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A SUMMABILITY FACTOR THEOREM FOR INFINITE SERIES

Abstract

We obtain sufficient conditions for the series $\sum a_n \lambda_n$ to be absolutely summable of order k by a triangular matrix.

The concept of absolute summability of order k was defined by Flett [2] as follows. A series $\sum a_n$ is summable $|C, \delta|_k, k \geq 1, \delta > -1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta \sigma_{n-1}^{\delta}|^k < \infty, \tag{1}$$

where σ_n^{δ} denotes the n-th Cesáro means of order δ, of the partial sums s_n of the series $\sum a_n$.

In defining absolute summability of order k for weighted mean methods Bor [1] and others, have used the definition

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta u_{n-1}|^k < \infty, \tag{2}$$

where u_n is the n-th term of the weighted mean transform of $\{s_n\}$.

In using (2) as the definition, it was apparently assumed that the n in (1) represented the reciprocal of the n-th main diagonal term of $(C, 1)$. But this interpretation can not be correct. For, if it were, then for the Cesáro methods of order δ with $\delta \neq 1$, one would have

$$\sum_{n=1}^{\infty} (n^{\delta})^{k-1} |\Delta \sigma_{n-1}^{\delta}|^k < \infty.$$

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However, Flett [2] stays with n for all values of $\delta > -1$.

Bor [1] obtained sufficient conditions for $\sum a_n\lambda_n$ to be summable $|\overline{N}, p_n|_k$, $k \geq 1$. Unfortunately he used an incorrect definition of absolute summability. In this paper we generalize this result by replacing the weighted mean matrix with a triangular matrix, and using the correct definition of absolute summability.

Let T be a lower triangular matrix and let $\{s_n\}$ a sequence. Then

$$T_n := \sum_{\nu=0}^{n} t_{n\nu} s_\nu.$$

A series $\sum a_n$ is said to be summable $|T|_k, k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty. \tag{3}$$

We may associate with T two lower triangular matrices $\overline{T}$ and $\hat{T}$ defined by

$$\bar{t}_{n\nu} = \sum_{r=\nu}^{n} t_{nr},\ n, \nu = 0, 1, 2, \dots,$$

and

$$\hat{t}_{n\nu} = \bar{t}_{n\nu} - \bar{t}_{n-1,\nu},\ n = 1, 2, 3, \dots.$$

We may write

$$T_n = \sum_{\nu=0}^{n} a_{n\nu} \sum_{i=0}^{\nu} a_i\lambda_i = \sum_{i=0}^{n} a_i\lambda_i \sum_{\nu=i}^{n} a_{n\nu} = \sum_{i=0}^{n} \bar{a}_{ni} a_i \lambda_i.$$

Thus

$$\begin{aligned} T_n - T_{n-1} &= \sum_{i=0}^{n} \bar{a}_{ni} a_i\lambda_i - \sum_{i=0}^{n-1} \bar{a}_{n-1,i} a_i\lambda_i = \sum_{i=0}^{n} \bar{a}_{ni} a_i\lambda_i - \sum_{i=0}^{n} \bar{a}_{n-1,i} a_i\lambda_i \\ &= \sum_{i=0}^{n} (\bar{a}_{ni} - \bar{a}_{n-1,i}) a_i\lambda_i = \sum_{i=0}^{n} \hat{a}_{ni} a_i\lambda_i = \sum_{i=1}^{n} \hat{a}_{ni}\lambda_i (s_i - s_{i-1}) \\ &= \sum_{i=1}^{n} \hat{a}_{ni}\lambda_i s_i - \sum_{i=1}^{n} \hat{a}_{ni}\lambda_i s_{i-1} \\ &= \sum_{i=1}^{n-1} \hat{a}_{ni}\lambda_i s_i + \hat{a}_{nn}\lambda_n s_n - \sum_{i=1}^{n} \hat{a}_{ni}\lambda_i s_{i-1} \end{aligned}$$

$$= \sum_{i=1}^{n-1} \hat{a}_{ni}\lambda_i s_i + a_{nn}\lambda_n s_n - \sum_{i=0}^{n-1} \hat{a}_{n,i+1}\lambda_{i+1}s_i$$
$$= \sum_{i=1}^{n} (\hat{a}_{ni}\lambda_i - \hat{a}_{n,i+1}\lambda_{i+1})s_i + a_{nn}\lambda_n s_n.$$

