals from $\mathcal{H}_m(J)$ are covered by an interval $\psi \star I$, where $I \in \mathcal{H}_m$, $I \not\subset J$. Thus we have

$$\alpha(\mathcal{Y}_m, J) \geq a_m - \psi c - \psi = a_m - \psi(c+1) \geq a_m - 2c\psi \geq a_k - 2c\psi > 0.$$

This shows that $\mathcal{Y}$ is good with the constant $2c\psi$. Clearly $\mathcal{Y} \prec \mathcal{D}$. The inclusion $D_{\psi,r}(\mathbf{F}(\mathcal{D})) \subset \mathbf{F}(\mathcal{Y})$ follows by the definition. Indeed, if $x \in \mathbf{F}(\mathcal{D}) \setminus \mathbf{F}(\mathcal{Y})$, then there exist $m \geq k$, $I \in \mathcal{D}_m \setminus \mathcal{Y}_m$, $K \in \mathcal{H}_m \setminus \mathcal{D}_m$ with $x \in I \subset \psi \star K$. It gives $\operatorname{Int} K \cap \mathbf{F}(\mathcal{D}) = \emptyset$, $\operatorname{diam} K = b_m \leq b_k < r$, and $x \in \psi \star K$. Thus $x \notin D_{\psi,r}(\mathbf{F}(\mathcal{D}))$.

Set $r^* = \frac{1}{2} b_k$. To prove the second inclusion choose an open ball $B(z, s)$ such that $s \leq r^*$ and $B(z, s) \cap \mathbf{F}(\mathcal{D}) = \emptyset$. Let $y \in \mathbf{F}(\mathcal{Y})$ and let $j \in \mathbb{N}$ be the smallest number such that $B(z, s)$ contains an interval from $\mathcal{H}_j$. Clearly $j > k$. The ball $B(z, s)$ intersects at most two elements of $\mathcal{H}_{j-1}$. We find $O_1, O_2 \in \mathcal{H}_{j-1}$ with $B(z, s) \subset O_1 \cup O_2$. We distinguish two possibilities.

1) The intervals O_1, O_2 are in $\mathcal{D}_{j-1}$. Since $a_j \geq \alpha(\mathcal{D}_j, O_i) \geq a_j - c$, $i = 1, 2$, we have $2s \leq (c+2)b_j$. Take $K \in \mathcal{H}_j$ with $K \subset B(z,s)$ such that $\operatorname{dist}(y, K)$ is minimal. Find $I \in \mathcal{Y}_j$ with $y \in I$. We have $I \not\subset \psi \star K$ by the definition of $\mathcal{Y}$. An easy computation gives $|z-y| \geq \operatorname{dist}(K, I) \geq \frac{1}{2}(\psi - 3)b_j$. We get

$$\frac{s}{|z-y|} \leq \frac{\frac{1}{2}(c+2)b_j}{\frac{1}{2}(\psi-3)b_j} \leq \frac{c+2}{9c\omega - 3} < \frac{1}{\omega}.$$

2) There is $i \in \{1, 2\}$ with $O_i \notin \mathcal{D}_{j-1}$. Find $I \in \mathcal{Y}_{j-1}$ with $y \in I$. We have $s \leq b_{j-1}$ and $I \not\subset \psi \star O_i$. It gives

$$|z-y| \geq \operatorname{dist}(O_i, I) - |O_i| > \frac{1}{2}(\psi-3)b_{j-1} - b_{j-1} = \frac{1}{2}(\psi-5)b_{j-1}$$

and

$$\frac{s}{|z-y|} \leq \frac{b_{j-1}}{\frac{1}{2}(\psi-5)b_{j-1}} \leq \frac{2}{\psi-5} \leq \frac{2}{9c\omega-5} < \frac{1}{\omega}.$$

In both cases, we have $y \notin B(z, \omega s)$. Since $\mathbf{F}(\mathcal{Y}) \subset \mathbf{F}(\mathcal{D})$, we obtain $\mathbf{F}(\mathcal{Y}) \subset D_{\omega, r^*}(\mathbf{F}(\mathcal{D}))$. □

Now suppose that $P(H)$ is contained in an $F_{\sigma\delta}$ set M. Our aim is to prove that M is non-σ-porous. The set M can be written as follows

$$M = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} M(n,m),$$

where

- $M(n,m)$ is closed for every $n, m \in \mathbb{N}$,
- for every $n, m \in \mathbb{N}$, $n > 1$, there exists $m' \in \mathbb{N}$ with $M(n,m) \subset M(n-1, m')$.

Definition 3.6. Let $n \in \mathbb{N}$ and $\mathcal{D} \in \mathfrak{S}$. The symbol $Z(n, \mathcal{D})$ stands for the set of all points $y \in \mathbf{F}(\mathcal{D})$ such that there exist $\mathcal{D}^* \in \mathfrak{S}$, $s > 0$, and $m \in \mathbb{N}$ such that

- $\mathcal{D}^*$ is good and $\mathcal{D}^* \prec \mathcal{D}$,
- $y \in W(\mathbf{F}(\mathcal{D}^*))$,
- $\overline{B}(y, s) \cap \mathbf{F}(\mathcal{D}^*) \subset M(n,m)$.

