Real Analysis Exchange
Vol. 30(2), 2004/2005, pp. 451–494

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AN n-th ORDER INTEGRAL AND ITS INTEGRATION BY PARTS WITH APPLICATIONS TO TRIGONOMETRIC SERIES

Abstract

An n-th order symmetric Perron type integral is defined and its properties are studied. An integration by parts formula is proved and applied to solve problems related to summable trigonometric series.

1 Introduction.

The P^n-integral introduced by James, [10], was defined to solve problems related to summable trigonometric series, [11]. The definition has some lacunæ that were removed in [13]. This integral is such that while the n-th primitive of an integrable function exists the previous primitives may not exist. The absence of the first primitive caused difficulty in expressing the coefficients of a trigonometric series by the usual Fourier formulæ and expressions for the coefficients of a trigonometric series take a different form; see [13, 7]. Also, because of this absence of the first primitive, an integration by parts formula could not be proved in its usual form and therefore these Fourier coefficients were obtained by formal multiplication of trigonometric series thus avoiding the need for integration by parts; see [11, 13, 7]. In addition additivity of the integral for abutting intervals was a problem for this integral; see[8, 9, 19].

In the present paper the definition of the P^n-integral is simplified so that a first primitive exists; see also [12]. This enables us to obtain an integration

Key Words: P^n-integral, integration by parts, summable trigonometric series
Mathematical Reviews subject classification: 26A39
Received by the editors July 24, 2003
Communicated by: Peter Bullen

*The author would like to dedicate this paper to Professor P. S. Bullen in recognition of his friendship and encouragement over many years.

by parts formula and then to get the usual Fourier formulæ when applied to trigonometric series. Finally additivity of the integral for abutting intervals holds with no additional conditions; see [8, 9, 19].

2 Preliminaries.

Let f be a real-valued function defined on some neighborhood of $x, x \in \mathbb{R}$. If there is a polynomial $P(t) = P_x(t)$ of degree at most k such that

$$\frac{1}{2}\big[f(x+t) + (-1)^k f(x-t)\big] = P(t) + o(t^k), \tag{2.1}$$

then f is said to possess a *k-th symmetric de la Vallée Poussin (d.l.V.P) derivative at x*, and if $a_k/k!$ is the coefficient of t^k in $P(t)$, then a_k is called the *k-th symmetric d.l.V.P. derivative of f at x*, denoted by $D^k f(x)$.[1] It is clear that $P(t)$ has only even or odd powers of t according as k is even or odd. Also if $D^k f(x)$ exists, then $D^{k-2} f(x)$ also exists, where we take $D^0 f(x) = f(x)$. Thus $P(t)$ in (2.1) has the form

$$P(t) = \begin{cases} \displaystyle\sum_{i=0}^{k/2} \frac{t^{2i}}{(2i)!} D^{2i} f(x) & \text{if } k \text{ is even} \\ \displaystyle\sum_{i=0}^{(k-1)/2} \frac{t^{2i+1}}{(2i+1)!} D^{2i+1} f(x) & \text{if } k \text{ is odd.} \end{cases} \tag{2.2}$$

Suppose that $D^k f(x)$ exists and write

$$\frac{t^{k+2}}{(k+2)!}\omega_{k+2}(f, x, t) = \frac{1}{2}\big(f(x+t) + (-1)^k f(x-t)\big) - P(t) \tag{2.3}$$

The *upper symmetric d.l.V.P. derivate of f at x of order $k+2$* is defined to be

$$\overline{D}^{k+2} f(x) = \limsup_{t \to 0} \omega_{k+2}(f, x, t). \tag{2.4}$$

Replacing $\limsup$ in (2.4) by $\liminf$ one gets the lower derivate $\underline{D}^{k+2} f(x)$. If $\overline{D}^{k+2} f(x) = \underline{D}^{k+2} f(x)$, the common value is the derivative $D^{k+2} f(x)$, possibly infinite in this case.

The function f is said to be *smooth at x of order $k+2$* if $D^k f(x)$ exists and $\lim_{t\to 0} t\omega_{k+2}(f, x, t) = 0$. If f is smooth at x of order $k+2$, we write $f \in \mathcal{S}_{k+2}(x)$, or $f \in \mathcal{S}_{k+2}$ at x.

[1] Here and elsewhere $o(t^k)$ denotes a quantity which when divided by t^k tends to 0 as $t \to 0$; and $O(t^k)$ denotes a quantity which when divided by t^k remains bounded as $t \to 0$.

If there is a polynomial $Q(t) = Q_x(t)$ of degree at most ℓ such that

$$f(x+t) = Q(t) + o(t^\ell), \tag{2.5}$$

then f is said to possess a *ℓ-th Peano derivative at x*, and if $a_\ell/\ell!$ is the coefficient of t^ℓ in $Q(t)$, then a_ℓ is called the *ℓ-th Peano derivative of f at x*, denoted by $f_{(\ell)}(x)$. It is clear that if $f_{(\ell)}(x)$ exists, then $f_{(\ell-1)}(x)$ and $D^\ell f(x)$ also exist; where $f_0(x) = f(x)$. Thus $Q(t)$ in (2.5) has the form

$$Q(t) = \sum_{i=0}^{\ell} \frac{t^i}{i!} f_{(i)}(x). \tag{2.6}$$

If $f_{(\ell)}(x)$, exists we write

$$\frac{t^{\ell+1}}{(\ell+1)!}\gamma_{\ell+1}(f,x,t) = f(x+t) - Q(t). \tag{2.7}$$

The *upper and lower Peano derivates of f at x of order $\ell+1$* ,which are denoted by $\overline{f}_{(\ell+1)}(x)$ and $\underline{f}_{(\ell+1)}(x)$, are obtained by taking upper and lower limits of $\gamma_{\ell+1}(f,x,t)$ respectively. By suitably restricting, the definitions of *unilateral Peano derivates* are obtained; the right, (respectively left) Peano derivate of f at x of order ℓ being denoted by $f^+_{(\ell)}(x)$, (respectively $f^-_{(\ell)}(x)$).

For the definition of *n-convex functions* we refer to [2]. Recall that a *function f is said to satisfy property $\mathcal{R}$ in an interval I*, written $f \in \mathcal{R}$ in I, if for every perfect set $P \subseteq I$ there is a portion of P on which f restricted to P is continuous; see [13]. The property $\mathcal{R}$ is also called the *Baire*-1 property*. The class of *Darboux functions* will be denoted by $\mathcal{D}$ and μ will denote *Lebesgue measure*.

3 Auxiliary Results.

Lemma 3.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous and let $D^n f$ exist in (a,b). If $f \in \mathcal{S}_{n+2}(x)$ for all $x \in (a,b)$, then $D^i f \in \mathcal{R}$ in (a,b) for $i = n, n-2, \dots$.*

Proof. Let i, as in the statement, be fixed and let $P \subset (a,b)$ be any perfect set. Choose $a < c < d < b$ such that $P \cap [c,d]$ is perfect. Then it follows from [4, Theorem 3.1] that there is a sequence of closed sets $\{Q_k\}$ such that $[c,d] = \bigcup_{k=1}^\infty Q_k$ and $D^i f|Q_k$, the restriction of $D^i f$ to Q_k, is continuous for all k. Since $P \cap [c,d] = \cup_{k=1}^\infty (Q_k \cap P)$, by Baire's theorem there is a k_0 and a portion $P \cap (\alpha,\beta)$ of $P \cap [c,d]$ such that $P \cap (\alpha,\beta) \subset Q_{k_0}$. Since $D^i f|Q_{k_0}$ is continuous, $D^i f|P \cap (\alpha,\beta)$ is continuous and so $D^i f \in \mathcal{R}$ in (a,b). $\square$

Theorem 3.2. *Let $f : [a.b] \to \mathbb{R}$ be continuous and assume that in (a, b):*

(i) $D^{n-2}f$ exists and $D^k f \in \mathcal{D}$ for $k = n-2, n-4, \ldots$;

(ii) $\overline{D}^n f \geq 0$ almost everywhere;

(iii) $\overline{D}^n f > -\infty$ nearly everywhere[2];

(iv) $f \in \mathcal{S}_n$.

Then the ordinary derivative of order $(n-2)$, $f^{(n-2)}$, exists, is continuous and convex in (a, b).

PROOF. By Lemma 3.1, $D^i f \in \mathcal{R}$ in (a, b) for $i = n-2, n-4, \ldots$. Hence f satisfies the hypotheses of [13, Theorem 3.2], or its analogue according as n is even or odd, and hence by that theorem $f^{(n-2)}$, exists, is continuous and convex in (a, b), completing the proof. □

Lemma 3.3. *If $f \in \mathcal{S}_{k+2}(x_0)$ and if $f_{(k)}(x_0)$ exists, then*

$$\overline{f}_{(k+1)}(x_0) = \overline{D}^{k+1} f(x_0), \text{ and } \underline{f}_{(k+1)}(x_0) = \underline{D}^{k+1} f(x_0).$$

PROOF. Since

$$\gamma_{k+1}(f, x_0, t) = \frac{t}{k+2}\omega_{k+2}(f, x_0, t) + \omega_{k+1}(f, x_0, t),$$

the result follows. □

Lemma 3.4. *If F is measurable and if $D^{r-2}F$ exists on a set E and if*

$$-\infty < \underline{D}^r F \leq \overline{D}^r F < \infty, \text{ for } x \in E, \tag{3.1}$$

then $F_{(r)}$ exists almost everywhere on E and is finite there.

PROOF. Write

$$\Delta_r(x, t, F) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} F(x + jt - \frac{1}{2}rt). \tag{3.2}$$

[2]That is except on a countable set.

Then since $\Delta_r(x,-t,F) = (-1)^r\Delta_r(x,t,F)$, we have from (3.2) and (2.3) that

$$\begin{aligned}\Delta_r(x,t,F) =& \frac{1}{2}\big(\Delta_r(x,t,F) + (-1)^r\Delta_r(x,-t,F)\big)\\ =& \sum_{j=0}^{r}\frac{(-1)^{r-j}}{2}\binom{r}{j}\Big(F(x+jt-\frac{1}{2}rt) + (-1)^r F(x-jt+\frac{1}{2}rt)\Big)\\ =& \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\Big(P\big((j-\frac{1}{2}r)t\big) + \frac{\big((j-\frac{1}{2}r)t\big)^r}{r!}\omega_r\big(F,x,(j-\frac{1}{2}r)t\big)\Big).\end{aligned} \tag{3.3}$$

where P is a polynomial of degree at most $(r-2)$ having the form (2.2). Since $\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}j^i = 0$ for $i = 0, 1, \dots, r-1$, we have from (3.3) that

$$\Delta_r(x,t,F) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\frac{\big((j-\frac{1}{2}r)t\big)^r}{r!}\omega_r\big(F,x,(j-\frac{1}{2}r)t\big). \tag{3.4}$$

Since by (3.1), $\omega_r(F,x,t) = O(1)$ as $t \to 0$, for each $x \in E$, we have $\Delta_r(x,t,F) = O(t^r)$, and so by [15, Theorem 3.1] the result follows. □

4 The T^n-Integral.

Definition 4.1. Let f be an extended real valued function defined on the interval $[a,b]$ and let $n \geq 2$ be a fixed positive integer. A function $Q : [a,b] \to \mathbb{R}$ is said to be a *T^n-major function of f* if:

(i) Q is continuous on $[a,b]$;

(ii) $Q_{(n-2)}$ exists, finitely, on $[a,b]$;

(iii) $Q_{(n-1)}$ exists, finitely, on $[a.b]$ except on a set of measure zero in (a,b);

(iv) $Q_{(r)}(a) = 0$ for $r = 0, 1, \dots, n-1$;

(v) $\underline{D}^n Q \geq f$ almost everywhere on (a,b);

(vi) $\underline{D}^n Q > -\infty$ nearly everywhere on (a,b);

(vii) $Q \in \mathcal{S}_n(x)$ for $x \in (a,b)$.

A function $q : [a,b] \to \mathbb{R}$ is said to be a *T^n-minor function of f* if $-q$ is a T^n-major function of $-f$.

If there is no confusion we shall simply say major or minor, omitting T^n.

Lemma 4.2. *If Q and q are major and minor functions of f, then for each $r, 1 < r \leq n, (Q-q)^{(n-r)}$ exists and is k-convex in $[a,b]$ for all k, $0 \leq k \leq r$; and so $Q_{(n-r)} - q_{(n-r)}$ is k-convex in $[a,b]$ for all k, $0 \leq k \leq r$.*

Proof. Let $\phi = Q - q$. Then ϕ is continuous and $\phi_{(n-2)}$ exists in $[a,b]$. So $\phi_{(k)} \in \mathcal{D}$, $1 \leq k \leq n-2$; [16]. Also $\phi \in \mathcal{S}_n(x)$ for $x \in (a,b)$ and on (a,b), $\underline{D}^n\phi \geq \underline{D}^n Q - \overline{D}^n q \geq 0$ almost everywhere and $\underline{D}^n\phi > -\infty$ nearly everywhere. Hence by Theorem 3.2 $\phi_{(n-2)}$ exists, is continuous and convex in $[a,b]$; so $\phi_{(n-2)}$ is the derivative $\phi^{(n-2)}$ on $[a,b]$, [16]. By convexity, the right-hand derivative of $\phi^{(n-2)}$ exists in [a,b), the left-hand derivative of $\phi^{(n-2)}$ exists in (a,b], and $\phi^{(n-1)}$ exists nearly everywhere in (a,b) and is non-decreasing on the set on which it exists. Also $\phi^{(n-2)}(x) = \int_a^x \phi^{(n-1)}$, $x \in [a,b]$, and so $\phi^{(n-2)}$ is non-negative and non-decreasing on $[a,b]$. Since $\phi^{(n-2)} = Q_{(n-2)} - q_{(n-2)}$, the result is proved for $r = 2$.