We may write

$$\begin{aligned}(\hat{a}_{ni}\lambda_i - \hat{a}_{n,i+1}\lambda_{i+1}) &= \hat{a}_{ni}\lambda_i - \hat{a}_{n,i+1}\lambda_{i+1} - \hat{a}_{n,i+1}\lambda_i + \hat{a}_{n,i+1}\lambda_i \\ &= (\hat{a}_{ni} - \hat{a}_{n,i+1})\lambda_i + \hat{a}_{n,i+1}(\lambda_i - \lambda_{i+1}) \\ &= \lambda_i \Delta_i \hat{a}_{ni} + \hat{a}_{n,i+1}\Delta\lambda_i.\end{aligned}$$

Therefore

$$\begin{aligned}T_n - T_{n-1} &= \sum_{i=1}^{n-1} \Delta_i \hat{a}_{ni}\lambda_i s_i + \sum_{i=1}^{n-1} \hat{a}_{n,i+1}\Delta\lambda_i s_i + a_{nn}\lambda_n s_n \\ &= T_{n1} + T_{n2} + T_{n3}, \text{ say.}\end{aligned}$$

A triangle is a lower triangular matrix with all nonzero main diagonal entries.

Theorem 1. *Let A be a lower triangular matrix satisfying:*

(i) $na_{nn} \asymp O(1)$

(ii) $a_{n-1,\nu} \geq a_{n\nu}$ *for* $n \geq \nu + 1$, *and*

(iii) $\bar{a}_{n0} = 1$ *for* n, *and*

(iv) $\sum_{\nu=1}^{n-1} a_{\nu\nu}|\hat{a}_{n,\nu+1}| = O(a_{nn})$.

If $\{X_n\}$ is a positive non-decreasing sequence and $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences such that:

(v) $|\Delta\lambda_n| \leq \beta_n$,

(vi) $\lim \beta_n = 0$,

(vii) $\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty$, *and*

(viii) $|\lambda_n|X_n = O(1)$

(ix) $\sum_{n=1}^{m} \frac{1}{n}|s_n|^k = O(X_m)$,

then the series $\sum a_n\lambda_n$ *is summable* $|A|_k, k \geq 1$.

We need the following lemma for the proof of our theorem.

Lemma 1 (Lemma. [2]). . *Under the conditions on* $(X_n), (\beta_n)$ *and* (λ_n) *as taken in the statement of the theorem the following conditions hold, when (vii) satisfies:*

(1) $\displaystyle\sum_{n=1}^{\infty} \beta_n X_n < \infty$ *and*

(2) $n\beta_n X_n = O(1)$.

PROOF. To prove the theorem it will be sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1}|T_{nr}|^k < \infty, \text{ for } r = 1, 2, 3.$$

Using Hölder's inequality and (i),

$$\begin{aligned} I_1 := \sum_{n=1}^{m} n^{k-1}|T_{n1}|^k &= \sum_{n=1}^{m} n^{k-1}\Big|\sum_{i=1}^{n-1} \Delta_i \hat{a}_{ni}\lambda_i s_i\Big|^k \\ &= O(1)\sum_{n=1}^{m+1} n^{k-1}\Big(\sum_{i=1}^{n-1} |\Delta_i \hat{a}_{ni}||\lambda_i||s_\nu|\Big)^k \\ &= O(1)\sum_{n=1}^{m+1} n^{k-1}\Big(\sum_{i=1}^{n-1} |\Delta_i \hat{a}_{ni}||\lambda_i|^k|s_i|^k\Big) \times \Big(\sum_{i=1}^{n-1} |\Delta_i \hat{a}_{ni}|\Big)^{k-1} \end{aligned}$$