Lemma 3.7. *Let $n \in \mathbb{N}$ and $\mathcal{D}, \mathcal{D}^* \in \mathfrak{S}$ be good. If $\mathcal{D}^* \prec \mathcal{D}$, then $Z(n, \mathcal{D}) \cap \mathbf{F}(\mathcal{D}^*)$ is dense in $\mathbf{F}(\mathcal{D}^*)$.*

PROOF. Lemma 3.4 and the Baire Category Theorem show that the set

$$O := \{y \in \mathbf{F}(\mathcal{D}^*);\ \exists m \in \mathbb{N}\, \exists s > 0 :\ \overline{B}(y,s) \cap \mathbf{F}(\mathcal{D}^*) \subset M(n,m)\}$$

is open and dense in $\mathbf{F}(\mathcal{D}^*)$. An easy computation yields that each portion of $\mathbf{F}(\mathcal{D}^*)$ has positive Lebesgue measure, in particular, each portion of $\mathbf{F}(\mathcal{D}^*)$ is not σ-porous. Lemma 2.12 gives that $O \cap W(\mathbf{F}(\mathcal{D}^*))$ is dense in $\mathbf{F}(\mathcal{D}^*)$. Since $O \cap W(\mathbf{F}(\mathcal{D}^*)) \subset Z(n,\mathcal{D})$, we get the conclusion. □

Setting 3.8. Now we fix real numbers $\varepsilon, \kappa, \alpha_n, \delta \in (0,1)$, $\omega_n > 1$ ($n \in \mathbb{N}$) such that $10\kappa < \varepsilon < 1/8$, $40\delta < \kappa$, $\alpha_n < \delta^{n+1}$, and $1/\omega_n < \alpha_n \kappa/10$.

Lemma 3.9. *Let $n \in \mathbb{N}$, $r > 0$, $\mathcal{D} \in \mathfrak{S}$ be good, and $x \in W(\mathbf{F}(\mathcal{D}))$. Then there exists $r^* \in (0,r)$ and a set $S \subset \mathbb{R}$ such that S has the $\mathcal{C}(n,\delta,\kappa,\alpha_n)$-property in $\overline{B}(x,r^*)$ and $S \setminus \{x\} \subset Z(n+1,\mathcal{D})$.*

PROOF. We may and do assume that $\mathcal{D} \in \mathfrak{S}$ is good with a constant $c \in \mathbb{N}$. Set $c_0 := c$, $\psi_0 := 9c_0\omega_n$, $c_j := 2c_{j-1}\psi_{j-1}$, and $\psi_j := 9c_j\omega_n$, where $j = 1,\dots,n$. Using $x \in W(\mathbf{F}(\mathcal{D}))$ we find $\tilde{r} \in (0,r)$ such that $x \in D^n_{\psi_{n-1},\tilde{r}}(\mathbf{F}(\mathcal{D}))$. According to Lemma 3.5 we find $\mathcal{Y}^0, \mathcal{Y}^1, \mathcal{Y}^2, \dots, \mathcal{Y}^n$ from $\mathfrak{S}$ and positive real numbers $r_0^* > r_1^* > \dots > r_n^*$ such that

- $\mathcal{Y}^0 = \mathcal{D}$, $r_0^* = \tilde{r}$,
- $\mathcal{Y}^j$ is good with the constant c_j, $j = 0,\dots,n-1$,
- $\mathcal{Y}^{j+1} \prec \mathcal{Y}^j$, $j = 0,\dots,n-1$,
- $D_{\psi_j,r_j^*}(\mathbf{F}(\mathcal{Y}^j)) \subset \mathbf{F}(\mathcal{Y}^{j+1}) \subset D_{\omega_n,r^*_{j+1}}(\mathbf{F}(\mathcal{Y}^j))$, $j = 0,\dots,n-1$.

Using the inequalities $\psi_0 < \psi_1 < \dots < \psi_{n-1}$ and $\tilde{r} = r_0^* > r_1^* > \dots > r_n^*$ we get

$$\begin{aligned} x \in D^n_{\psi_{n-1},\tilde{r}}(\mathbf{F}(\mathcal{D})) &\subset D_{\psi_{n-1},\tilde{r}} \circ D_{\psi_{n-2},\tilde{r}} \circ \dots \circ D_{\psi_1,\tilde{r}} \circ D_{\psi_0,\tilde{r}}(\mathbf{F}(\mathcal{D})) \\ &\subset D_{\psi_{n-1},\tilde{r}} \circ D_{\psi_{n-2},\tilde{r}} \circ \dots \circ D_{\psi_1,\tilde{r}}(\mathbf{F}(\mathcal{Y}^1)) \\ &\ \ \vdots \\ &\subset D_{\psi_{n-1},\tilde{r}}(\mathbf{F}(\mathcal{Y}^{n-1})) \subset \mathbf{F}(\mathcal{Y}^n). \end{aligned}$$