Suppose that the result is true for a fixed r, $1 < r < n$. Since $(Q-q)^{(n-r)}$ is k-convex in $[a,b]$ for $0 \leq k \leq r$, we have $\phi^{(n-r-1)}(x) = \int_a^x \phi^{(n-r)}$, $x \in [a,b]$, and hence $\phi^{(n-r-1)}$is k-convex on $[a,b]$ for $0 \leq k \leq r+1$; that is, $Q_{(n-r-1)} - q_{(n-r-1)}$ is k-convex on $[a,b]$ for $0 \leq k \leq r+1$, proving the result for $r+1$; and so the proof has been completed by induction. □

Let $\overline{\mathbb{M}}$, respectively $\underline{\mathbb{M}}$, be the family of all major, respectively minor, functions of f. Let

$$U = \inf_{Q \in \overline{\mathbb{M}}} Q_{(n-1)}(b) \text{ and } V = \sup_{q \in \underline{\mathbb{M}}} q_{(n-1)}(b).$$

If $Q \in \overline{\mathbb{M}}$ and $q \in \underline{\mathbb{M}}$, then by Lemma 4.2 $(Q-q)^{(n-2)}$has a right-hand derivative at a and a left-hand derivative

at b, and

$$\begin{aligned} 0 = Q_{(n-1)}(a) - q_{(n-1)}(a) &= (Q-q)^{(n-1)}(a) \\ &\leq (Q-q)^{(n-1)}(b) = Q_{(n-1)}(b) - q_{(n-1)}(b). \end{aligned}$$

Hence $Q_{(n-1)}(b) \geq q_{(n-1)}(b)$ which shows that $U \geq V$.

If $U = V \neq \pm\infty$, then f *is said to be T^n-integrable on* $[a,b]$ and the common value is called the *T^n-integral of f* and is denoted by $(T^n)\int_a^n f$ or $(T^n)\int_a^n f(t)\mathrm{d}t$.

Let f be T^n-integrable on $[a,b]$ and let $\epsilon > 0$ be arbitrary. Then there is a $Q \in \overline{\mathbb{M}}$ and a $q \in \underline{\mathbb{M}}$ such that $Q_{(n-1)}(b) - q_{(n-1)}(b) < \epsilon$. Hence from Lemma 4.2 and from the definitions of Q and q we have almost everywhere in (a,b) that

$$0 \leq Q_{(n-1)}(x) - q_{(n-1)}(x) \leq Q_{(n-1)}(b) - q_{(n-1)}(b) < \epsilon. \tag{4.1,1}$$

Since $Q_{(n-2)} - q_{(n-2)}$ is convex,

$$Q_{(n-2)}(x) - q_{(n-2)}(x) = \int_a^x (Q_{(n-1)} - q_{(n-1)}),\ x \in [a,b].$$

Hence from (4.1,1) and Lemma 4.2 for $x \in [a,b]$ we have

$$0 \le Q_{(n-2)}(x) - q_{(n-2)}(x) \le Q_{(n-2)}(b) - q_{(n-2)}(b) < \epsilon(b-a). \qquad (4.1,2)$$

So, since ϵ is arbitrary, for each $x \in [a,b]$,

$$\inf_{Q\in\overline{\mathbb{M}}} Q_{(n-2)}(x) = \sup_{q\in\underline{\mathbb{M}}} q_{(n-2)}(x) = F_2(x), \text{ say.}$$

The function F_2 is called *the second primitive of* f. Suppose that *the* r*-th primitive of* f is defined, F_r, $2 \le r < n$, and that the relation

$$0 \le Q_{(n-r)}(x) - q_{(n-r)}(x) \le Q_{(n-r)}(b) - q_{(n-r)}(b) < \epsilon(b-a)^{r-1}, \qquad (4.1,\mathrm{r})$$

for $x \in [a,b]$ is obtained. Since $Q_{(n-r-1)} - q_{(n-r-1)}$ is convex,

$$Q_{(n-r-1)}(x) - q_{(n-r-1)}(x) = \int_a^x (Q_{(n-r)} - q_{(n-r)}),\ x \in [a,b].$$

Hence from (4.1,r) and Lemma 4.2 we have for $x \in [a,b]$

$$\begin{aligned} 0 &\le Q_{(n-r-1)}(x) - q_{(n-r-1)}(x) \\ &\le Q_{(n-r-1)}(b) - q_{(n-r-1)}(b) < \epsilon(b-a)^r, \end{aligned} \qquad (4.1,\mathrm{r+1})$$

and so, since ϵ is arbitrary, for each $x \in [a,b]$,

$$\inf_{Q\in\overline{\mathbb{M}}} Q_{(n-r-1)}(x) = \sup_{q\in\underline{\mathbb{M}}} q_{(n-r-1)}(x) = F_{r+1}(x), \text{ say,}$$

the $(r+1)$-th primitive of f. So all of the primitives F_r, $2 \le r \le n$ of f have been defined and it remains to define the first primitive of f, F_1.

Henceforth we shall, where there is no confusion, write integrable and integral instead of T^n-integrable and T^n-integral, and omit the prefix (T_n) in the notation $(T^n)\int_a^n f$.

Lemma 4.3. *Let f be integrable on $[a,b]$ with F_r, $2 \le r \le n$, its r-th primitive. Then there is a sequence of major functions $\{Q_i\}$, and a sequence of minor functions $\{q_i\}$ such that $\{(Q_i)_{(n-r)}\}$ and $\{(q_i)_{(n-r)}\}$ converge uniformly in $[a,b]$ to F_r, $2 \le r \le n$.*

Proof. Let i be a positive integer. Since f is integrable, there is a major function Q_i and a minor function q_i such that $(Q_i)_{(n-1)}(b) - (q_i)_{(n-1)}(b) < \frac{1}{i}$. From this we get, as in (4.1,1)–(4.1.r), for $x \in [a,b]$ that

$$0 \leq (Q_i)_{(n-r)}(x) - (q_i)_{(n-r)}(x) \leq \frac{1}{i}(b-a)^{r-1}.$$

Hence from the definition of F_r we have for $x \in [a,b]$ that

$$0 \leq (Q_i)_{(n-r)}(x) - F_r(x) \leq (Q_i)_{(n-r)}(x) - (q_i)_{(n-r)}(x) \leq \frac{1}{i}(b-a)^{r-1}.$$

This shows that the sequence $\{Q_i)_{(n-r)}\}$ converges uniformly to F_r in $[a,b]$. The rest is clear. $\square$

Lemma 4.4. *Let f be integrable on $[a,b]$ with F_r, $2 \leq r \leq n$, its r-th primitive. Then for any major function Q and any minor function q the functions $Q_{(n-r)} - F_r$ and $F_r - q_{(n-r)}$ are k-convex, $0 \leq k \leq r$, on $[a,b]$.*

Proof. By Lemma 4.3 there is a sequence of minor functions of $f, \{q_i\}$, such that $\{(q_i)_{(n-r)}\}$ converges uniformly to F_r on $[a,b]$, for $2 \leq r \leq n$. Let Q be any major function of f. Then by Lemma 4.2, $Q_{(n-r)} - (q_i)_{(n-r)}$ is k-convex on $[a,b]$ for $0 \leq k \leq n$, and for each i, $i = 1, 2, 3, \dots$. Hence

$$Q_{(n-r)} - F_r = Q_{(n-r)} - \lim_{n\to\infty} (q_i)_{(n-r)} = \lim_{n\to\infty} \big(Q_{(n-r)} - (q_i)_{(n-r)}\big)$$

is k-convex on $[a,b]$. A similar argument can be given for $F_r - q_{(n-r)}$. $\square$

Theorem 4.5. *Let f be integrable on $[a,b]$ and let F_n be its n-th primitive. Then $(F_n)_{(n-2)}$ exists, finitely, on $[a,b]$ and there is a set $B \subseteq [a,b]$ such that $a \in B$, $b \in B$, $\mu(B) = b-a$ and $(F_n)_{n-1}$ exists, finitely, on B. Also $(F_n)_{n-1}(a) = 0$ and $(F_n)_{n-1}(b) = \int_a^b f$.*

Proof. Let Q be any major function of f. Then by Lemma 4.4 the function $\phi = Q - F_n$ is k-convex, $0 \leq k \leq n$, on $[a,b]$; in particular, ϕ is n-convex on $[a,b]$. So $\phi_{(n-2)}$ exists on $[a,b]$. Hence since $Q_{(n-2)}$ exists in $[a,b]$, so does $(F_n)_{(n-2)}$. Also $\phi^+_{(n-1)}(a)$ and $\phi^-_{(n-1)}(b)$ exist and $\phi_{(n-1)}$ exists , finitely, nearly everywhere on (a,b). Since $Q_{(n-1)}(a)$ and $Q_{(n-1)}(b)$ exist and $Q_{(n-1)}$ exists almost everywhere on (a,b), we have that $(F_n)_{(n-1)}$ exists, finitely at a and b, and almost everywhere in (a,b). Let

$$B = \{x \in [a,b]; (F_n)_{(n-1)}(x) \,\text{exists finitely}\}.$$

Then B satisfies the requirements.

Let $\epsilon > 0$ be arbitrary. Then there is major function $\tilde{Q}$ of f and a minor function $\tilde{q}$ of f such that

$$0 \leq \tilde{Q}_{(n-1)}(b) - \tilde{q}_{(n-1)}(b) < \epsilon. \tag{4.2}$$

By Lemma 4.4 if Q is any major function of F and q any minor function of f, $Q - F_n$ and $F_n - q$ are k-convex, $0 \leq k \leq n$, and hence

$$Q_{(n-1)}(b) - (F_n)_{(n-1)}(b) \geq 0 \text{ and } (F_n)_{(n-1)}(b) - q_{(n-1)}(b) \geq 0, \tag{4.3}$$

for $Q \in \overline{\mathbb{M}}$, $q \in \underline{\mathbb{M}}$. Hence from (4.2) and (4.3)

$$0 \leq \tilde{Q}_{(n-1)}(b) - (F_n)_{(n-1)}(b) < \epsilon. \tag{4.4}$$

From (4.3) and (4.4)

$$\inf_{Q \in \overline{\mathbb{M}}} Q_{(n-1)}(b) = (F_n)_{(n-1)}(b). \tag{4.5}$$

Since (4.3) holds if b is replaced by a, it follows from the definition of Q and q that $(F_n)_{(n-1)}(a) = 0$. □

Definition 4.6. The set B in Theorem 4.5 is called the base of the integral and $(F_n)_{(n-1)}$ is called the first primitive of f, denoted by F_1.

Clearly the first primitive F_1 is only defined on the set B.

Corollary 4.7. *If f is integrable on $[a,b]$ and if B is the base of the integral, then for any $c \in B$ f is integrable on $[a,c]$ and $(F_n)_{(n-1)}(c) = \int_a^c f$.*

Proof. By k-convexity the relations (4.2)–(4.5) hold if b is replaced by c and the rest is clear. □

Theorem 4.8. *If Q is a major function of f and q a minor function of f, then $Q_{(n-1)}$ and $q_{(n-1)}$ exist on B.*

Proof. By Lemma 4.4, if F_n is the n-th primitive of f, then $Q - F_n$ is n-convex. So the unilateral derivatives $(Q - F_n)^+_{(n-1)}$ and $(Q - F_n)^-_{(n-1)}$ exist in $[a, b)$ and $(a, b]$ respectively. Let $\xi \in B \cap (a, b)$. Then since $(F_n)_{(n-1)}(\xi)$ exists both $Q^+_{(n-1)}(\xi)$ and $Q^-_{(n-1)}(\xi)$ exist. Since $Q \in \mathcal{S}_n(\xi)$, it follows that $Q_{(n-1)}(\xi)$ exists by [13, Lemma 2.1]. The proof for the case of the minor function is similar. □

Theorem 4.9. *If f is integrable on $[a,b]$ with F_r its r-th primitive, $1 \leq r \leq n$, then:*

(i) $(F_n)_{(n-r)}(x) = F_r(x)$ *for* $x \in [a,b]$, $1 \le r \le n$*;*

(ii) $D^n F_n = f$ *almost everywhere in* (a,b)*;*

(iii) $F_n \in \mathcal{S}_n(x)$ *for all* $x \in (a,b)$*.*

PROOF. *(i)* We may suppose that $1 < r < n$ since the case $r = n$ is trivial and the case $r = 1$ is Theorem 4.5. By Lemma 4.4 $Q - F_n$ and $F_n - q$ are k-convex for $0 \le k \le n$ and so $(Q - F_n)^{(n-r)}$ and $(F_n - q)^{(n-r)}$ are k-convex for $0 \le k \le r$. Hence for $x \in [a,b]$ and all $Q \in \overline{\mathbb{M}}$, $q \in \underline{\mathbb{M}}$,

$$Q_{(n-r)}(x) - (F_n)_{(n-r)}(x) \ge 0, \text{ and } (F_n)_{(n-r)}(x) - q_{(n-r)}(x) \ge 0. \tag{4.6}$$

Let $\epsilon > 0$ be arbitrary. By (4.1,r) there is a major function of f, $\tilde{Q}$, and a minor function of f, $\tilde{q}$, such that

$$0 \le \tilde{Q}_{(n-r)}(x) - \tilde{q}_{(n-r)}(x) \le \epsilon(b-a)^{r-1} \text{ for } x \in [a,b]. \tag{4.7}$$

From (4.6) and (4.7)

$$0 \le \tilde{Q}_{(n-r)}(x) - (F_n)_{(n-r)}(x) \le \epsilon(b-a)^{r-1} \text{ for } x \in [a,b]. \tag{4.8}$$

From (4.6) and (4.8)

$$\inf_{Q \in \mathbb{M}} Q_{(n-r)}(x) = (F_n)_{(n-r)}(x) \text{ for } x \in [a,b]. \tag{4.9}$$

But by definition the left-hand side of (4.9) is $F_r(x)$, so the proof of *(i)* is complete.