From (ii)

$$\begin{aligned} \Delta_i \hat{a}_{ni} &= \hat{a}_{ni} - \hat{a}_{n,i+1} = \bar{a}_{ni} - \bar{a}_{n-1,i} - \bar{a}_{n,i+1} + \bar{a}_{n-1,i+1} \\ &= a_{ni} - a_{n-1,i} \leq 0. \end{aligned}$$

Thus, using (iii),

$$\sum_{i=0}^{n-1} |\Delta_i \hat{a}_{ni}| = \sum_{i=0}^{n-1}(a_{n-1,i} - a_{ni}) = 1 - 1 + a_{nn} = a_{nn}.$$

Using the fact that, from (viii), $\{\lambda_n\}$ is bounded, and condition (1) of the Lemma,

$$I_1 = O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\sum_{i=1}^{n-1} |\Delta_i \hat{a}_{ni}||\lambda_i|^k|s_i|^k$$

$$
\begin{aligned}
&= O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\sum_{i=1}^{n-1}|\Delta_i\hat{a}_{ni}||\lambda_i||\lambda_i|^{k-1}|s_i|^k\\
&= O(1)\sum_{i=1}^{m}|\lambda_i||s_i|^k\sum_{n=i+1}^{m+1}(na_{nn})^{k-1}|\Delta_i\hat{a}_{ni}|\\
&= O(1)\sum_{i=1}^{m}|\lambda_i||s_i|^k a_{ii}\\
&= O(1)\sum_{i=1}^{m}|\lambda_i|\Big(\sum_{r=1}^{i}|s_r|^k a_{rr}-\sum_{r=1}^{i-1}|s_r|^k a_{rr}\Big)\\
&= O(1)\Big[\sum_{i=1}^{m}|\lambda_i|\sum_{r=1}^{i}|s_r|^k a_{rr}-\sum_{j=0}^{m-1}|\lambda_{j+1}|\sum_{r=1}^{j}|s_r|^k a_{rr}\Big]\\
&= O(1)\Big[\sum_{i=1}^{m-1}|\Delta\lambda_i|\sum_{r=1}^{i}|s_r|^k a_{rr}+|\lambda_m|\sum_{r=1}^{m}|s_r|^k a_{rr}\Big]\\
&= O(1)\Big[\sum_{i=1}^{m-1}|\Delta\lambda_i|\sum_{r=1}^{i}|s_r|^k \frac{1}{r}+|\lambda_m|\sum_{r=1}^{m}|s_r|^k \frac{1}{r}\Big]\\
&= O(1)\Big[\sum_{i=1}^{m-1}\beta_i X_i+O(1)|\lambda_m|X_m=O(1).
\end{aligned}
$$

Using (i), (v) and Hölder's inequality, and condition (2) of the Lemma

$$
\begin{aligned}
I_2 :&= \sum_{n=2}^{m+1}n^{k-1}|T_{n2}|^k=\sum_{n=2}^{m+1}n^{k-1}\Big|\sum_{i=1}^{n-1}\hat{a}_{n,i+1}(\Delta\lambda_i)s_i\Big|^k\\
&\le \sum_{n=2}^{m+1}n^{k-1}\Big[\sum_{i=1}^{n-1}|\hat{a}_{n,i+1}||\Delta\lambda_i||s_i|\Big]^k\\
&= O(1)\sum_{n=2}^{m+1}n^{k-1}\Big[\sum_{i=1}^{n-1}|\hat{a}_{n,i+1}|\beta_i|s_\nu|\Big]^k\\
&= O(1)\sum_{n=2}^{m+1}n^{k-1}\Big[\sum_{i=1}^{n-1}(i\beta_i)|s_i|a_{ii}|\hat{a}_{n,i+1}|\Big]^k\\
&= O(1)\sum_{n=2}^{m+1}n^{k-1}\sum_{i=1}^{n-1}(i\beta_i)^k|s_i|^k a_{ii}|\hat{a}_{n,i+1}|\Big]\times\Big[\sum_{i=1}^{n-1}a_{ii}|\hat{a}_{n,i+1}|\Big]^{k-1}
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}(na_{nn})^{k-1}\sum_{i=1}^{n-1}(i\beta_i)^k|s_i|^k a_{ii}|\hat{a}_{n,i+1}| \\
&= O(1)\sum_{n=2}^{m+1}(na_{nn})^{k-1}\sum_{i=1}^{n-1}(i\beta_i)^{k-1}(i\beta_i)a_{ii}|\hat{a}_{n,i+1}||s_i|^k \\
&= O(1)\sum_{i=1}^{m}(i\beta_i)a_{ii}|s_i|^k\sum_{n=i+1}^{m+1}(na_{nn})^{k-1}|\hat{a}_{n,i+1}| \\
&= O(1)\sum_{i=1}^{m}(i\beta_i)a_{ii}|s_i|^k\sum_{n=i+1}^{m+1}|\hat{a}_{n,i+1}| = O(1)\sum_{i=1}^{m}(i\beta_i)a_{ii}|s_i|^k.
\end{aligned}
$$