Hence we have $x \in \mathbf{F}(\mathcal{Y}^j)$ for $j = 0,\dots,n$. Set $r^* := r_n^*$ and $P_j := (\mathbf{F}(\mathcal{Y}^{n-j}) \cap Z(n+1,\mathcal{D})) \cup \{x\}$, $j = 0,\dots,n$. Then we have

$$\begin{aligned} P_j \subset \mathbf{F}(\mathcal{Y}^{n-j}) &\subset D_{\omega_n,r^*_{n-j}}(\mathbf{F}(\mathcal{Y}^{n-j-1})) \\ &\subset D_{\omega_n,r^*}(\mathbf{F}(\mathcal{Y}^{n-j-1})),\ j = 0,\dots,n-1. \end{aligned}$$

This and the inclusion $P_j \subset P_{j+1}$ give $P_j \subset D_{\omega_n, r^*}(P_{j+1})$, since P_{j+1} is dense in $\mathbf{F}(\mathcal{Y}^{n-j-1})$ by Lemma 3.7. We see that the assumptions of Lemma 2.8 are satisfied and so we get the desired set S with the $\mathcal{C}(n,\delta,\kappa,\alpha_n)$-property in $\overline{B}(x, r^*)$ and with $S \setminus \{x\} \subset P_n \setminus \{x\} \subset Z(n+1, \mathcal{D})$. □

Now we will construct inductively a sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$ of countable B-systems such that

(i) $\mathcal{U}_0 = \{U_0\}$, where $U_0 \subset \mathbb{R}$ is a closed ball,

(ii) $\mathcal{U}_n = \bigcup\{\mathcal{U}_n(C);\ C \in \mathcal{U}_{n-1}\}$, where $\mathcal{U}_n(C)$ has the $\mathcal{P}(n,\delta,\alpha_n,\varepsilon)$-property in C, $n \in \mathbb{N}$.

Moreover, for every $n \in \mathbb{N}$ and $C \in \mathcal{U}_{n-1}$, we will construct a set $S_n(C)$ and a good sequence $\mathcal{D}(n, C) \in \mathfrak{S}$ such that

(iii) $c(C) \in W(\mathbf{F}(\mathcal{D}(n, C)))$,

(iv) $\forall C^* \in \mathcal{U}_n(C):\ \mathcal{D}(n+1, C^*) \prec \mathcal{D}(n, C)$,

(v) $\forall C^* \in \mathcal{U}_n(C), c(C^*) \neq c(C)\ \exists m \in \mathbb{N}:\ \mathbf{F}(\mathcal{D}(n+1, C^*)) \cap C^* \subset M(n+1, m)$,

(vi) $S_n(C)$ has the $\mathcal{C}(n,\delta,\kappa,\alpha_n)$-property in C,

(vii) $S_n(C) \setminus \{c(C)\} \subset Z(n+1, \mathcal{D}(n, C))$.

Using Lemma 3.7 we have $W(\mathbf{F}(\mathcal{J})) = W(H) \neq \emptyset$. Choose $x_0 \in W(H)$. According to Lemma 3.9 there exist $r_0 > 0$ and a set $S_1 \subset \mathbb{R}$ such that

- S_1 has the $\mathcal{C}(1,\delta,\kappa,\alpha_1)$-property in $\overline{B}(x_0, r_0)$,
- $S_1 \setminus \{x_0\} \subset Z(2, \mathcal{J})$.

We set $U_0 = \overline{B}(x_0, r_0)$, $\mathcal{U}_0 = \{U_0\}$, $\mathcal{D}(1, U_0) = \mathcal{J}$, $S_1(U_0) = S_1$. Assume that we have constructed a countable B-system $\mathcal{U}_{n-1}$ and the corresponding $S_n(C)$ and $\mathcal{D}(n, C)$ for $C \in \mathcal{U}_{n-1}$. We will construct $\mathcal{U}_n$ and the corresponding $S_{n+1}(C)$'s and $\mathcal{D}(n+1, C)$'s.

Take $C \in \mathcal{U}_{n-1}$. Using (vi) and Lemma 2.9 we find a function $r_1 : S_n(C) \to (0, +\infty)$ such that, for every function $r : S_n(C) \to (0, +\infty)$ with $r \leq r_1$, we have that the set $\{\overline{B}(z, r(z));\ z \in S_n(C)\}$ is a B-system with the $\mathcal{P}(n,\delta,\alpha_n,\varepsilon)$-property in C. Take $y \in S_n(C) \setminus \{c(C)\}$. Since $y \in Z(n+1, \mathcal{D}(n, C))$ by (vii), we can find $t(y) \in (0, r_1(y))$, a good sequence $\mathcal{D}^y \in \mathfrak{S}$, and $m \in \mathbb{N}$ such that $\mathcal{D}^y \prec \mathcal{D}(n, C)$, $y \in W(\mathbf{F}(\mathcal{D}^y))$, and $\overline{B}(y, t(y)) \cap \mathbf{F}(\mathcal{D}^y) \subset M(n+1, m)$. Using Lemma 3.9 we find $r(y) \in (0, t(y))$ and a set S^y such that

- S^y has the $\mathcal{C}(n+1,\delta,\kappa,\alpha_{n+1})$-property in $\overline{B}(y,r(y))$,
- $S^y\setminus\{y\}\subset Z(n+2,\mathcal{D}^y)$.