(ii) For any positive integer k, let

$$E_k = \Big\{x : x \in (a,b); f(x) > \underline{D}^n F_n(x) + \frac{1}{k}\Big\}.$$

Suppose that E_k has positive outer measure, p say, and choose ϵ such that $0 < \epsilon < \dfrac{p}{2k}$. Let Q be a major function of f such that

$$0 \le Q_{(n-1)}(b) - (F_n)_{(n-1)}(b) < \epsilon; \tag{4.10}$$

such a Q exists by (4.5). Let $R = Q_{(n-1)} - (F_n)_{(n-1)}$ on B. Then R is non-decreasing since $Q - F_n$ is n-convex. Extend R to the whole of $[a,b]$ as a non-decreasing function. Then by (4.10)

$$\int_a^b R' \le R(b) \le \epsilon. \tag{4.11}$$

Let

$$G_k = \Big\{x;\, x \in B \cap E_k; 0 \le R'(x) \le \frac{1}{2k}\Big\} \text{ and } H_k = \Big\{x;\, x \in B; R'(x) > \frac{1}{2k}\Big\}.$$

Then H_k is measurable and $\mu(H_k) < p$ by (4.11). Since $\mu(B) = b - a$, the set $B \cap E_k$ has outer measure p and, since $B \cap E_k \subset G_k \cup H_k$, G_k has positive outer measure. Since $Q_n - F_n$ is n-convex, $Q_{(n-2)} - (F_n)_{(n-2)}$ is convex and so by [21, Vol. I, p. 328, Lemma 3.16] $D^2(Q_{(n-2)} - (F_n)_{(n-2)})$ exists, finitely, almost everywhere in (a, b). Hence $D^n(Q - F_n)$ exists, finitely, almost everywhere in (a, b), and also, by Lemma 3.4, so does the Peano derivative $(Q - F_n)_{(n)}$. Since $(Q - F_n)_{(n-1)} = R$ almost everywhere in (a, b) by [21, Vol. II, p. 77, Theorem 4.26] $D^n(Q - F_n) = (Q - F_n)_{(n)} = R'_{ap}$ almost everywhere in (a, b). Since R is monotonic, we deduce that $D^n(Q - F_n) = R'$ almost everywhere in (a, b). So almost everywhere on G_k

$$f \le \underline{D}^n Q = \underline{D}^n F_n + D^n(Q - F_n) = \underline{D}^n F_n + R' \le \underline{D}^n F_n + \frac{1}{2k}.$$

But this is a contradiction since $G_k \subset E_k$. Therefore $\mu(E_k) = 0$ and since

$$\Big\{x \in (a, b); f(x) > \underline{D}^n F_n(x)\Big\} = \cup_{k=1}^{\infty} E_k,$$

we have that $f \le \underline{D}^n F_n$ almost everywhere in (a, b). Similarly $f \ge \overline{D}^n F_n$ almost everywhere in (a, b). This completes the proof of *(ii)*.

(iii) Let $\epsilon > 0$ be arbitrary and Q a major function of f, q a minor function of f such that

$$Q_{(n-1)}(b) - q_{(n-1)}(b) < \epsilon. \tag{4.12}$$

Let $c \in (a, b)$ and choose h, $0 < h < \min\{c - a, b - c\}$. By the mean value theorem there is a θ, $0 < \theta < 1$ such that

$$\begin{aligned} h\omega_n(Q, c, h) - h\omega_n(F_n, c, h) =& h\omega_n(Q - F_n, c, h) = \theta h \omega_2\big((Q - F_n)_{(n-2)}, c, \theta h\big) \\ =& \theta h \omega_2\big(Q_{(n-2)} - (F_n)_{(n-2)}, c, \theta h\big) \qquad (4.13) \\ =& \theta h \omega_2\big(Q_{(n-2)}, c, \theta h\big) - \theta h \omega_2\big((F_n)_{(n-2)}, c, \theta h\big). \end{aligned}$$

Since $(Q - F_n)_{(n-2)}$ is non-decreasing,

$$Q_{(n-2)}(c - \theta h) - (F_n)_{(n-2)}(c - \theta h) - Q_{(n-2)}(c) + (F_n)_{(n-2)}(c) \le 0 \tag{4.14}$$

and since $(Q - F_n)_{(n-2)}$ is also convex,

$$\begin{aligned} & Q_{(n-2)}(c + \theta h) - (F_n)_{(n-2)}(c + \theta h) - Q_{(n-2)}(c) + (F_n)_{(n-2)}(c) \\ =& \int_c^{c+\theta h} (Q - F_n)_{(n-1)} \le \big(Q_{(n-1)}(b) - (F_n)_{(n-1)}(b)\big)\theta h \qquad (4.15) \\ \le& \big(Q_{(n-1)}(b) - q_{(n-1)}(b)\big)\theta h < \epsilon \theta h, \end{aligned}$$

by (4.12). Adding (4.14) and (4.15) and dividing by θh,

$$\theta h\omega_2\big(Q_{(n-2)}, c, \theta h\big) - \theta h\omega_2\big((F_n)_{(n-2)}, c, \theta h\big) < \epsilon. \tag{4.16}$$

From (4.13) and (4.16)

$$h\omega_n(Q, c, h) - h\omega_n(F_n, c, h) < \epsilon. \tag{4.17}$$

Since $Q \in \mathcal{S}_n(c)$, from (4.17) $\liminf_{h\to 0} h\omega_n(F_n, c, h) \geq -\epsilon$ and so since ϵ is arbitrary, $\liminf_{h\to 0} h\omega_n(F_n, c, h) \geq 0$.

By a similar argument using q we get that $\limsup_{h\to 0} h\omega_n(F_n, c, h) \leq 0$, which completes the proof of *(iii)*. □

Theorem 4.10. *If f is T^n-integrable, then f is T^m-integrable for any $m > n$ and the two integrals are equal.*

PROOF. Let f be T^n-integrable and let Q be any T^n-major function of f with $\tilde{Q}$ the $(m-n)$-th indefinite integral of Q with $\tilde{Q}_{(r)}(a) = 0$, $r = 0, 1, \ldots, m-n-1$. Since $\tilde{Q}_{(m-n)} = Q$ in $[a, b]$, by the mean value theorem there is for each x, a $\theta = \theta_x$, $0 < \theta < 1$, such that $\omega_m(\tilde{Q}, x, t) = \omega_n(Q, x, \theta t)$. Hence $\underline{D}^m\tilde{Q}(x) \geq \underline{D}^n Q(x)$ for all x and $\tilde{Q} \in \mathcal{S}_m(x)$ for $x \in (a, b)$. So $\tilde{Q}$ is a T^m-major function of f. Also $Q_{(n-1)}(b) = \tilde{Q}_{(m-1)}(b)$ by the mean value theorem. The rest is clear. □

Theorem 4.11. *If f is integrable, then f is measurable and finite almost everywhere.*

PROOF. Let F_n be the n-th primitive of f. Then F_n is continuous and since $\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}j^i = 0$ for $i = 0, 1, \ldots, n-1$ and $\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}j^n = n!$ we have by (3.4) that for each $x \in (a, b)$ for which $D^n F_n(x)$ exists, $\lim\limits_{t\to 0}\dfrac{\Delta_n(x, t, F_n)}{t^n} = D^n F_n(x)$, where $\Delta_n(x, t, F_n)$ is given by (3.2). Therefore $D^n F_n$ is measurable on the set where it exists. By Theorem 4.9 *(ii)*, f is measurable. Suppose that $f = \infty$ on a set of positive measure. Then by Theorem 4.9 *(ii)*, $D^n F_n = \infty$ on a set of positive measure. Let q be any minor function of f. Then $F_n - q$ is n-convex and as in the proof of Theorem 4.9 *(ii)* it can be shown that $D^n(F_n - q)$ exists, finitely, almost everywhere in (a, b). Since $\underline{D}^n F_n = \underline{D}^n q + D^n(F_n - q)$, we have that $D^n q = \infty$ on a set of positive measure, which is a contradiction. Thus $f < \infty$ almost everywhere. Similarly $f > -\infty$ almost everywhere. □

Theorem 4.12. *If f is integrable on $[a, b]$ and if $c \in B$, where B is the base of the integral, then f is integrable on $[a, c]$ and $[c, b]$. Conversely if f is integrable* *on both $[a,c]$ and $[c,b]$ for some c, $a<c<b$, then f is integrable on $[a,b]$. In both cases*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{4.18}$$

Proof. Let f be integrable on $[a,b]$ and let $c \in B$. Then, by Corollary 4.7, f is integrable on $[a,c]$ and

$$\int_a^c f = (F_n)_{(n-1)}(c), \tag{4.19}$$

where F_n is the n-th primitive of f on $[a,b]$. Also as remarked in the proof of that corollary (4.5) holds if b is replaced by c and so

$$\inf_{Q\in\overline{\mathbb{M}}} Q_{(n-1)}(c) = (F_n)_{(n-1)}(c). \tag{4.20}$$

For each $Q \in \overline{\mathbb{M}}$ let

$$\tilde{Q}(x) = Q(x) - \sum_{i=0}^{n-1} \frac{(x-c)^i}{i!} Q_i(c), \text{ for } c \le x \le b.$$

Then $\tilde{Q}$ is a major function of f on $[c,b]$. Also

$$\tilde{Q}_{(n-1)}(b) + Q_{(n-1)}(c) = Q_{(n-1)}(b). \tag{4.21}$$

Hence, if $\overline{\mathbb{U}}$ is the family of major functions of f on $[c,b]$, we have from (4.21) that

$$\begin{aligned} \inf_{U\in\overline{\mathbb{U}}} U_{(n-1)}(b) + \inf_{Q\in\overline{\mathbb{M}}} Q_{(n-1)}(c) &\le \inf_{Q\in\overline{\mathbb{M}}} \tilde{Q}_{(n-1)}(b) + \inf_{Q\in\overline{\mathbb{M}}} Q_{(n-1)}(c) \\ &\le \inf_{Q\in\overline{\mathbb{M}}} \big(\tilde{Q}_{(n-1)}(b) + Q_{(n-1)}(c)\big) \\ &= \inf_{Q\in\overline{\mathbb{M}}} Q_{(n-1)}(b). \end{aligned} \tag{4.22}$$

From (4.19), (4.20) and (4.22)

$$\inf_{U\in\overline{\mathbb{U}}} U_{(n-1)}(b) \le \int_a^b f - \int_a^c f. \tag{4.23}$$

Similarly if $\underline{\mathbb{U}}$ is the family of minor functions of f on $[c,b]$,

$$\sup_{u\in\underline{\mathbb{U}}} u_{(n-1)}(b) \ge \int_a^b f - \int_a^c f. \tag{4.24}$$

From (4.23) and (4.24) f is integrable on $[c, b]$ and (4.18) holds.

Conversely, let f be integrable on $[a, c]$ and $[c, b]$ and let $\epsilon > 0$ be arbitrary. Let Q, respectively q, be a major, respectively a minor, function of f on $[a, c]$, and let U, respectively u, be a major, respectively a minor, function of f on $[c, b]$, chosen so that

$$Q_{(n-1)}(c) - q_{(n-1)}(c) < \epsilon, \text{ and } U_{(n-1)}(b) - u_{(n-1)}(b) < \epsilon. \tag{4.25}$$

Let

$$\tilde{Q}(x) = \begin{cases} Q(x) & \text{if } a \leq x \leq c, \\ U(x) + \sum_{i=0}^{n-1} \frac{(x-c)^i}{i!} Q_i(c) & \text{if } c \leq x \leq b. \end{cases} \tag{4.26}$$

and

$$\tilde{q}(x) = \begin{cases} q(x) & \text{if } a \leq x \leq c \\ u(x) + \sum_{i=0}^{n-1} \frac{(x-c)^i}{i!} q_i(c) & \text{if } c \leq x \leq b. \end{cases} \tag{4.27}$$

Then $\tilde{Q}$ is a major function of f on $[a, b]$, and $\tilde{q}$ is a minor function of f on $[a, b]$. Also by (4.25) $\tilde{Q}_{(n-1)}(b) - \tilde{q}_{(n-1)}(b) < 2\epsilon$. Hence f is integrable on $[a, b]$.

If F_n is the n-th primitive of f on $[a, b]$ and if $\tilde{Q}$ is as in (4.26), then let $\Psi = \tilde{Q} - F_n$. Since Ψ is n-convex on $[a, b]$, the unilateral derivatives $\Psi^+_{(n-1)}(c)$, and $\Psi^-_{(n-1)}(c)$ exist and are finite. Further $\Psi \in \mathcal{S}_n(c)$ since $\tilde{Q} \in \mathcal{S}_n(c)$, and by Theorem 4.9 *(iii)* $F_n \in \mathcal{S}_n(c)$. Since

$$\frac{1}{2}\big(\gamma_{n-1}(\Psi, c, h) - \gamma_{n-1}(\Psi, c, -h)\big) = \frac{h}{n}\omega_n(\Psi, c, h)$$

we get, by letting $h \to 0+$ that $\Psi^+_{(n-1)}(c) = \Psi^-_{(n-1)}(c)$. Thus $\Psi_{(n-1)}(c)$ exists, finitely, and from (4.26) $\tilde{Q}_{(n-1)}(c)$ exists, finitely, so then does $(F_n)_{(n-1)}(c)$. Hence $c \in B$ and the proof is complete by the first part. $\square$

Theorem 4.13. *Suppose that:*

(i) F *is continuous on* $[a, b]$*;*

(ii) $F_{(n-2)}$ *exists finitely on* $[a, b]$*;*

(iii) $F_{(n-1)}$ *exists finitely on* $[a, b]$ *except on a set of measure zero in* (a, b)*;*

(iv) $D^n F = f$ *almost everywhere in* (a, b)*;*

(v) $-\infty < \underline{D}^n F \leq \overline{D}^n F < \infty$ *nearly everywhere in* (a, b)*;*

(vi) $F \in \mathcal{S}_n(x)$ *for* $x \in (a.b)$.

Then f is T^n-integrable on $[a,b]$ and $(T^n)\int_a^b f = F_{(n-1)}(b) - F_{(n-1)}(a)$.

PROOF. The function $\Phi(x) = F(x) - \sum_{i=0}^{n-1}\frac{(x-a)^i}{i!}F_{(i)}(a)$ for $a \le x \le b$, is both a T^n-major function and a T^n-minor function of f on $[a,b]$. So f is T^n-integrable on $[a,b]$ and

$$(T^n)\int_a^b f = \Phi_{(n-1)}(b) = F_{(n-1)}(b) - F_{(n-1)}(a). \qquad \square$$

5 Integration by Parts.

In what follows we need two theorems on the C_rP-integral introduced in [5] and which is equivalent to the Z_r-integral defined in [1]. A major function for the Z_r-integral of a function f defined on $[a,b]$ is required to satisfy the following conditions on $[a,b]$:

(i) M is continuous;

(ii) $M_{(r)}$ exists finitely;

(iii) $\underline{M}_{(r+1)} \ge f$;

(iv) $\underline{M}_{(r+1)} > -\infty$.

The conditions for a minor function are similar. We note that the conditions *(iii)* and *(iv)* can be relaxed to

(iii)′ $\underline{M}_{(r+1)} \ge f$ almost everywhere;

(iv)′ $\underline{M}_{(r+1)} > -\infty$ nearly everywhere.