Using summation by parts,

$$
\begin{aligned}
I_2 &= O(1)\sum_{i=1}^{m}(i\beta_i)\Big[\sum_{r=1}^{i}a_{rr}|s_r|^k-\sum_{r=1}^{i-1}a_{rr}|s_r|^k\Big] \\
&= O(1)\Big[\sum_{i=1}^{m}(i\beta_i)\sum_{r=1}^{i}a_{rr}|s_r|^k-\sum_{i=0}^{m-1}(i+1)\beta_{i+1}\sum_{r=1}^{i}a_{rr}|s_r|^k\Big] \\
&= O(1)\Big[\sum_{i=1}^{m}\Delta(i\beta_i)\sum_{r=1}^{i}a_{rr}|s_r|^k+m\beta_i\sum_{r=1}^{i}a_{rr}|s_r|^k\Big] \\
&= O(1)\Big[\sum_{i=1}^{m}\Delta(i\beta_i)\sum_{r=1}^{i}\frac{1}{r}|s_r|^k+m\beta_i\sum_{r=1}^{i}\frac{1}{r}|s_r|^k\Big] \\
&= O(1)\sum_{i=1}^{m}i|\Delta\beta_i|X_i+O(1)\sum_{i=1}^{m}\beta_{i+1}X_i+O(1)\beta_iX_i \\
&= O(1),
\end{aligned}
$$

by virtue of the hypothesis and the Lemma.

Using (i),

$$
\begin{aligned}
\sum_{n=1}^{m+1}n^{k-1}|T_{n3}|^k &\le \sum_{n=1}^{m+1}n^{k-1}|a_{nn}\lambda_n s_n|^k \\
&= O(1)\sum_{n=1}^{m}(na_{nn})^{k-1}a_{nn}|\lambda_n|^k|s_n|^k \\
&= O(1)\sum_{n=1}^{m}a_{nn}|\lambda_n|^{k-1}|\lambda_n||s_n|^k
\end{aligned}
$$

$$= O(1)\sum_{n=1}^{m} a_{nn}|\lambda_n||s_n|^k = O(1),$$

as in the proof of T_{n1}. □

Corollary 1. *Let $\{p_n\}$ be a positive sequence such that $P_n := \sum_{k=0}^{n} p_k \to \infty$, and satisfies*
(i) $np_n \asymp O(P_n)$.
If $\{X_n\}$ is a positive non-decreasing sequence and if $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences such that

(ii) $|\Delta\lambda_n| \leq \beta_n$,

(iii) $\lim \beta_n = 0$,

(iv) $\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty$, *and*

(v) $|\lambda_n|X_n = O(1)$

(vi) $\sum_{n=1}^{m} \frac{1}{n}|s_n|^k = O(X_m)$,

then the series $\sum a_n\lambda_n$ *is summable* $|\overline{N}, p_n|_k, k \geq 1$.

Proof. Conditions (ii)-(vi) of Corollary 1 are, respectively, conditions (v) - (ix) of Theorem 1.

Conditions (ii), (iii) and (iv) of Theorem 1 are automatically satisfied for any weighted mean method. Condition (i) of Theorem 1 becomes condition (i) of Corollary 1. □

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