For $y=c(C)$ we have $y\in W(\mathbf{F}(\mathcal{D}(n,C)))$ by (iii). Using Lemma 3.9 we find $r(y)\in(0,r_1(y))$, S^y with the $\mathcal{C}(n+1,\delta,\kappa,\alpha_{n+1})$-property in $\overline{B}(y,r(y))$, and $S^y\setminus\{y\}\subset Z(n+2,\mathcal{D}(n,C))$. Set $\mathcal{D}^y=\mathcal{D}(n,C)$.

We set $\mathcal{U}_n(C)=\{\overline{B}(y,r(y));\ y\in S_n(C)\}$ and $S_{n+1}(C^*)=S^y$, $\mathcal{D}(n+1,C^*)=\mathcal{D}^y$, where $y=c(C^*)$, $C^*\in\mathcal{U}_n(C)$. The system $\mathcal{U}_n(C)$ is a B-system with the $\mathcal{P}(n,\delta,\alpha_n,\varepsilon)$-property in C by Lemma 2.9. Set

$$\mathcal{U}_n=\bigcup\{\mathcal{U}_n(C);\ C\in\mathcal{U}_{n-1}\}.$$

The system $\mathcal{U}_n$ is a B-system according to Lemma 2.3. The sets $S_n(C)$, $C\in\mathcal{U}_{n-1}$, are countable (Observation 2.6) and $\mathcal{U}_{n-1}$ is also countable, thus $\mathcal{U}_n$ is countable as well. This finishes the construction of $\mathcal{U}_n$'s. Conditions (i) – (vii) are clearly satisfied.

Using Theorem 2.10 we have that the set

$$L_0=\bigcap_{n=0}^{\infty}\bigcup\mathcal{U}_n$$

is a closed non-σ-porous set.

Set $L_1=L_0\setminus\bigcup_{n=0}^{\infty}c(\mathcal{U}_n)$. The set $\bigcup_{n=0}^{\infty}c(\mathcal{U}_n)$ is countable and therefore L_1 is non-σ-porous.

Take $x\in L_1$ and consider the following tree of sequences of balls

$$\mathcal{T}=\{(U_1,\dots,U_k);\ x\in U_i\in\mathcal{U}_i(U_{i-1}),\ i=1,\dots,k\}\cup\{\emptyset\}.$$

The tree $\mathcal{T}$ is clearly infinite. Since the $\mathcal{U}_n$'s are point finite, the tree $\mathcal{T}$ is finite splitting. Using König's Lemma ([K, 4.12]), we get a sequence $\{U_k\}_{k=1}^{\infty}$ such that $x\in U_k\in\mathcal{U}_k(U_{k-1})$ for every $k\in\mathbb{N}$. Choose $n\in\mathbb{N}$. Since $x\notin\bigcup_{j=0}^{\infty}c(\mathcal{U}_j)$ and $\lim_{j\to\infty}\operatorname{diam}U_j=0$, there exists $k_0\in\mathbb{N}$, $k_0\geq n$, with $c(U_{k_0-1})\neq c(U_{k_0})$. Using (v) we find $m\in\mathbb{N}$ such that $\mathbf{F}(\mathcal{D}(k_0+1,U_{k_0}))\cap U_{k_0}\subset M(k_0+1,m)$. Since $\mathcal{D}(j+1,U_j)\prec\mathcal{D}(j,U_{j-1})$ by (iv), we have $\mathbf{F}(\mathcal{D}(j+1,U_j))\subset\mathbf{F}(\mathcal{D}(j,U_{j-1}))$. This and $\lim_{j\to\infty}\operatorname{diam}U_j=0$ imply that $x\in\mathbf{F}(\mathcal{D}(k_0+1,U_{k_0}))$. Thus $x\in M(k_0+1,m)$ and therefore $x\in M(n,m')$ for some $m'\in\mathbb{N}$. This shows that $L_1\subset M$. Hence M is a non-σ-porous set and the proof is complete.

4 Proof of Theorem 1.2.

To prove Theorem 1.2 we employ the technique of construction of non-σ-porous sets developed by Zajíček in [Z$_2$].

Definition 4.1. Let $\varepsilon \in (0,1)$ and $G \subset \mathbb{R}$, $\emptyset \neq G \neq \mathbb{R}$, be an open set. We say that a system $\mathcal{B}$ of open nonempty intervals is a $[G,\varepsilon]$-*system* if the following conditions hold:

(a) the system $\{\overline{B};\ B \in \mathcal{B}\}$ does not cover G and is discrete in G (i.e., for each $x \in G$ there exists a neighborhood of x which intersects at most one member of $\{\overline{B};\ B \in \mathcal{B}\}$),

(b) if $y \in G$, $r > 0$, and $B(y, \frac{1}{\varepsilon}r) \setminus G \neq \emptyset$, then $B(y,r)$ contains a member of $\mathcal{B}$,

(c) if $x \in \partial G$ and J is a set intersecting each member of $\mathcal{B}$, then $p(x, J \cup (\mathbb{R} \setminus G)) = 0$,

(d) for every $B \in \mathcal{B}$ we have $(2 \star B \setminus B) \cap (\bigcup \mathcal{B}) = \emptyset$,

(e) for every $B \in \mathcal{B}$ we have $\operatorname{dist}(B, G^c) > \operatorname{diam} B$.