This modification defines an integral, say the Z_r^*-integral, that clearly includes the Z_r-integral. It can be verified that all the properties of the Z_r-integral remain true for the Z_r^*-integral; see [6].

The theorems that we need are the following:

Theorem 5.1. *If $F_{(r)}$ exists finitely in $[a,b]$ and if $F_{(r+1)}$ exist almost everywhere in $[a,b]$ and if $\underline{F}_{(r+1)}$ and $\overline{F}_{(r+1)}$ are finite nearly everywhere on $[a,b]$, then $F_{(r+1)}$ is Z_r^*-integrable and $(Z_r^*)\int_a^b F_{(r+1)} = F_{(r)}\Big|_a^b$.*

Theorem 5.2. *Let f be Z_r^*-integrable on $[a,b]$ and let $F(x) = (Z_r^*)\int_a^x f$ for $a \le x \le b$. If g is of bounded variation on $[a,b]$ and if*

$$G(x) = \frac{1}{(r-1)!}\int_a^x (x-t)^{r-1}g(t)\mathrm{d}t,\ a \le x \le b,$$

then fG is Z_r^*-integrable on $[a,b]$ and

$$(Z_r^*)\int_a^b fG = FG\Big|_a^b - (Z_{r-1}^*)\int_a^b FG'.$$

These are analogues for the Z_r^*-integral of [1, Propositions 3.4 and 5.1].

Lemma 5.3. *For any positive integer $n \geq 2$ we have:*

(i) $\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}r^i = \begin{cases} 0 & \text{for } i = 0, 1, \dots, n-1, \\ n & \text{for } i = n; \end{cases}$

(ii) $\sum_{r=0}^{n-k}(-1)^{r}\binom{n}{r}\binom{n-r}{k} = \begin{cases} 0 & \text{for } k = 0, 1, \dots, n-1, \\ 1 & \text{for } k = n; \end{cases}$

(iii) $\sum_{r=1}^{n-k-1}(-1)^{r}\binom{n-1}{r-1}\binom{n-r-1}{k} = (-1)^{n-k-1}$ *for* $k = 0, 1, \dots, n-2;$

(iv) $\sum_{r=1}^{n-k-1}(-1)^{r}\binom{n}{r-1}\binom{n-r-1}{k} = (-1)^{n-k-1}(n-k-1)$ *for* $k = 0, 1, \dots, n-2.$

PROOF. *(i)* This is a well known result.

(ii) If $0 \leq k \leq n$ then

$$\begin{aligned}
I_1 &= \sum_{r=0}^{n-k}(-1)^r\binom{n}{r}\binom{n-r}{k} \\
&= \frac{1}{k!}\sum_{r=0}^{n-k}(-1)^r\binom{n}{r}(n-r)(n-r-1)\cdots(n-r-k+1) \\
&= \frac{1}{k!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}(n-r)(n-r-1)\cdots(n-r-k+1) \\
&= \frac{(-1)^{n-k}}{k!}\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\sum_{i=0}^{k}p_i r^{k-i},
\end{aligned}$$

where $p_0 = 1$ and the remaining p's depend on n and k. So by *(i)* $I_1 = 0$ for $k = 0, \dots, n-1$ and $I_1 = 1$ for $k = n$, as had to be proved.

(iii) If $0 \leq k \leq n-2$ then

$$\begin{aligned}
I_2 = &\sum_{r=1}^{n-k-1} (-1)^r \binom{n-1}{r-1}\binom{n-r-1}{k} \\
=&\frac{1}{k!}\sum_{r=1}^{n-k-1} (-1)^r \binom{n-1}{r-1}(n-r-1)(n-r-2)\cdots(n-r-k) \\
=&\frac{1}{k!}\sum_{r=1}^{n-1}(-1)^r \binom{n-1}{r-1}(n-r-1)(n-r-2)\cdots(n-r-k) \qquad (5.1)\\
=&\frac{(-1)^{n-k}}{k!}\sum_{r=1}^{n-1}(-1)^{n-r}\binom{n-1}{r-1}\big((r-1)-(n-2)\big)\cdots\big((r-1)-(n-k-1)\big)\\
=&\frac{(-1)^{n-k}}{k!}\sum_{r=1}^{n-1}(-1)^{n-r}\binom{n-1}{r-1}\sum_{i=0}^{k} q_i (r-1)^{k-i},
\end{aligned}$$

where $q_0 = 1$ and $q_1, \ldots, q_k$ are given by

$$x^k + q_1 x^{k-1} + \cdots + q_k = \big(x-(n-2)\big)\big(x-(n-3)\big)\cdots\big(x-(n-k-1)\big). \quad (5.2)$$

Since $0 \leq k \leq n-2$, we have by *(i)* that $\sum_{\nu=0}^{n-1}(-1)^{n-\nu-1}\binom{n-1}{\nu}\nu^i = 0$, $i = 0, 1, \ldots, k$, and so

$$\sum_{\nu=0}^{n-2}(-1)^{n-\nu-1}\binom{n-1}{\nu}\nu^i = -(n-1)^i, \text{ for } i = 0, 1, \ldots, k. \qquad (5.3)$$

From (5.1) and (5.3)

$$\begin{aligned}
I_2 =&\frac{(-1)^{n-k}}{k!}\sum_{\nu=0}^{n-2}(-1)^{n-\nu-1}\binom{n-1}{\nu}\sum_{i=0}^{k} q_i \nu^{k-i}\\
=&\frac{(-1)^{n-k-1}}{k!}\sum_{i=0}^{k} q_i (n-1)^{k-i} = (-1)^{n-k-1},
\end{aligned}$$

by (5.2) completing the proof of *(iii)*.

(iv) As in (5.1) we have

$$\begin{aligned} I_3 = & \sum_{r=1}^{n-k-1}(-1)^r\binom{n}{r-1}\binom{n-r-1}{k} \qquad (5.4)\\ =&\frac{(-1)^{n-k}}{k!}\sum_{r=1}^{n-1}(-1)^{n-r}\binom{n}{r-1}\sum_{i=0}^{k}q_i(r-1)^{k-i}\\ =&\frac{(-1)^{n-k}}{k!}\sum_{\nu=0}^{n-2}(-1)^{n-\nu-1}\binom{n}{\nu}\sum_{i=0}^{k}q_i\nu^{k-i}\end{aligned}$$

where $q_0 = 1$ and the remaining q are given in equation (5.2). Since $0 \le k \le n-2$, we have by *(i)* $\sum_{\nu=0}^{n}(-1)^{n-\nu}\binom{n}{\nu}\nu^i = 0$ for $i = 0, 1, \ldots, k$, which gives $\sum_{\nu=0}^{n-2}(-1)^{n-\nu-1}\binom{n}{\nu}\nu^i = n^i - n(n-1)^i$ for $i = 0, 1, \ldots, k$. Hence by (5.2) and (5.4)

$$\begin{aligned} I_3 =&\frac{(-1)^{n-k}}{k!}\Big(\sum_{i=0}^{k}q_in^{k-i} - n\sum_{i=0}^{k}q_i(n-1)^{k-i}\Big)\\ =&\frac{(-1)^{n-k}}{k!}\big((k+1)! - n(k!)\big) = (-1)^{n-k-1}(n-k-1). \qquad \square\end{aligned}$$

Lemma 5.4. *Let $F^{(n)}$ exist in $[a, b]$, for some $n \ge 2$, and let g be of bounded variation in $[a, b]$. If $G(x) = \frac{1}{(n-2)!}\int_a^x (x-t)^{n-2}g(t)\,\mathrm{d}t$ for $a \le x \le b$, then the function S defined by*

$$\begin{aligned} S(x) = F(x)G(x) &+ \sum_{r=1}^{n-2}(-1)^r\binom{n}{r}\frac{1}{(r-1)!}\int_a^x (x-t)^{r-1}F(t)G^{(r)}(t)\mathrm{d}t\\ &+ (-1)^{n-1}\frac{n}{(n-2)!}\int_a^x (x-t)^{n-2}F(t)g(t)\,dt \qquad (5.5)\\ &+ (-1)^{n}\frac{1}{(n-2)!}\int_a^x (x-t)^{n-2}\Big(\int_a^t F\mathrm{d}g\Big)\mathrm{d}t,\end{aligned}$$

is such that $S^{(n-1))}$ exists in $[a, b]$ and for all $x \in [a, b]$

$$\begin{aligned} S^{(n-1)}(x) =&\Big(\int_a^x F^{(n)}G\Big) + (-1)^{n-1}F(a)g(a)\\ =&\sum_{k=0}^{n-2}(-1)^kF^{(n-k-1)}(x)G^{(k)}(x) + (-1)^{n-1}\Big(F(x)g(x) - \int_a^x F\mathrm{d}g\Big).\end{aligned}$$

Proof. Integrating by parts

$$\frac{1}{(r-1)!}\int_a^x (x-t)^{r-1}F(t)G^{(r)}(t)\mathrm{d}t = \int_a^x \mathrm{d}\xi_1 \int_a^{\xi_1} \mathrm{d}\xi_2 \cdots \int_a^{\xi_{r-1}} FG^{(r)},$$

and taking the derivative of order $(n-2)$, for $0 \le r \le n-2$ we have

$$\Big(\frac{1}{(r-1)!}\int_a^x (x-t)^{r-1}F(t)G^{(r)}(t)\mathrm{d}t\Big)^{(n-2)} = \big(FG^{(r)}\big)^{(n-r-2)}(x).$$

Similarly

$$\Big(\frac{1}{(n-2)!}\int_a^x (x-t)^{n-2}F(t)g(t)\mathrm{d}t\Big)^{(n-2)} = \int_a^x Fg,$$

and

$$\Big(\frac{1}{(n-2)!}\int_a^x (x-t)^{n-2}\big(\int_a^t F\mathrm{d}g\big)\mathrm{d}t\Big)^{(n-2)} = \int_a^x \big(\int_a^t F\mathrm{d}g\big)\mathrm{d}t.$$

Hence from (5.5)

$$\begin{aligned} S^{(n-2)}(x) = \big(FG\big)^{(n-2)}(x) + \sum_{r=1}^{n-2}(-1)^r\binom{n}{r}\big(F(x)G^{(r)}(x)\big)^{(n-r-2)} \\ + (-1)^{n-1}n\int_a^x Fg + (-1)^n\int_a^x\Big(\int_a^t F\mathrm{d}g\Big)\mathrm{d}t. \end{aligned}$$

Since $\binom{n}{r} = \binom{n-2}{r} + \binom{n-2}{r-1} + \binom{n-1}{r-1}$, we have

$$\begin{aligned} S^{(n-2)}(x) &= \sum_{k=0}^{n-2}\binom{n-2}{k}F^{(k)}(x)G^{(n-k-2)}(x) \\ &+ \sum_{r=1}^{n-2}(-1)^r\binom{n-2}{r}\Big(\sum_{k=0}^{n-r-2}\binom{n-r-2}{k}F^{(k)}(x)G^{(n-k-2)}(x)\Big) \\ &+ \sum_{r=1}^{n-2}(-1)^r\Big(\binom{n-2}{r-1}+\binom{n-1}{r-1}\Big)\Big(\sum_{k=0}^{n-r-2}\binom{n-r-2}{k}F^{(k)}(x)G^{(n-k-2)}(x)\Big) \\ &+ (-1)^{n-1}n\int_a^x Fg + (-1)^n\int_a^x\Big(\int_a^t F\mathrm{d}g\Big)\mathrm{d}t. \end{aligned} \tag{5.6}$$

By Lemma 5.3 *(ii)*

$$\begin{aligned}
\sum_{k=0}^{n-2}&\binom{n-2}{k}F^{(k)}G^{(n-k-2)}\\
&+\sum_{r=1}^{n-2}(-1)^r\binom{n-2}{r}\left(\sum_{k=0}^{n-r-2}\binom{n-r-2}{k}F^{(k)}G^{(n-k-2)}\right)\\
&=\sum_{r=0}^{n-2}(-1)^r\binom{n-2}{r}\left(\sum_{k=0}^{n-r-2}\binom{n-r-2}{k}F^{(k)}G^{(n-k-2)}\right) \qquad (5.7)\\
&=\sum_{k=0}^{n-2}\sum_{r=0}^{(n-k-2)}(-1)^r\binom{n-2}{r}\binom{n-r-2}{k}F^{(k)}G^{(n-k-2)}=F^{(n-2)}G.
\end{aligned}$$

Also by Lemma 5.3 *(iii)*

$$\begin{aligned}
\sum_{r=1}^{n-3}&(-1)^r\binom{n-2}{r-1}\left(\sum_{k=1}^{n-r-2}\binom{n-r-2}{k}F^{(k)}G^{(n-k-2)}\right) \qquad (5.8)\\
&=\sum_{k=1}^{n-3}\left(\sum_{r=1}^{n-k-2}(-1)^r\binom{n-2}{r-1}\binom{n-r-2}{k}F^{(k)}G^{(n-k-2)}\right)\\
&=\sum_{k=1}^{n-3}(-1)^{n-k-2}F^{(k)}G^{(n-k-2)},
\end{aligned}$$

and by Lemma 5.3 *(iv)*

$$\begin{aligned}
\sum_{r=1}^{n-3}&(-1)^r\binom{n-1}{r-1}\left(\sum_{k=1}^{n-r-2}\binom{n-r-2}{k}F^{(k)}G^{(n-k-2)}\right) \qquad (5.9)\\
&=\sum_{k=1}^{n-3}\left(\sum_{r=1}^{n-k-2}(-1)^r\binom{n-1}{r-1}\binom{n-r-2}{k}F^{(k)}G^{(n-k-2)}\right)\\
&=\sum_{k=1}^{n-3}(-1)^{n-k-2}(n-k-2)F^{(k)}G^{(n-k-2)}.
\end{aligned}$$

Since $\sum_{r=1}^{n-2}(-1)^r\binom{n-2}{r-1}=(-1)^{n-2}$ and $\sum_{r=1}^{n-2}(-1)^r\binom{n-1}{r-1}=(-1)^{n-2}(n-2)$, we have

$$\sum_{r=1}^{n-2}(-1)^r\left(\binom{n-2}{r-1}+\binom{n-1}{r-1}\right)=(-1)^{n-2}(n-1). \qquad (5.10)$$