The next lemma relates the quantity $\Gamma(x, B_1, B_2, M)$ (Definition 2.2(iii)) to the porosity index $p(x, M)$.

Lemma 4.2. ([ZP, Lemma 2.15]) *Let $\varepsilon \in (0,1)$, $M \subset \mathbb{R}$, $x \in M$, and $(B_n)_{n=1}^{\infty}$ be a sequence of closed balls in $\mathbb{R}$ such that for every $n \in \mathbb{N}$ we have*

(i) $x \in B_n$,

(ii) $\operatorname{dist}(B_{n+1}, B_n^c) \geq \operatorname{diam} B_{n+1}$,

(iii) $\Gamma(x, B_n, B_{n+1}, M) < \varepsilon$,

(iv) $\operatorname{diam} B_{n+1} \leq \frac{1}{2} \operatorname{diam} B_n$.

Then $p(x, M) < 4\varepsilon$.

The following observation, which can be verified by an easy calculation, enables us to use Zajíček's result from [Z$_2$].

Observation 4.3. *Let $\varepsilon \in (0,1)$ and $G \subset \mathbb{R}$, $\emptyset \neq G \neq \mathbb{R}$, be an open set. If $\mathcal{B}$ is a G-system (see* [Z$_2$] *for the definition) with respect to the function*
$g(t) = \max\{\sqrt{t}, \frac{1}{\varepsilon}t\}$, $t \in [0, +\infty)$, *then $\mathcal{B}$ satisfies* (a) – (c) *of Definition 4.1.*

Lemma 4.4. *Let $\eta > 0$, $\varepsilon \in (0,1)$, and $G \subset \mathbb{R}$, $\emptyset \neq G \neq \mathbb{R}$, be an open set. Then there exists a $[G,\varepsilon]$-system $\mathcal{B}$ such that* $\operatorname{diam} B < \eta$ *for every $B \in \mathcal{B}$.*

PROOF. Using Lemma 2 in [Z_2] and Observation 4.3 we find a G-system $\mathcal{B}_0$ satisfying (a) – (c) of the definition of $[G,\varepsilon]$-system. Then for every $B \in \mathcal{B}_0$ we find a nonempty open interval $C(B) \subset B$ such that $2 \star C(B) \subset B$, $\operatorname{diam} C(B) < \eta$, and $\operatorname{dist}(C(B), B^c) > \operatorname{diam} C(B)$. We set $\mathcal{B} := \{C(B); B \in \mathcal{B}\}$. Using [$Z_2$, Note 1(iii)] we see that $\mathcal{B}$ has the desired properties. □

Observation 4.5. *Let $\mathcal{B}$ be a $[G,\varepsilon]$-system.*

(i) *We have $\partial(\bigcup \mathcal{B}) = \partial G \cup \bigcup\{\partial B;\ B \in \mathcal{B}\}$.*

(ii) *If $B \in \mathcal{B}$, then $\operatorname{diam} B < \frac{1}{2} \operatorname{diam} G$.*

Lemma 4.4 makes the following constructions possible.

Construction 4.6. (cf. [Z_2, Construction 1]) *Let $m \in \mathbb{N}$ and let $G \subset \mathbb{R}$, $\emptyset \neq G \neq \mathbb{R}$, be an open set. Then we choose a system $\tilde{\mathcal{D}}(G,m)$ such that*

- *$\tilde{\mathcal{D}}(G,m)$ is a $[G, 1/(m+1)]$-system,*
- *if $B \in \tilde{\mathcal{D}}(G,m)$, then $\operatorname{diam} B < 1/m$.*

Further set

- *$\tilde{R}(G,m) = G \setminus \bigcup\{\overline{B};\ B \in \tilde{\mathcal{D}}(G,m)\}$.*

Construction 4.7. (cf. [Z_2, Construction 2]) *Let $m \in \mathbb{N}$ and let $G \subset \mathbb{R}$, $\emptyset \neq G \neq \mathbb{R}$, be an open set. Then we define a sequence of nonempty systems of nonempty open intervals*

$$\tilde{\mathcal{S}}_1(G,m),\ \tilde{\mathcal{S}}_2(G,m), \dots$$

and a sequence of nonempty open sets

$$G \supset \tilde{R}_1(G,m) \supset \tilde{R}_2(G,m) \supset \dots$$

inductively in the following way:

(i) *$\tilde{\mathcal{S}}_1(G,m) = \tilde{\mathcal{D}}(G,m)$ and $\tilde{R}_1(G,m) = \tilde{R}(G,m)$,*

(ii) *if $\tilde{\mathcal{S}}_k(G,m)$ and $\tilde{R}_k(G,m)$ are defined, then we set*

$$\tilde{\mathcal{S}}_{k+1}(G,m) = \tilde{\mathcal{D}}(\tilde{R}_k(G,m), m) \quad \text{and} \quad \tilde{R}_{k+1}(G,m) = \tilde{R}(\tilde{R}_k(G,m), m).$$

Construction 4.8. (cf. [Z_2, Construction 3])

(i) *Set* $U = (-1,1)$ *and* $\tilde{\mathcal{K}}_0 = \tilde{\mathcal{K}}_0^0 = \{U\}$.