From (5.8), (5.9) and (5.10)

$$\begin{aligned}\sum_{r=1}^{n-2}&(-1)^r\left(\binom{n-2}{r-1}+\binom{n-1}{r-1}\right)\left(\sum_{k=0}^{n-r-2}\binom{n-r-2}{k}F^{(k)}G^{(n-k-2)}\right)\\ &=\sum_{r=1}^{n-2}(-1)^r\left(\binom{n-2}{r-1}+\binom{n-1}{r-1}\right)FG^{(n-2)} \qquad (5.11)\\ &\quad+\sum_{r=1}^{n-3}(-1)^r\left(\binom{n-2}{r-1}+\binom{n-1}{r-1}\right)\left(\sum_{k=1}^{n-r-2}\binom{n-r-2}{k}F^{(k)}G^{(n-k-2)}\right)\\ &=(-1)^{n-2}(n-1)FG^{(n-2)}+\sum_{k=1}^{n-3}(-1)^{n-k-2}(n-k-1)F^{(k)}G^{(n-k-2)}.\end{aligned}$$

Finally, integrating by parts yields

$$\begin{aligned}(-1)^{n-1}n\int_a^x Fg&+(-1)^n\int_a^x(\int_a^t F\mathrm{d}g)\mathrm{d}t\\ &=(-1)^{n-1}\big(n\int_a^x Fg-\int_a^x\big(F(t)g(t)-F(a)g(a)-\int_a^t F'(u)g(u)\mathrm{d}u\big)\mathrm{d}t\big)\\ &=(-1)^{n-1}\Big((n-1)F(x)G^{(n-2)}(x)-(n-1)\int_a^x F'G^{(n-2)}\\ &\quad+\int_a^x\big(F(a)g(a)+(\int_a^t F'g)\big)\mathrm{d}t\Big) \qquad (5.12)\end{aligned}$$

Using (5.7), (5.11) and (5.12) we get from (5.6)

$$\begin{aligned}S^{(n-2)}(x)=&F^{(n-2)}(x)G(x)+\sum_{k=1}^{(n-3)}(-1)^{n-k-2}(n-k-1)F^{(k)}(x)G^{(n-k-2)}(x)\\ &+(-1)^n(n-1)\int_a^x F'G^{(n-2)}+(-1)^{n-1}\int_a^x\big(F(a)g(a)+(\int_a^t F'g)\big)\mathrm{d}t\end{aligned}$$

Hence

$$\begin{aligned}S^{(n-1)}(x)=&F^{(n-2)}(x)G'(x)+F^{(n-1)}(x)G(x)\\ &+\sum_{k=1}^{(n-3)}(-1)^{n-k-2}(n-k-1)\big(F^{(k+1)}(x)G^{(n-k-2)}(x)\\ &+F^{(k)}(x)G^{(n-k-1)}(x)\big)+(-1)^n(n-1)F'(x)G^{(n-2)}(x) \qquad (5.13)\\ &+(-1)^{n-1}\big(F(a)g(a)+\int_a^x F'g\big)\end{aligned}$$

Now

$$
\begin{aligned}
&\sum_{k=1}^{(n-3)}(-1)^{n-k-2}(n-k-1)\big(F^{(k+1)}G^{(n-k-2)}+F^{(k)}G^{(n-k-1)}\big)\\
&=\sum_{k=2}^{n-2}(-1)^{n-k-1}(n-k)F^{(k)}G^{(n-k-1)}\\
&\quad+\sum_{k=1}^{n-3}(-1)^{n-k-2}(n-k-1)F^{(k)}G^{(n-k-1)} \qquad (5.14)\\
&=-2F^{(n-2)}G'+\sum_{k=2}^{n-3}(-1)^{n-k-1}\big((n-k)-(n-k-1)\big)F^{(k)}G^{(n-k-1)}\\
&\quad+(-1)^{n-3}(n-2)F'G^{(n-2)}\\
&=-F^{(n-2)}G'+\sum_{k=1}^{n-2}(-1)^{n-k-1}F^{(k)}G^{(n-k-1)}+(-1)^{n-3}(n-1)F'G^{(n-2)}.
\end{aligned}
$$

Also integrating by parts successively leads to

$$
\int_a^x F'g=\sum_{k=1}^{n-1}(-1)^{k-1}F^{(k)}(x)G^{(n-k-1)}(x)+(-1)^{n-1}\int_a^x F^{(n)}G. \qquad (5.15)
$$

Using (5.14) and (5.15) we get from (5.13)

$$
S^{(n-1)}(x)=\int_a^x F^{(n)}G+(-1)^{n-1}F(a)g(a), \qquad (5.16)
$$

which proves the first part.

Integrating the right-hand side of (5.16) by parts we get

$$
S^{(n-1)}(x)=\sum_{k=0}^{n-2}(-1)^kF^{(n-k-1)}(x)G^{(k)}(x)+(-1)^{n-1}\Big(F(x)g(x)-\int_a^x F\mathrm{d}g\Big),
$$

which completes the proof. □

Lemma 5.5. *Let $M:[a,b]\to\mathbb{R}$ be continuous, $g:[a,b]\to\mathbb{R}$ be of bounded variation, and, for an $n\geq 2$, let $M_{(n-2)}$ exist on $[a,b]$. If G is defined by*

$G(x) = \frac{1}{(n-2)!}\int_a^x (x-t)^{n-2} g(t)\,\mathrm{d}t,\ a \le x \le b$, *then the function* S *defined by*

$$\begin{aligned} S(x) =& M(x)G(x) + \sum_{r=1}^{n-2} (-1)^r \binom{n}{r} \frac{1}{(r-1)!} \int_a^x (x-t)^{r-1} M(t) G^{(r)}(t)\mathrm{d}t \\ &+ \frac{(-1)^{n-1} n}{(n-2)!} \int_a^x (x-t)^{n-2} M(t) g(t)\mathrm{d}t \\ &+ \frac{(-1)^n}{(n-2)!} \int_a^x (x-t)^{n-2} \Big(\int_a^t M\mathrm{d}g \Big)\mathrm{d}t \end{aligned} \tag{5.17}$$

is such that:

(i) S *is continuous and* $S_{(n-2)}$ *exists in* $[a,b]$*;*

(ii) $t\omega_n(S, x_0, t) = tG(x_0)\omega_n(M, x_0, t) + o(1)$ *for all* $x_0,\ a < x_0 < b$*;*

(iii) if $n > 2$, *then*

$$\begin{aligned} \omega_n(S, x_0, t) =& G(x_0)\omega_n(M, x_0, t) + nG'(x_0)\Big(\omega_{n-1}(M, x_0, t) \\ &- \frac{n}{t^n} \int_a^t \xi^{n-1} \omega_{n-1}(M, x_0, \xi)\mathrm{d}\xi\Big) + o(1), for\ a < x_0 < b; \end{aligned}$$

(iii)′ if $n = 2$, *then*[3]

$$\begin{aligned} \omega_2(S, x_0, t) =& G(x_0)\omega_2(M, x_0, t) + 2g(x_0)\big[\omega_1(M, x_0, t) \\ &- \frac{2}{t^2} \int_a^t \xi \omega_1(M, x_0, \xi)\mathrm{d}\xi\big] + o(1), \end{aligned}$$

for those $x_0, a < x_0 < b$, *for which*

$$\overset{x_0+h}{\underset{x_0}{V}}(g) = O(h) \text{ and } \overset{x_0}{\underset{x_0-h}{V}}(g) = O(h). \tag{5.18}$$

(iv) for those $x_0, a < x_0 < b$, *for which* $M_{(n-1)}(x_0)$ *exists*

$$\begin{aligned} S_{(n-1)}(x_0) =& \sum_{r=1}^{n-1} (-1)^{n-r-1} M_{(r)}(x_0) G^{(n-r-1)}(x_0) \\ &+ (-1)^{n-1} M(x_0) g(x_0) + (-1)^n \int_a^x M\mathrm{d}g. \end{aligned}$$

[3]The total variation of g in $[c,d]$ is written $\overset{d}{\underset{c}{V}}(g)$.

Proof. Suppose that $x_0 \in [a,b]$, and without loss in generality assume $x_0 = 0$. Let:

$$\begin{aligned}
P(t) =& M(0) + tM_{(1)}(0) + \cdots + \frac{t^{n-2}}{(n-2)!}M_{(n-2)}(0);\\
L(t) =& M(t) - P(t);\\
R(t) =& P(t)G(t) + \sum_{r=1}^{n-2} \frac{(-1)^r}{(r-1)!}\binom{n}{r}\int_0^t (t-\xi)^{r-1}P(t)G^{(r)}(t)\mathrm{d}\xi\\
&+ \frac{(-1)^{n-1}n}{(n-2)!}\int_0^t (t-\xi)^{n-2}P(\xi)g(\xi)\mathrm{d}\xi \qquad (5.19)\\
&+ \frac{(-1)^n}{(n-2)!}\int_0^t (t-\xi)^{n-2}\Big(\int_0^\xi P\mathrm{d}g\Big)\mathrm{d}\xi;\\
T(t) =& L(t)G(t) + \sum_{r=1}^{n-2} \frac{(-1)^r}{(r-1)!}\binom{n}{r}\int_0^t (t-\xi)^{r-1}L(\xi)G^{(r)}(\xi)\mathrm{d}\xi\\
&+ \frac{(-1)^{n-1}n}{(n-2)!}\int_0^t (t-\xi)^{n-2}L(\xi)g(\xi)\mathrm{d}\xi \qquad (5.20)\\
&+ \frac{(-1)^n}{(n-2)!}\int_0^t (t-\xi)^{n-2}\Big(\int_0^\xi L\mathrm{d}g\Big)\mathrm{d}\xi;\\
U(t) =& \frac{(-1)^n}{(n-2)!}\int_0^t (t-\xi)^{n-2}\Big(\int_a^0 M\mathrm{d}g\Big)\mathrm{d}\xi; \qquad (5.21)\\
V(t) =& \sum_{r=1}^{n-2}\binom{n}{r}\frac{(-1)^r}{(r-1)!}\int_a^0 (t-\xi)^{r-1}M(\xi)G^{(r)}(\xi)\mathrm{d}\xi \qquad (5.22)\\
&+ (-1)^{n-1}\frac{n}{(n-2)!}\int_a^0 (t-\xi)^{n-2}M(\xi)g(\xi)\mathrm{d}\xi\\
&+ \frac{(-1)^n}{(n-2)!}\int_a^0 (t-\xi)^{n-2}\Big(\int_a^\xi M\mathrm{d}g\Big)\mathrm{d}\xi
\end{aligned}$$

Then, summing (5.19)–(5.22), we get from (5.17)

$$S = R + T + U + V. \tag{5.23}$$

Since P is a polynomial of degree at most $(n-2)$, by Lemma 5.4, $R^{(n-1)}$ is constant and so R is a polynomial of degree at most $(n-1)$. Also from (5.21), (5.22)

$$U^{(n-1)} = (-1)^n \int_a^0 M\mathrm{d}g \text{ and } V^{(n-1)} = 0; \tag{5.24}$$

so $R+U+V$ is a polynomial of degree at most $(n-1)$.

Since
$$L(t) = M(t) - P(t) = o(t^{n-2}) \tag{5.25}$$
we have
$$\frac{1}{(r-1)!}\int_0^t (t-\xi)^{r-1} L(\xi)G(\xi)\mathrm{d}\xi = o(t^{n+r-2}) \text{ for } 1 \le r \le n-2; \tag{5.26}$$
$$\frac{1}{(n-2)!}\int_0^t (t-\xi)^{n-2} L(\xi)g(\xi)\mathrm{d}\xi = o(t^{2n-3}); \tag{5.27}$$
$$\frac{1}{(n-2)!}\int_0^t (t-\xi)^{n-2}\Big(\int_0^\xi L\,\mathrm{d}g\Big)\,\mathrm{d}\xi = o(t^{2n-3}). \tag{5.28}$$

From (5.20) and from (5.25)–(5.28) we get
$$T(t) = L(t)G(t) + o(t^{n-1}). \tag{5.29}$$

Since g is bounded
$$G(t) = G(0) + tg(0) + O(t) \text{ for } n = 2, \tag{5.30}$$
and
$$G(t) = G(0) + tG'(0) + o(t) \text{ for } n > 2; \tag{5.31}$$
and so from (5.23), (5,25), (5.29) and from (5.30) or (5.31)
$$S(t) = R(t) + U(t) + V(t) + G(0)L(t) + o(t^{n-1}), \tag{5.32}$$

Since R is a polynomial, from (5.24), (5.25) and (5.32) S is continuous and $S_{(n-2)}$ exists in $[a,b]$ proving *(i)*. Further $R+U+V$ is a polynomial of degree $(n-1)$ so from (5.32)
$$\omega_n(S,0,t) = G(0)\omega_n(L,0,t) + o(t^{-1}) = G(0)\omega_n(M,0,t) + o(t^{-1}),$$
proving *(ii)*.