(ii) *If* $\tilde{\mathcal{K}}_n$ *is defined, then we set*

$$\tilde{\mathcal{K}}_{n+1}^k = \bigcup\{\tilde{\mathcal{S}}_k(B, n+1);\ B \in \tilde{\mathcal{K}}_n\},\ k \in \mathbb{N}, \quad \textit{and}\ \ \tilde{\mathcal{K}}_{n+1} = \bigcup_{k=1}^{n+1} \tilde{\mathcal{K}}_{n+1}^k.$$

We set $L = \overline{\bigcap_{n=1}^\infty \bigcup \tilde{\mathcal{K}}_n}$. Lemma 5 of [$Z_2$] shows that the set $\bigcap_{n=1}^\infty \bigcup \tilde{\mathcal{K}}_n$ is nowhere dense since $\bigcap_{n=1}^\infty \bigcup \tilde{\mathcal{K}}_n = A(U, \{n\}_{n=1}^\infty)$ (see [Z_2] for the definition of $A(U, \{n\}_{n=1}^\infty)$). Thus the set L is a closed nowhere dense set.

By definition we have that each $[G, 1/(m+1)]$-system, $m \in \mathbb{N}$, is a G-system with respect to the function $g(t) = 2t$, $t \in [0, \infty)$. According to Lemma 1 and Proposition from [Z_2], we have that $\bigcap_{n=1}^\infty \bigcup \tilde{\mathcal{K}}_n$ is a non-σ-$\langle g \rangle$-porous set, hence L is a non-σ-$\langle g \rangle$-porous. Since the notions of σ-porosity and σ-$\langle g \rangle$-porosity coincides ([Z_4, Lemma E]), we have that L is not σ-porous.

The next lemmas summarize properties of the systems $\tilde{\mathcal{K}}_n$'s, which we will need in the sequel. The first lemma deals with topological and metric properties and the second one captures some properties related to porosity.

Lemma 4.9. (i) *For every* $n \in \mathbb{N}_0$ *we have*

$$\overline{\bigcup \tilde{\mathcal{K}}_n} = \bigcup \tilde{\mathcal{K}}_n \cup \bigcup_{j=0}^n \bigcup\{\partial B;\ B \in \tilde{\mathcal{K}}_j\}.$$

(ii) *For every* $n \in \mathbb{N}_0$, $B_1 \in \tilde{\mathcal{K}}_n$, *and* $B_2 \in \tilde{\mathcal{K}}_{n+1}$ *with* $B_2 \subset B_1$, *we have* $\operatorname{diam} B_2 < \frac{1}{2} \operatorname{diam} B_1$.

(iii) *The set* L *intersects each interval from* $\bigcup_{j=0}^\infty \tilde{\mathcal{K}}_j$.

(iv) *We have* $L = \bigcap_{n=0}^\infty \overline{\bigcup \tilde{\mathcal{K}}_n}$.

(v) *For every* $n \in \mathbb{N}_0$, $B_1 \in \tilde{\mathcal{K}}_n$, *and* $B_2 \in \tilde{\mathcal{K}}_{n+1}$ *with* $B_2 \subset B_1$, *we have* $\operatorname{dist}(B_2, B_1^c) \geq \operatorname{diam} B_2$.

PROOF. (i) Using Observation 4.5(i) we see that for every $C \in \tilde{\mathcal{K}}_j$, $j \in \mathbb{N}_0$, we have

$$\partial C \subset \overline{\bigcup\{\partial B;\ B \in \tilde{\mathcal{K}}_{j+1}\}}.$$

This yields

$$\bigcup\{\partial C;\ C \in \tilde{\mathcal{K}}_j\} \subset \overline{\bigcup\{\partial B;\ B \in \tilde{\mathcal{K}}_{j+1}\}},\ j \in \mathbb{N}_0.$$

Then we get

$$\bigcup_{j=0}^{n}\bigcup\{\partial B;\ B \in \tilde{\mathcal{K}}_j\} \subset \overline{\bigcup\{\partial B;\ B \in \tilde{\mathcal{K}}_n\}} \subset \overline{\bigcup \tilde{\mathcal{K}}_n}.$$

Thus we have

$$\bigcup \tilde{\mathcal{K}}_n \cup \bigcup_{j=0}^{n}\bigcup\{\partial B;\ B \in \tilde{\mathcal{K}}_j\} \subset \overline{\bigcup \tilde{\mathcal{K}}_n}.$$

To prove the inverse inclusion we proceed by induction over n. For $n = 0$ the assertion clearly holds. Assume that the assertion holds for n. Using Observation 4.5(i) we have for every $C \in \tilde{\mathcal{K}}_n$

$$\overline{\bigcup \tilde{\mathcal{K}}_{n+1}} \cap C = \bigcup\{\overline{B};\ B \in \tilde{\mathcal{K}}_{n+1}, B \subset C\}.$$

It implies

$$\overline{\bigcup \tilde{\mathcal{K}}_{n+1}} \cap \bigcup \tilde{\mathcal{K}}_n = \bigcup\{\partial B;\ B \in \tilde{\mathcal{K}}_{n+1}\} \cup \bigcup \tilde{\mathcal{K}}_{n+1}.$$

Since

$$\overline{\bigcup \tilde{\mathcal{K}}_n} \setminus \bigcup \tilde{\mathcal{K}}_n \subset \bigcup_{j=0}^{n}\bigcup\{\partial B;\ B \in \tilde{\mathcal{K}}_j\},$$

(by the induction hypothesis) and

$$\overline{\bigcup \tilde{\mathcal{K}}_{n+1}} \subset \overline{\bigcup \tilde{\mathcal{K}}_n},$$

we conclude

$$\begin{aligned}\overline{\bigcup \tilde{\mathcal{K}}_{n+1}} &= \left(\overline{\bigcup \tilde{\mathcal{K}}_{n+1}} \cap \bigcup \tilde{\mathcal{K}}_n\right) \cup \left(\overline{\bigcup \tilde{\mathcal{K}}_{n+1}} \cap \left(\overline{\bigcup \tilde{\mathcal{K}}_n} \setminus \bigcup \tilde{\mathcal{K}}_n\right)\right)\\ &\subset \bigcup \tilde{\mathcal{K}}_{n+1} \cup \bigcup_{j=0}^{n+1}\bigcup\{\partial B;\ B \in \tilde{\mathcal{K}}_j\}.\end{aligned}$$

(ii) The construction and Observation 4.5(ii) give (ii).

(iii) Let $B \in \tilde{\mathcal{K}}_n$, $n \in \mathbb{N}_0$. Then there are intervals $B_1, B_2, \ldots$ such that $B_j \in \tilde{\mathcal{K}}_{n+j}$, $B \supset \overline{B_1}$, and $B_j \supset \overline{B_{j+1}}$ for every $j \in \mathbb{N}$. Then we have $\emptyset \neq \bigcap_{j=1}^{\infty} B_j \subset L \cap B$.

(iv) The inclusion $L \subset \bigcap_{n=0}^{\infty} \overline{\bigcup \tilde{\mathcal{K}}_n}$ obviously holds. To prove the inverse inclusion consider $x \in \bigcap_{n=0}^{\infty} \overline{\bigcup \tilde{\mathcal{K}}_n}$ and $\varepsilon > 0$. Using Lemma 4.9(ii) we find $n \in \mathbb{N}$ and $B \in \tilde{\mathcal{K}}_n$ such that $B(x, \varepsilon) \cap B \neq \emptyset$ and $\operatorname{diam} B < \varepsilon$. According to (iii) we have $\emptyset \neq B \cap L \subset B(x, 2\varepsilon) \cap L$. It gives $x \in \overline{L} = L$, since ε can be chosen arbitrarily small.

(v) This property follows directly from the construction and from the definition of $[G, \varepsilon]$-system. □

Lemma 4.10. (i) *Let $n \in \mathbb{N}_0$, $B_1 \in \tilde{\mathcal{K}}_n$, $j \in \{1, \dots, n\}$, $B_2 \in \tilde{\mathcal{K}}_{n+1}^j$, $B_2 \subset B_1$, and $x \in B_2$. Then $\Gamma(x, B_1, B_2, L) \leq 1/(n+2)$.*

(ii) *If $n \in \mathbb{N}$, $n > 1$, $j \in \{1, \dots, n-1\}$, $B \in \tilde{\mathcal{K}}_n^j$, and $x \in \partial B$, then $p(x, L) = 0$.*

(iii) *If $n \in \mathbb{N}_0$ and $B \in \tilde{\mathcal{K}}_n^n$, then $(2 \star B \setminus \overline{B}) \cap L = \emptyset$.*

PROOF. (i) Take an interval $B(y, r) \subset B_1$ such that $B(y, r) \cap L = \emptyset$ and $y \in B_1 \setminus B_2$. The set L intersects each interval from $\tilde{\mathcal{K}}_{n+1}^{j+1}$ (Lemma 4.9(iii)) and it implies $B(y, (n+2)r) \cap B_2 = \emptyset$. Thus we have

$$\frac{r}{\operatorname{dist}(y, x)} < \frac{r}{(n+2)r} = \frac{1}{n+2}.$$

Consequently, $\Gamma(x, B_1, B_2, L) \leq 1/(n+2)$.

(ii) There exists $C \in \tilde{\mathcal{K}}_{n-1}$ with $B \subset C$. By Lemma 4.9(iii) the set L intersects each interval of $\tilde{\mathcal{S}}_{j+1}(C, n)$. We have also $x \in \partial \tilde{R}_j(C, n)$ (Observation 4.5(i)). Bearing property (c) of Definition 4.1 in mind, we get $p(x, L \cup (\mathbb{R} \setminus \tilde{R}_j(C, n))) = 0$. There exists a neighborhood U of x such that $U \cap (\mathbb{R} \setminus \tilde{R}_j(C, n)) = U \cap B$. Thus we have $p(x, L \cup B) = 0$. We have that L intersects each element of $\tilde{\mathcal{S}}_1(B, n+1)$. Since $x \in \partial B$ we get $p(x, L \cup (\mathbb{R} \setminus B)) = 0$. The set B is an interval and therefore we can conclude that $p(x, L) = 0$.