To prove *(iii)* note that in this case G' exists and
$$G'(t) - G'(0) = O(t). \tag{5.33}$$

Hence
$$G(t) = G(0) + tG'(0) + O(t^2), \tag{5.34}$$

and so from (5.20), (5.26)–(5.28), (5.33) and (5.34)

$$\begin{aligned} T(t) =& L(t)G(t) - n\int_0^t LG' + o(t^n) \\ =& G(0)L(t) + tG'(0)L(t) - nG'(0)\int_0^t L + o(t^n). \end{aligned}$$

Hence

$$\begin{aligned} T(t) + (-1)^n T(-t) =& G(0)\big(L(t) + (-1)^n L(-t)\big) \\ &+ G'(0)\big(L(t) + (-1)^{n-1} L(-t)\big)t \\ &- nG'(0)\Big(\int_0^t \big(L(\xi) + (-1)^{n-1} L(-\xi)\big)\,\mathrm{d}\xi\Big) + o(t^n). \end{aligned} \tag{5.35}$$

Since $R+U+V$ is a polynomial of degree $(n-1)$, from (5.23) and (5.35)

$$\begin{aligned} \omega_n(S,0,t) =& G(0)\omega_n(L,0,t) + nG'(0)\big(\omega_{n-1}(L,0,t) \\ &- \frac{n}{t^n}\int_0^t \xi^{n-1}\omega_{n-1}(L,0,\xi)\mathrm{d}\xi\big) + O(1) \\ =& G(0)\omega_n(M,0,t) + nG'(0)\big(\omega_{n-1}(M,0,t) \\ &- \frac{n}{t^n}\int_0^t \xi^{n-1}\omega_{n-1}(M,0,\xi)\mathrm{d}\xi\big) + O(1), \end{aligned}$$

proving *(iii)*. For *(iii)'* note from (5.18) that

$$g(t) = g(0) + O(t) \tag{5.36}$$

and hence

$$G(t) = G(0) + tg(0) + O(t^2). \tag{5.37}$$

Also, from (5,18) and (5.25)

$$\int_0^t (\int_0^\xi L\mathrm{d}g)\mathrm{d}\xi = o(t^2). \tag{5.38}$$

From (5.36)–(5.38)

$$\begin{aligned} T(t) =& L(t)G(t) - 2\int_o^t Lg + \int_0^t \big(\int_0^\xi L\,\mathrm{d}g\big)\,\mathrm{d}\xi \\ =& G(0)L(t) + tg(0)L(t) - 2\int_0^t g(0)L(\xi)\,\mathrm{d}\xi + o(t^2), \end{aligned}$$

and hence

$$
\begin{aligned}
T(t)+T(-t)=&G(0)\big(L(t)+L(-t)\big)+g(0)\big(L(t)-L(-t)\big)t\\
&-2g(0)\int_0^t\big(L(\xi)-L(-\xi)\big)\,\mathrm{d}\xi+o(t^2).
\end{aligned}
\tag{5.39}
$$

Since $R+U+V$ is linear, we have from (5.23) and (5.39)

$$
\begin{aligned}
\omega_2(S,0,t)=&G(0)\omega_2(L,0,t)+2g(0)\frac{L(t)-L(-t)}{2t}\\
&-\frac{2g(0)}{t^2}\int_0^t\big(L(\xi)-L(-\xi)\big)\,\mathrm{d}\xi+o(1)\\
=&G(0)\omega_2(M,0,t)+2g(0)\Big(\omega_1(M,0,t)-\frac{2}{t^2}\int_0^t\xi\omega_1(M,0,\xi)\,\mathrm{d}\xi\Big)+o(1),
\end{aligned}
$$

proving *(iii)* $'$.

To prove *(iv)* we have by Lemma 5.4

$$
\begin{aligned}
R^{(n-1)}(x)=&\sum_{k=0}^{n-2}(-1)^kP^{(n-k-1)}(x)G^{(k)}(x)\\
&+(-1)^{n-1}P(x)g(x)+(-1)^n\int_0^x P\,\mathrm{d}g
\end{aligned}
$$

and hence

$$
\begin{aligned}
R^{(n-1)}(0)&=\sum_{k=0}^{n-2}(-1)^kP^{(n-k-1)}(0)G^{(k)}(0)+(-1)^{n-1}P(0)g(0)\\
&=\sum_{k=1}^{n-2}(-1)^kP^{(n-k-1)}(0)G^{(k)}(0)+(-1)^{n-1}P(0)g(0),
\end{aligned}
\tag{5.40}
$$

since P is a polynomial of degree $(n-2)$. Since $M_{(n-1)}(0)$ exists, $L_{(n-1)}(0)$ exists and $L_{(n-1)}(0)=M_{(n-1)}(0)$. Hence, from (5.32), (5.24) and (5.40) we have

$$\begin{aligned}S_{(n-1)}(0) &= \sum_{r=1}^{n-2}(-1)^{n-r-1}P^{(r)}(0)G^{(n-r-1)}(0) + (-1)^{n-1}P(0)g(0)\\ &\quad + (-1)^n\int_a^0 M\,\mathrm{d}g + M_{(n-1)}(0)G(0)\\ &= \sum_{r=1}^{n-1}(-1)^{n-r-1}M_{(r)}(0)G^{(n-r-1)}(0)\\ &\quad + (-1)^{n-1}M(0)g(0) + (-1)^n\int_a^0 M\,\mathrm{d}g. \qquad \square\end{aligned}$$

Theorem 5.6 (Integration by Parts). *Let f be T^n integrable on $[a,b]$, $n \geq 2$, and let $F(x) = (T^n)\int_a^x f$, $x \in B$. Let ϕ be the n-th primitive of f and let ϕ satisfy*

$$\omega_{n-1}(\phi,x,t) - \frac{n}{t^n}\int_0^t \xi^{n-1}\omega_{n-1}(\phi,x,\xi)\,\mathrm{d}\xi = O(1) \tag{5.41}$$

nearly everywhere on (a,b). Let g be of bounded variation in $[a,b]$ and let $G(x) = \frac{1}{(n-2)!}\int_a^x (x-t)^{n-2}g(t)\,\mathrm{d}t$, $a \leq x \leq b$. Then:

(i) if $n > 2$, then fG is T^n-integrable on $[a,b]$ and

$$(T^n)\int_a^b fG = \big[\phi_{(n-1)}(x)G(x) - \phi_{(n-2)}(x)G'(x)\big]\Big|_a^b + (Z^*_{(n-3)})\int_a^b \phi_{(n-2)}G^{(2)};$$

the last integral exists by Theorems 5.1 and 5.2. Moreover, if F is Z^*_{n-2}-integrable on $[a,b]$, then

$$(T^n)\int_a^b fG = F(x)G(x)\Big|_a^b - (Z^*_{n-2})\int_a^b FG';$$

(ii) if $n = 2$ and if g satisfies the conditions (5.18) nearly everywhere on (a,b), then fG is T^n-integrable on $[a,b]$ and

$$(T^2)\int_a^b fG = \big[\phi_{(1)}(x)G(x) - \phi(x)G'(x)\big]\Big|_a^b + \int_a^b \phi\,\mathrm{d}g.$$

If moreover F is D^*-integrable on $[a,b]$, then

$$(T^2)\int_a^b fG = F(x)G(x)\Big|_a^b - (D^*)\int_a^b Fg.$$

Proof. *(i)* We first suppose that $g \geq 0$. Then $G \geq 0$ and $G^{(r)} \geq 0$ for $1 \leq r \leq n-2$. Let Q be any major function of f and let $L = Q - \phi$. Then L is n-convex and so $L_{(n-1)}$ exists, finitely, nearly everywhere in (a,b) and so, nearly everywhere in (a,b), $\omega_{n-1}(L,x,t)$ tends to a finite limit as $t \to 0$. Since by the mean value theorem $\frac{n}{t^n}\int_0^t \xi^{n-1}\omega_{n-1}(L,x,\xi)\,\mathrm{d}\xi = \omega_{n-1}(L,x,\theta t)$ for $0 < \theta < 1$, we have nearly everywhere on (a,b) that

$$\omega_{n-1}(L,x,t) - \frac{n}{t^n}\int_0^t \xi^{n-1}\omega_{n-1}(L,x,\xi)\,\mathrm{d}\xi = o(1).$$

So by (5.41)

$$\omega_{n-1}(Q,x,t) - \frac{n}{t^n}\int_0^t \xi^{n-1}\omega_{n-1}(Q,x,\xi)\,\mathrm{d}\xi = O(1), \tag{5.42}$$

nearly everywhere on (a,b). Let, $a \leq x \leq b$,

$$\begin{aligned} S(x) =& Q(x)G(x) + \sum_{r=1}^{n-2}\frac{(-1)^r}{(r-1)!}\binom{n}{r}\int_a^x (x-t)^{r-1}Q(t)G^{(r)}(t)\,\mathrm{d}t \\ &+ \frac{(-1)^{n-1}n}{(n-2)!}\int_a^x (x-t)^{n-2}Q(t)g(t)\,\mathrm{d}t + \frac{(-1)^n}{(n-2)!}\int_a^x (x-t)^{n-2}\Big(\int_a^t Q\,\mathrm{d}g\Big)\,\mathrm{d}t. \end{aligned} \tag{5.43}$$

Then by Lemma 5.5 *(i), (ii)*, in $[a,b]$, S is continuous and $S_{(n-2)}$ exists, and in (a,b) S is smooth of order n, and by Lemmas 5.5 *(iv)* $S_{(n-1)}$ exists at a and b and almost everywhere in (a,b). Also by (5.42) and Lemma 5.5 *(iii)* $\underline{D}^n S > -\infty$ nearly everywhere on (a,b), and since the existence of $Q_{(n-1)}(x)$ implies

$$\omega_{n-1}(Q,x,t) - \frac{n}{t^n}\int_0^t \xi^{n-1}\omega_{n-1}(Q,x,\xi)\,\mathrm{d}\xi = o(1),$$

$\underline{D}^n S \geq fG$ almost everywhere in (a,b). let

$$U(x) = S(x) - \sum_{r=0}^{n-1}\frac{(x-a)^r}{r!}S_{(r)}(a).$$

Then U has the above properties of S and moreover $U_{(r))}(a) = 0$ for $r = 0, 1, \ldots, (n-1)$. So U is a major function of fG.

Similarly if q is a minor function of f, then u is a minor function of fG where

$$u(x) = s(x) - \sum_{r=0}^{n-1}\frac{(x-a)^r}{r!}s_{(r)}(a).$$

and

$$\begin{aligned} s(x) =& q(x)G(x) + \sum_{r=1}^{n-2} \frac{(-1)^r}{(r-1)!}\binom{n}{r}\int_a^x (x-t)^{r-1} q(t) G^{(r)}(t)\,\mathrm{d}t \\ &+ \frac{(-1)^{n-1}n}{(n-2)!}\int_a^x (x-t)^{n-2} q(t) g(t)\,\mathrm{d}t + \frac{(-1)^n}{(n-2)!}\int_a^x (x-t)^{n-2} (\int_a^t q\,\mathrm{d}g)\,\mathrm{d}t. \end{aligned} \tag{5.44}$$

Applying Lemma 5.5 *(iv)* in (5.43) and (5.44) we get

$$\begin{aligned} &\big|S_{(n-1)}(x_0) - s_{(n-1)}(x_0)\big| \le \\ &\quad \sum_{r=1}^{n-1} \big|Q_{(r)}(x_0) - q_{(r)}(x_0)\big| G^{(n-r-1)}(x_0) \\ &\quad + |Q(x_0) - q(x_0)| g(x_0) + \big|\int_a^{x_0} (Q-q)\,\mathrm{d}g\big|, \end{aligned} \tag{5.45}$$

for those x_0 for which $Q_{(n-1)}(x_0)$ and $q_{(n-1)}(x_0)$ exist.

Let $\epsilon > 0$ be arbitrary. Since f is integrable, there is a major function Q and a minor function q which satisfy the conditions (4.1,1)–(4.1,n). For these Q and q we have from (5.45)

$$\big|S_{(n-1)}(b) - s_{(n-1)}(b)\big| \le K\epsilon, \tag{5.46}$$

where K is a constant. Since $S_{(n-1)}(a) = s_{(n-1)}(a) = 0$, by Lemma 5.5 *(iv)*, we have from (5.46) and from the definitions of U and u that $\big|U_{(n-1)}(b) - u_{(n-1)}(b)\big| \le K\epsilon$, showing that fG is integrable on $[a, b]$.

Now using integration by parts for the D^*-integral [17, p. 246], and then successively for the Z_r^*- integrals by Theorem 5.2

$$\begin{aligned} \int_a^b Q\,\mathrm{d}g =& Qg\Big|_a^b - \int_a^b Q_{(1)} g = Qg\Big|_a^b - Q_{(1)}G^{(n-2)}\Big|_a^b + \int_a^b Q_{(2)} G^{(n-2)} = \cdots \\ =& Q(b)g(b) + \sum_{r=1}^{n-3} (-1)^r Q_{(r)}(b) G^{(n-r-1)}(b) \\ &+ (-1)^{n-2} (Z_{n-3}^*) \int_a^b Q_{(n-2)} G^{(2)}. \end{aligned} \tag{5.47}$$

Applying Lemma 5.5 *(iv)* to the function S in (5.43) we have

$$\begin{aligned} S_{(n-1)}(b) =& \sum_{r=1}^{n-1} (-1)^{n-r-1} Q_{(r)}(b) G^{(n-r-1)}(b) \\ &+ (-1)^{n-1} Q(b) g(b) + (-1)^n \int_a^b Q\,\mathrm{d}g. \end{aligned} \tag{5.48}$$

From (5.47) and (5.48)

$$\begin{aligned} U_{(n-1)}(b) =& S_{(n-1)}(b) = Q_{(n-1)}(b)G(b) - Q_{(n-2)}(b)G'(b) \\ &+ \int_a^b Q_{(n-2)}G^{(2)}. \end{aligned} \tag{5.49}$$

Let $\epsilon > 0$ be arbitrary. Then as in (4.6) and (4.9) there is a major function Q of f such that

$$\begin{aligned} 0 \leq& Q_{(n-1)}(b) - \phi_{(n-1)}(b) < \epsilon, \\ 0 \leq& Q_{(n-2)}(x) - \phi_{(n-2)}(x) < \epsilon(b-a),\ a \leq x \leq b. \end{aligned}$$

Hence

$$\begin{aligned} &Q_{(n-1)}(b)G(b) - Q_{(n-2)}(b)G'(b) + \int_a^b Q_{(n-2)}G^{(2)} \\ &\qquad < \phi_{(n-1)}(b)G(b) - \phi_{(n-2)}(b)G'(b) + \int_a^b \phi_{(n-2)}G^{(2)} + \epsilon K, \end{aligned} \tag{5.50}$$

where K is a constant.

Since (5.49) holds for any major function Q of f, we have from (5.49) and (5.50)

$$U_{(n-1)}(b) < \phi_{(n-1)}(b)G(b) - \phi_{(n-2)}(b)G'(b) + \int_a^b \phi_{(n-2)}G^{(2)} + \epsilon K.$$

Since ϵ is arbitrary,

$$\int_a^b fG \leq \phi_{(n-1)}(b)G(b) - \phi_{(n-2)}(b)G'(b) + \int_a^b \phi_{(n-2)}G^{(2)}. \tag{5.51}$$

In a similar manner, considering a minor function q of f we get

$$\int_a^b fG \geq \phi_{(n-1)}(b)G(b) - \phi_{(n-2)}(b)G'(b) + \int_a^b \phi_{(n-2)}G^{(2)}. \tag{5.52}$$

From (5.51) and (5.52)

$$\int_a^b fG = \phi_{(n-1)}(b)G(b) - \phi_{(n-2)}(b)G'(b) + \int_a^b \phi_{(n-2)}G^{(2)}, \tag{5.53}$$

which completes the proof of the first part of *(i)*.

Now if F is Z^*_{n-2}-integrable on $[a,b]$, then since $\phi_{(n-2)}$ is an indefinite Z^*_{n-2}-integral of F, we get using integration by parts for the Z^*_{n-2}-integral, by Theorem 5.2,

$$(Z^*_{n-2})\int_a^b FG' = \phi_{(n-2)}(x)G'(x)\big|_a^b - (Z^*_{n-3})\int_a^b \phi_{(n-2)}G^{(2)}. \tag{5.54}$$

From (5.53) and (5,54) we get

$$\int_a^b fG = \phi_{(n-1)}(b)G(b) - (Z^*_{n-2})\int_a^b FG',$$

which completes the proof of *(i)*.

The proof of *(ii)* is similar but uses Lemma 5.5 *(iii)* $'$ instead of Lemma 5.5 *(iii)*. □

6 Applications to Trigonometric Series.

We write for convenience:

$$\begin{aligned}
&A_0(x) = \frac{1}{2}a_0,\ A_n(x) = a_n\cos nx + b_n \sin nx,\ n\geq 1,\\
&B_0(x) = 0,\ B_n(x) = b_n\cos nx - a_n\sin nx,\ n\geq 1,\\
&C_0(x) = 0,\ C_n(x) = -\frac{B_n(x)}{n},\ n\geq 1,\\
&A_n^0(x) = \sum_{r=0}^{n} A_r(x),\ A_n^k(x) = \sum_{r=0}^{n} A_r^{k-1}(x),\ k\geq 1;
\end{aligned}$$

with similar meaning for $B_n^k(x)$ and $C_n^k(x)$. The upper and lower (C,k) sums of the series

$$\sum_{n=0}^{\infty} A_n(x) \tag{6.1}$$

will be denoted by $S^k(x)$ and$s^k(x)$ respectively.

The following theorem includes an extension of a result of Zygmund, [21, II, p. 66, Theorem 2.1], and a result of Wolf, [20, Theorem B].

Theorem 6.1. *Let k be a fixed positive integer and let the series (6.1) integrated term-by-term r times, $r > k+1$, converge to a function F in some neighborhood of x_0. Then:*

(i) if $A_n^{k-1}(x_0) = o(n^k)$, then $D^{r-2}F(x_0)$ exists and $F \in \mathcal{S}_r(x_0)$;

(ii) *if* $A_n^k(x_0) = O(n^k)$, *then* $\overline{D}^r F(x_0)$ *and* $\underline{D}^r F(x_0)$ *are finite;*

(iii) *if* $A_n^{k-1}(x_0) = o(n^k)$ *and if* $C_n^{k-2}(x_0) = o(n^{k-1})$, *then* $F_{(r-2)}(x_0)$ *exists finitely, where for* $k = 1$ *we take* $C_n^{k-2}(x_0)$ *to be* $C_n^0(x_0)$.

(iv) *if* $A_n^{k-1}(x_0) = o(n^k)$, $C_n^{k-2}(x_0) = o(n^{k-1})$ *and* $C_n^{k-1}(x_0) = O(n^{k-1})$, *then* $\overline{F}_{(r-1)}(x_0)$ *and* $\underline{F}_{(r-1)}(x_0)$ *are finite.*

Proof. We may assume that $r = k+2$, $x_0 = 0$, $a_0 = 0$ and at first we suppose that r is even. Let

$$\gamma(t) = \frac{\cos t}{t^r},\ t \neq 0; \quad P(t) = \sum_{\nu=0}^{k/2} (-1)^\nu \frac{t^{2\nu}}{(2\nu)!};$$

$$\lambda(t) = \gamma(t) - \frac{P(t)}{t^r} = \sum_{\nu=k/2+1}^{\infty} (-1)^\nu \frac{t^{2\nu-r}}{(2\nu)!}.$$

Then

$$\begin{aligned}\frac{F(t)+F(-t)}{2} &= (-1)^{r/2}\sum_{n=1}^{\infty}\frac{A_n(t)+A_n(-t)}{2n^r} = (-1)^{r/2}\sum_{n=1}^{\infty}\frac{a_n \cos nt}{n^r} \\ &= (-1)^{r/2} t^r \sum_{n=1}^{\infty} a_n \gamma(nt) \\ &= (-1)^{r/2} t^r \sum_{n=1}^{\infty} \frac{a_n P(nt)}{(nt)^r} + (-1)^{r/2} t^r \sum_{n=1}^{\infty} a_n \lambda(nt) \\ &= \sum_{\nu=0}^{k/2} \frac{t^{2\nu}}{(2\nu)!}\beta_{2\nu} + \frac{t^r}{r!}\omega_r(t),\end{aligned} \tag{6.2}$$

where

$$\beta_{2\nu} = (-1)^{r/2+\nu} \sum_{n=1}^{\infty} a_n n^{2\nu - r}; \tag{6.3}$$

$$\omega_r(t) = (-1)^{r/2} r! \sum_{n=1}^{\infty} a_n \lambda(nt). \tag{6.4}$$

Consider the difference operator $\Delta^j u_n$, for any sequence $\{u_n\}$ defined by $\Delta^1 u_n = u_n - u_{n+1}$ and $\Delta^j u_n = \Delta^1(\Delta^{j-1} u_n)$ for $j > 1$. It can be proved by induction that

$$\Delta^j u_n = \sum_{i=0}^{j} (-1)^i \binom{j}{i} u_{n+i}. \tag{6.5}$$

Now, summing by parts k times and writing $\sigma_n^{k-1} = A_n^{k-1}(0)$, we get from (6.3) and (6.4)

$$\beta_{2\nu} = (-1)^{r/2+\nu} \sum_{n=1}^{\infty} \sigma_n^{k-1} \Delta^k n^{2\nu-r}; \tag{6.6}$$

$$\omega_r(t) = (-1)^{r/2} r! \sum_{n=1}^{\infty} \sigma_n^{k-1} \Delta^k \lambda(nt). \tag{6.7}$$

Since λ is infinitely differentiable, for each i, $0 \le i \le k$, there is a θ_i, $0 < \theta_i < 1$, such that

$$\lambda(nt + it) = \sum_{j=0}^{k-1} \frac{(it)^j}{j!} \lambda^{(j)}(nt) + \frac{(it)^k}{k!} \lambda^{(k)}(nt + \theta_i it). \tag{6.8}$$

Using (6.5) and (6.8)

$$\begin{aligned}
\Delta^k \lambda(nt) &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \Big(\sum_{j=0}^{k-1} \frac{(it)^j}{j!} \lambda^{(j)}(nt) \Big) \\
&\quad + \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{(it)^k}{k!} \lambda^{(k)}(nt + \theta_i it) \\
&= \sum_{j=0}^{k-1} \frac{t^j}{j!} \lambda^{(j)}(nt) \Big(\sum_{i=0}^{k} (-1)^i \binom{k}{i} i^j \Big) \\
&\quad + \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{(it)^k}{k!} \lambda^{(k)}(nt + \theta_i it) \\
&= \frac{t^k}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} i^k \lambda^{(k)}(nt + \theta_i it).
\end{aligned} \tag{6.9}$$

Since $\lambda^{(k)}$ remains bounded, we have

$$\Delta^k \lambda(nt) = O(t^k). \tag{6.10}$$

Since $\Delta^k n^{2\nu-r} = O(n^{2\nu-r-k})$ and $\sigma_n^{k-1} = o(n^k)$, and since $2\nu - r \le -2$, the series in (6.6) is absolutely convergent and so $\beta_{2\nu}$ is finite for $\nu = 0, 1, \dots k/2$

Let $0 < t < 1$ and choose a positive integer N such that $N \le \dfrac{1}{t} < N + 1$.

Then from (6.7)

$$\begin{aligned}\left|t\omega_r(t)\right| &= r!t\Big|\sum_{n=1}^{\infty}\sigma_n^{k-1}\Delta^k\lambda(nt)\Big| \\ &\le r!t\Big(\sum_{n=1}^{N}\left|\sigma_n^{k-1}\Delta^k\lambda(nt)\right| + \sum_{n=N+1}^{\infty}\left|\sigma_n^{k-1}\Delta^k\lambda(nt)\right|\Big) \\ &= r!t\big(U(t)+V(t)\big),\ \text{say.}\end{aligned}\tag{6.11}$$

From (6.10) there is a constant C such that

$$\begin{aligned}r!tU(t) \le& Ct^{k+1}\sum_{n=1}^{N}\left|\sigma_n^{k-1}\right| = Ct^{k+1}\sum_{n=1}^{N}o(n^k) \\ =& Ct^{k+1}o(N^{k+1}) = o(1).\end{aligned}\tag{6.12}$$

Again if $x \ge 1$, then there are constants C_1 and C_2 such that $\left|\gamma^{(k)}(x)\right| \le C_1x^{-r}$ and $\left|\big(P(x)x^{-r}\big)^{(k)}\right| \le C_2x^{-r}$ and so $\left|\lambda^{(k)}(x)\right| \le (C_1+C_2)x^{-r}$. Therefore if $n \ge N+1$, then $nt \ge (N+1)t > 1$ and so

$$\left|\lambda^{(k)}(nt+\theta_i it)\right| \le \frac{C_1+C_2}{(nt+\theta_i it)^r} \le \frac{C_1+C_2}{(nt)^r},\ 0 \le i \le k. \tag{6.13}$$

From (6.9) and (6.13) there is a constant C_3 such that $\left|\Delta^k\lambda(nt)\right| \le C_3t^k/(nt)^r$, and therefore

$$\begin{aligned}r!tV(t) &= r!t\Big(\sum_{n=N+1}^{\infty}\left|\sigma_n^{k-1}\Delta^k\lambda(nt)\right|\Big) \le C_3r!t^{k+1}\sum_{n=N+1}^{\infty}\frac{\left|\sigma_n^{k-1}\right|}{(nt)^r} \\ &= C_3\frac{r!}{t}\sum_{n=N+1}^{\infty}\frac{o(n^k)}{n^r} = C_3\frac{r!}{t}\sum_{n=N+1}^{\infty}o(n^{-2}) = o(1).\end{aligned}\tag{6.14}$$

From (6.11), (6.12) and (6.14) we get

$$t\omega_r(t) = o(1). \tag{6.15}$$

Hence $(t^r/r!)\omega_r(t) = o(t^{k+1})$, that is, $(t^r/r!)\omega_r(t) = o(t^k)$. So from (6.2), (2.1) and (2.3) it follows that $\beta_{2\nu}$ is the symmetric d.l.V.P. derivative of F at 0 of order 2ν, $0 \le \nu \le k/2$, and $\omega_r(t)$ is $\omega_r(F,0,t)$ defined in (2.3). It also follows from (6.15) that F is smooth at 0 of order r. This completes the proof of *(i)*.

To prove *(ii)*, summing by parts $(k+1)$ times and writing $\sigma_n^k = A_n^k(0)$ we get from (6.4) that

$$\omega_r(t) = (-1)^{r/2} r! \sum_{n=1}^{\infty} \sigma_n^k \Delta^{k+1}\lambda(nt). \tag{6.16}$$

Taking $0 < t < 1$ and $N \leq t^{-1} < N+1$ we get,as above, from (6.16)

$$\begin{aligned} \left|\omega_r(t)\right| &\leq r! \sum_{n=1}^{N} \left|\sigma_n^k \Delta^{k+1}\lambda(nt)\right| + r! \sum_{n=N+1}^{\infty} \left|\sigma_n^k \Delta^{k+1}\lambda(nt)\right| \\ &= G(t) + H(t), \text{ say.} \end{aligned} \tag{6.17}$$

As in (6.10) we have $\Delta^{k+1}(nt) = O(t^{k+1})$ and form this we have, as in (6.12)

$$G(t) \leq C_1 t^{k+1} \sum_{n=1}^{N} |\sigma_n^k| = O(1), \tag{6.18}$$

where C_1 is a constant. Also if $n \geq N+1$, then we have as in (6.14) that

$$H(t) = O(1). \tag{6.19}$$

From (6.17)–(6.19) $\omega_r(t) = O(1)$, completing the proof of *(ii)*.

Now since $a_0 = 0$, F is obtained from $\sum C_n(x)$ by integrating this series term-by-term $(r-1)$ times. So replacing k, r and $A_n^{k-1}(x_0)$ by $k-1, r-1$ and $C_n^{k-2}(x_0)$ respectively, we get from *(i)* that F is smooth at x_0 of order $r-1$. Also by *(i)* F is smooth at x_0 of order r. So, by [13, Lemma 2.1], $F_{(r-2)}(x_0)$ exists. This proves *(iii)*.

Now note that since F is obtained from $\sum C_n(x)$ we apply *(ii)* to get that $\overline{D}^{r-1}F(x_0)$ and $\underline{D}^{r-1}F(x_0)$ are finite. Also by *(i)* F is smooth at x_0 of order r and by *(iii)* $F_{(r-2)}(x_0)$ exists. So $D^iF(x_0)$ exists and $D^iF(x_0) = F_{(i)}(x_0)$ for $i = 1, 2, \ldots r-2$. Hence by a simple calculation applying (2.2), (2.3), (2.6) and (2.7) we have

$$\omega_{r-1}(F, x_0, t) + \frac{t}{r}\omega_r(F, x_0, t) = \gamma_{r-1}(F, x_0, t);$$

and so F being smooth at x_0 of order r, we have that $\underline{F}_{(r-1)}(x_0) = \underline{D}^{r-1}(x_0)$ and $\overline{F}_{(r-1)}(x_0) = \overline{D}^{r-1}(x_0)$, completing the proof of *(iv)* and the consideration of the case r even.