(iii) This property is obvious for $n = 0$. For $n > 0$ it easily follows by the construction using properties (d) and (e) of Definition 4.1. □

To finish the proof of Theorem 1.2 it remains to show that $P(L)$ is G_δ. We define

$$Q_n = \bigcup_{j=n}^{\infty} \bigcup \{\overline{B};\ B \in \tilde{\mathcal{K}}_j^j\} \cup \bigcup_{j=0}^{n-1} \bigcup \{\partial B;\ B \in \tilde{\mathcal{K}}_j^j\}, \qquad n \in \mathbb{N}.$$

We claim that

(i) $P(L) = \bigcap_{n=1}^{\infty} Q_n$,

(ii) $Q_n \cap L$ is G_δ for every $n \in \mathbb{N}$.

These two facts imply that $P(L)$ is G_δ.

Proof of (i). Since $Q_n \subset \overline{\bigcup \tilde{\mathcal{K}}_n}$ by Lemma 4.9(i), we have $\bigcap_{n=1}^{\infty} Q_n \subset L$ by Lemma 4.9(iv). Suppose that $x \in \bigcap_{n=1}^{\infty} Q_n$. If moreover $x \in \partial B$ for some $B \in \tilde{\mathcal{K}}_{j_0}^{j_0}$, $j_0 \in \mathbb{N}_0$, then $(2\star B \setminus \overline{B}) \cap L = \emptyset$ by Lemma 4.10(iii) and so x is clearly a point of porosity of L. Now assume that $x \in \bigcap_{n=1}^{\infty} Q_n \setminus \bigcup_{j=0}^{\infty} \bigcup \{\partial B;\ B \in \tilde{\mathcal{K}}_j^j\}$. Then there is an increasing sequence $\{j_k\}_{k=1}^{\infty}$ of natural numbers and a sequence $\{B_k\}_{k=1}^{\infty}$ of open intervals such that $x \in B_k \in \tilde{\mathcal{K}}_{j_k}^{j_k}$ for every $k \in \mathbb{N}$. We have $(2 \star B_k \setminus \overline{B_k}) \cap L = \emptyset$ for every $k \in \mathbb{N}$ by Lemma 4.10(iii). Since $\lim_{k\to\infty} \operatorname{diam} B_k = \emptyset$ (Lemma 4.9(ii)), we get $x \in P(L)$. Thus $\bigcap_{n=1}^{\infty} Q_n \subset P(L)$.

Now suppose that $x \in L \setminus \bigcap_{n=1}^{\infty} Q_n$. Thus there exists $n_0 \in \mathbb{N}$ such that $x \in L \setminus Q_{n_0}$. It implies that $x \notin \bigcup_{j=0}^{\infty} \bigcup \{\partial B;\ B \in \tilde{\mathcal{K}}_j^j\}$. Using (i) and (iv) of Lemma 4.9 we see that there are two possibilities.

1) There exist $n \in \mathbb{N}$, $n > 1$, $j \in \{1, \dots, n-1\}$, and $B \in \tilde{\mathcal{K}}_n^j$ such that $x \in \partial B$. Then Lemma 4.10 (ii) gives $p(x, L) = 0$.

2) There exists a sequence $\{B_n\}_{n=n_0+1}^{\infty}$ of open intervals and a sequence $\{j_n\}_{n=n_0+1}^{\infty}$ of natural numbers such that $x \in B_n \in \tilde{\mathcal{K}}_n^{j_n}$ and $j_n < n$. Then we have for every $n > n_0$:

- $\operatorname{dist}(B_{n+1}, B_n^c) \geq \operatorname{diam} B_{n+1}$ (Lemma 4.9(v)),
- $\Gamma(x, B_n, B_{n+1}, L) \leq 1/(n+2)$ (Lemma 4.10(i)),
- $\operatorname{diam} B_{n+1} < \frac{1}{2} \operatorname{diam} B_n$ (Lemma 4.9(ii)).

Now Lemma 4.2 gives $p(x, L) = 0$. This finishes the proof of (i). □

Proof of (ii). If $B \in \tilde{\mathcal{K}}_j^j$, $j \in \mathbb{N}$, then the set $\overline{B} \cap L$ is open in L (Lemma 4.10 (iii)). Thus the set $\left(\bigcup_{j=n}^{\infty} \bigcup \{\overline{B};\ B \in \tilde{\mathcal{K}}_j^j\}\right) \cap L$ is open in L. The system $\{\partial B;\ B \in \tilde{\mathcal{K}}_j^j\}$ is discrete in some open set. Thus the set $\bigcup_{j=0}^{n-1} \bigcup \{\partial B;\ B \in \tilde{\mathcal{K}}_j^j\}$ is G_δ. Consequently, $Q_n \cap L$ is G_δ for every $n \in \mathbb{N}$. □

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