If r is odd let

$$\gamma(t) = \frac{\sin t}{t^{r+1}},\ t \neq 0; \quad P(t) = \sum_{\nu=0}^{(k-1)/2} (-1)^\nu \frac{t^{2\nu}+1}{(2\nu+1)!};$$
$$\lambda(t) = \gamma(t) - \frac{P(t)}{t^{r+1}} = \sum_{\nu=(k+1)/2}^{\infty} (-1)^\nu \frac{t^{2\nu-r}}{(2\nu+1)!}.$$

and compute

$$\frac{F(t)-F(-t)}{2} = \sum_{\nu=0}^{(k-1)/2} \frac{t^{2\nu+1}}{(2\nu+1)!}\beta_{2\nu+1} + \frac{t^r}{r!}\omega_r(t);$$

where

$$\beta_{2\nu+1} = (-1)^{(r-1)/2+\nu} \sum_{n=1}^{\infty} a_n n^{2\nu+1-r};$$
$$\omega_r(t) = (-1)^{(r-1)/2} r! \sum_{n=1}^{\infty} a_n nt\lambda(nt).$$

The rest of the proof of this case is similar to the above and is omitted. □

Theorem 6.2. *Let the series (6.1) be such that*

(i) $-\infty < s^k(x) \leq S^k(x) < \infty$ *nearly everywhere;*

(ii) $A_n^{k-1}(x) = o(n^k)$ *for all* x*;*

(iii) $C_n^{k-2}(x) = o(n^{k-1})$ *for all* x*;*

(iv) $C_n^{k-1}(x) = O(n^{k-1})$ *nearly everywhere.*

Then the series obtained by integrating (6.1) term-by-term $(k+2)$ *times converges to a continuous function* G *such that* $G_{(k+2)}$ *exists almost everywhere and is* T^{k+2}*-integrable and (6.1) is the* T^{k+2}*-Fourier series of* $G_{(k+2)}$*. Moreover, for each* j*,* $1 \leq j \leq k+1$*, the* $(k+2-j)$ *times integrated series of (6.1) is the* Z^*_{j-1}*-Fourier series of* $G_{(j)}(x) - \big(a_0 x^{k+2-j}/2(k+2-j)!\big)$*.*

Proof. Condition *(i)* implies that $a_n = O(n^k)$, $b_n = O(n^k)$; [21, Volume I, p. 317, Theorem 1.4] and therefore the series obtained by integrating (6.1) term-by-term $(k+2)$ times converges uniformly to a continuous function, G say. Let

$$H(x) = G(x) - \frac{a_0}{2}\frac{x^{k+2}}{(k+2)!}. \tag{6.20}$$

By Theorem 6.1 $G_{(k)}$ exists, G is smooth of order $(k+2)$ everywhere and $\underline{G}_{(k+1)}, \overline{G}_{(k+1)}, \underline{D}^{k+2}G, \overline{D}^{k+2}G$ are all finite nearly everywhere. So from (6.20) $H_{(k)}$ exists, H is smooth of order $(k+2)$ everywhere and nearly everywhere

$$-\infty < \underline{H}_{(k+1)} \le \overline{H}_{(k+1)} < \infty, \tag{6.21}$$

$$-\infty < \underline{D}^{k+2}H \le \overline{D}^{k+2}H < \infty. \tag{6.22}$$

From (6.22) and Lemma 3.4 $H_{(k+2)}$ exists finitely almost everywhere. Let B be the set where $H_{(k+1)}$ exists finitely. By Theorem 4.13 $H_{(k+2)}$ is T^{k+2}-integrable and

$$(T^{k+2})\int_{x_1}^{x_2} H_{(k+2)} = H_{(k+1)}(x_2) - H_{(k+1)}(x_1), \text{ for } x_1, x_2 \in B. \tag{6.23}$$

Let $\alpha \in B$. Then $\alpha + 2\pi \in B$. So from (6.23) and (6.20) we get that $(T^{k+2})\int_\alpha^{\alpha+2\pi}(G_{(k+2)} - \frac{a_0}{2}) = 0$ giving

$$a_0 = \frac{1}{\pi}(T^{k+2})\int_\alpha^{\alpha+2\pi} G_{(k+2)}. \tag{6.24}$$

Let k be even. Then from (6.20)

$$H(x) = (-1)^{k/2+1}\sum_{n=1}^{\infty}\frac{A_n(x)}{n^{k+2}}. \tag{6.25}$$

Since the series in (6.25) converges uniformly, it is the Lebesgue-Fourier series of H. Hence

$$(-1)^{k/2+1}\frac{a_n}{n^{k+2}} = \frac{1}{\pi}(L)\int_\alpha^{\alpha+2\pi} H(x)\cos nx \,\mathrm{d}x. \tag{6.26}$$

Since $H_{(k)}$ exists in $[\alpha, \alpha+2\pi]$ and $H_{(k+1)}$ exists almost everywhere in the same interval and since (6.21) is satisfied nearly everywhere, we have by Theorem 5.1 that the function $H_{(r)}$ is Z^*_{r-1}-integrable with $H_{(r-1)}$ its indefinite Z^*_{r-1}-integral, $r = 1, 2, \dots, k+1$. Hence applying Theorem 5.2 successively we get

$$(Z^*_k)\int_\alpha^{\alpha+2\pi} H_{(k+1)}(x)\sin nx \,\mathrm{d}x = (-1)^{k/2+1}n^{k+1}(L)\int_\alpha^{\alpha+2\pi} H(x)\cos nx \,\mathrm{d}x. \tag{6.27}$$

Writing

$$F(x) = H_{(k+1)}(x) - H_{(k+1)}(\alpha),\ x \in B, \tag{6.28}$$

we have from (6.23) that $F(x) = (T^{k+2}) \int_\alpha^x H_{(k+2)},\ x \in B$. Also from (6.21) and Lemma 3.3 $\omega_{k+1}(H, x, t) = O(1)$ nearly everywhere, and so nearly everywhere we have

$$\omega_{k+1}(H, x, t) - \frac{k+2}{t^{k+2}} \int_0^t \xi^{k+1} \omega_{k+1}(H, x, \xi)\, \mathrm{d}\xi = O(1).$$

Integrating by parts by Theorem 5.6

$$\begin{aligned}(T^{k+2}) &\int_\alpha^{\alpha+2\pi} H_{(k+2)}(x) \cos nx\, \mathrm{d}x \\ &= F(x) \cos nx \Big|_\alpha^{\alpha+2\pi} + n(Z_k^*) \int_\alpha^{\alpha+2\pi} F(x) \sin nx\, \mathrm{d}x.\end{aligned} \tag{6.29}$$

Now $F(\alpha) = 0$ and since by (6.25) $H_{(k+1)}$ is periodic, by (6.28), $F(\alpha+2\pi) = 0$. So from (6.20), (6.29), (6.27) and (6.26)

$$\begin{aligned}(T^{k+2}) &\int_\alpha^{\alpha+2\pi} G_{(k+2)}(x) \cos nx\, \mathrm{d}x \\ &= (T^{k+2}) \int_\alpha^{\alpha+2\pi} H_{(k+2)}(x) \cos nx\, \mathrm{d}x \\ &= n(Z_k^*) \int_\alpha^{\alpha+2\pi} H_{(k+1)}(x) \sin nx\, \mathrm{d}x \\ &= (-1)^{k/2+1} n^{k+2} (L) \int_\alpha^{\alpha+2\pi} H(x) \cos nx\, \mathrm{d}x = \pi a_n.\end{aligned} \tag{6.30}$$

Similarly

$$(T^{k+2}) \int_\alpha^{\alpha+2\pi} G_{(k+2)}(x) \sin nx\, \mathrm{d}x = \pi b_n. \tag{6.31}$$

The first part of the theorem now follows from (6.24), (6.30) and (6.31).

To prove the second part note that since $G_{(j)}$ is Z_{j-1}^*-integrable for $j = 1, \ldots, k+1$ we have, when j is even, by Theorem 5.2, (6.25) and (6.26)

$$\begin{aligned}\frac{1}{\pi}(Z_{j-1}^*) \int_\alpha^{\alpha+2\pi} H_{(j)}(x) \cos nx\, \mathrm{d}x =& (-1)^{j/2} n^j \frac{1}{\pi} (D^*) \int_\alpha^{\alpha+2\pi} H(x) \cos nx\, \mathrm{d}x \\ =& (-1)^{(k+2-j)/2} \frac{a_n}{n^{k+2-j}}.\end{aligned}$$

When j is odd an analogous relation holds.

This completes the proof of the theorem in the case of even k; the proof when k is odd is similar. □

Remark. If in Theorem 6.2 the (C, k) summability almost everywhere of the series (6.1) is assumed, in addition to *(i)*, and if the conditions *(iii)* and *(iv)* are replaced by the single condition $B_n^{k-1}(x) = o(n^k)$ for all x, then it can be proved that (6.1) is the Fourier series of f where f is its (C, k) sum. However now we have to apply formal multiplication of trigonometric series since the integration by parts formula cannot be used.

7 Concluding Remarks.

In [3] we gave a proof for the integration by parts formula for the SCP-integral in the following form; [3, Theorem 1].

Theorem 7.1. *Assume that f is (SCP, B)-integrable on $[a, b]$ and let $F(x) = (SCP, B)\int_a^x f,\ x \in B$. Let g be a continuous function of bounded variation on $[a, b]$ with $G(x) = \int_a^x g,\ a \le x \le b$. If*

$$\omega_1(\phi, x, h) = O(h) \text{ nearly everywhere,} \tag{7.1}$$

where $\phi(x) = (D^*)\int_a^x F,\ a \le x \le b$. Then fG is (SCP, B)-integrable on $[a, b]$ and $(SCP, B)\int_a^b fG = FG\big|_a^b - (D^*)\int_a^b Fg$.

We also remarked at the end of [3] that this integration by parts formula could not be applied to solve the so-called 'coefficient problem' for convergent trigonometric series because the condition (7.1) need not be satisfied by every ϕ that is the sum of the twice integrated series of a convergent trigonometric series. However Skljarenko, [18], proved the above theorem without assuming the condition (7.1) and we have shown, [14], that not only is the condition (7.1) redundant but also the requirement made above that the function g be continuous. Since the proofs in [18] and [14] are long and involved whereas the proof of Theorem 5.6 is considerably simpler when $n = 2$ we will now discuss how Theorem 5.6, in the case $n = 2$, which is the same as Theorem 7.1 except that (7.1) is replaced by

$$\omega_1(\phi, x, h) - \frac{2}{h^2}\int_0^h \xi\omega_1(\phi, x, \xi)\,\mathrm{d}\xi = O(1); \tag{7.2}$$

$$\mathop{V}_{x}^{x+h}(g) = O(h);\ \mathop{V}_{x-h}^{x}(g) = O(h), \tag{7.3}$$

and omitting the condition of continuity of g, helps to solve the coefficient problem. All we need to show is that for every convergent trigonometric series the function ϕ satisfies the condition (7.2), since (7.3) is satisfied trivially as $g(x)$ is either $\cos nx$ or $\sin nx$. This we do in the following theorem.

Theorem 7.2. *Let (6.1) be such that $\sum_{k=1}^{n} k\rho_k = O(n^2)$, where $\rho_n = \sqrt{a_n^2 + b_n^2}$; and let ϕ and ψ be the sum of the twice and thrice integrated series of (6.1). Then for all x*

$$\frac{\phi(x+h)-\phi(x-h)}{2h} - \frac{\psi(x+h)-2\psi(x)+\psi(x-h)}{h^2} = O(1). \tag{7.4}$$

PROOF. Let $0 < h < 1$ and N be a positive integer such that $N \le \frac{1}{h} < N+1$ and suppose that $a_0 = 0$. Since ϕ and ψ are the sums of once and twice integrated series of $-\sum_{n=1}^{\infty} \frac{B_n(x)}{n}$, we have

$$\frac{\phi(x+h)-\phi(x-h)}{2h} = -\sum_{n=1}^{\infty} \frac{B_n(x)}{n} \frac{\sin nh}{nh};$$

$$\frac{\psi(x+h)-2\psi(x)+\psi(x-h)}{h^2} = -\sum_{n=1}^{\infty} \frac{B_n(x)}{n} \Big(\frac{\sin nh}{nh}\Big)^2.$$

So the left-hand side of (7.4) is

$$-\sum_{n=1}^{N} \frac{B_n(x)}{n} \left(\frac{\sin nh}{nh} - \Big(\frac{\sin nh}{nh}\Big)^2\right) - \sum_{n=N+1}^{\infty} \frac{B_n(x)}{n} \frac{\sin nh}{nh}$$
$$+ \sum_{n=N+1}^{\infty} \frac{B_n(x)}{n} \Big(\frac{\sin nh}{nh}\Big)^2 = P + Q + R, \text{ say;}$$

see [21, Volume I, pp. 319–322]. Since $\frac{\sin u}{u} - \Big(\frac{\sin u}{u}\Big)^2 = O(u^2)$ writing

$\tau_n = \sum_{k=1}^{n} k\rho_k$, we have:

$$
\begin{aligned}
|P| &\le \sum_{n=1}^{N} \frac{\rho_n}{n} O(n^2h^2) = O\big((h^2 \sum_{n=1}^{N} n\rho_n\big) = O(1);\\
|Q| &\le \frac{1}{h} \sum_{n=N+1}^{\infty} \frac{\rho_n}{n^2} = \frac{1}{h} \sum_{n=N+1}^{\infty} \frac{n\rho_n}{n^3} = \frac{1}{h} \sum_{n=N+1}^{\infty} \frac{\tau_n - \tau_{n-1}}{n^3}\\
&\le \frac{1}{h} \sum_{n=N+1}^{\infty} \tau_n \Big(\frac{1}{n^3} - \frac{1}{(n+1)^3}\Big) = \frac{1}{h} \sum_{n=N+1}^{\infty} O(n^2)O(n^{-4})\\
&= O(N)O(N^{-1}) = O(1);\\
|R| &\le \frac{1}{h^2} \sum_{n=N+1}^{\infty} \frac{\rho_n}{n^3} = \frac{1}{h^2} \sum_{n=N+1}^{\infty} \frac{n\rho_n}{n^4} = \frac{1}{h^2} \sum_{n=N+1}^{\infty} \frac{\tau_n - \tau_{n-1}}{n^4}\\
&\le \frac{1}{h^2} \sum_{n=N+1}^{\infty} \tau_n \Big(\frac{1}{n^4} - \frac{1}{(n+1)^4}\Big) = O(N^2) \sum_{n=N+1}^{\infty} O(n^2)O(n^{-5})\\
&= O(N^2) \sum_{n=N+1}^{\infty} O(n^{-3}) = O(N^2)O(N^{-2}) = O(1). \qquad \square
\end{aligned}
$$

Now if (6.1) is convergent, then $a_n = o(1)$, $b_n = o(1)$ and so $\rho_n = o(1)$. Hence $\sum_{k=1}^{n} k\rho_k = \sum_{k=1}^{n} o(k) = o(n^2)$ and so by Theorem 7.2 the condition (7.4) is satisfied and hence the condition (7.2) is also satisfied. Thus the problem raised in our remark in [3] is solved.

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