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ON SOME PROPERTIES OF ESSENTIAL DARBOUX RINGS OF REAL FUNCTIONS DEFINED ON TOPOLOGICAL SPACES

Abstract

This paper deals with rings of real Darboux functions defined on some topological spaces. Results are given concerning the existence of essential, as well as prime Darboux rings. We prove that, under some assumptions connected with the domain X of the functions, the equalities: $D(X) = S_{lf}(X), S(X) = \dim(\Re)$ hold, where $D(X)$ is a $\mathcal{D}$-number of X, $S(X)$ ($S_{lf}(X)$) denotes the Souslin (lf-Souslin) number of X and $\dim(\Re)$ is a Goldie dimension of an arbitrary prime Darboux ring $\Re$.

1 Introduction.

Although the properties of operations performed on Darboux functions have been studied for a long time (e.g., in [1], [3] [4], [5], [7], [11]) many articles concerning these problems are still appearing, among others in reference to algebraic structures of some subsets of the family of Darboux functions. By analogy to continuous functions, one can investigate some properties of rings of Darboux functions ([12], [10], [6]).

In this paper we shall consider some rings of real Darboux functions defined on a topological space. This topic seems to be especially interesting with respect to the problems connected with rings of Darboux functions different from the ring of all continuous functions.

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Let $f : X \to Y$, where X and Y are topological spaces. We say that f is *a Darboux function* if the image $f(C)$ is a connected set, for each connected set $C \subset X$.

We say that a ring $\Re$ of real Darboux functions defined on a topological space X is *an essential Darboux ring* if $\Re \setminus \mathcal{C}(X, \mathbb{R}) \neq \emptyset$, where $\mathcal{C}(X, \mathbb{R})$ denotes the family of all continuous functions mapping X into $\mathbb{R}$.

We say that an essential Darboux ring $\Re$ of real Darboux functions defined on a topological space X is *a prime Darboux ring* if $D(f) \subset Z(f)$, for any $f \in \Re$, where $Z(f) = f^{-1}(0)$ and $D(f)$ is the set of all discontinuity points of f.

The purpose of this paper is to present theorems connected with the existence of essential and prime rings of functions. We shall show (under some additional assumptions) that the Goldie dimension of any prime Darboux ring is equal to the Souslin number of the domain of the functions considered.

2 Preliminaries.

We use mostly standard notations. In particular, by the symbols $\mathbb{R}$ ($\mathbb{Q}$) we denote the set of *all real* (*rational*) *numbers*.

The closure and *interior* of a set A we denote by $\overline{A}$ and $\text{Int}(A)$, respectively. Let us denote $\text{Fr}(A) = \overline{A} \setminus \text{Int}(A)$. The cardinality of a set A we denote by the symbol $\text{card}(A)$.

A topological space X is called *a perfectly normal* space (or T_6-space) if X is a normal space and every closed subset of X is a G_δ-set.

The smallest cardinal number $\mathbf{m} \geq \aleph_0$ such that every (locally finite[1]) family of pairwise disjoint non-empty open subsets of X has cardinality less than or equal to $\mathbf{m}$, is called *the Souslin number* (*the lf-Souslin number*) of the space X and is denoted by $S(X)$ ($S_{lf}(X)$).

The composition of f and g we denote by $f \circ g$. By the symbol const_0, we denote the constant function assuming the value zero.

The set of all continuity (discontinuity) points of f we denote by $C(f)$ ($D(f)$). By the symbol $Z(f)$ we denote the zero set of f; i.e., $Z(f) = f^{-1}(0)$. If $\mathcal{A}$ is a family of functions $f : X \to Y$, then we let $D(\mathcal{A}) = \bigcup_{f \in \mathcal{A}} D(f)$ and $C(\mathcal{A}) = X \setminus D(\mathcal{A})$.

[1]A family $\{A_s\}_{s \in S}$ of subsets of topological space X is locally finite if for every point $x \in X$ there exists an open neighborhood U such that the set $\{s \in S : U \cap A_s \neq \emptyset\}$ is finite [2]. For every locally finite family $\{A_s\}_{s \in S}$ we have equality $\overline{\bigcup_{s \in S} A_s} = \bigcup_{s \in S} \overline{A_s}$.

If $\Re$ is some ring and $f \in \Re$, then the symbol $(f)_{\Re}$ stands for the ideal of the ring $\Re$ generated by f.

A family $\{A_t : t \in T\}$ of subsets of topological space X is locally discrete if there exists a locally finite family $\{V_t\}_{t\in T}$ of open sets such that $A_t \subset V_t$ $(t \in T)$ and

$$\overline{V_{t_0}} \cap \bigcup_{t\in T\setminus\{t_0\}} A_t = \emptyset, \text{ for each } t_0 \in T.$$

A family $\{\Im_t : t \in T\}$ of nonzero ideals of some ring $\Re$ is said to be *independent* if $\Im_{t_0} \cap \left(\sum_{t\neq t_0} \Im_t\right) = (\text{const}_0)_{\Re}$, for any $t_0 \in T$. *The Goldie dimension* of a ring $\Re$ (we denote by $\dim(\Re)$) is the smallest cardinal number $\mathbf{m}$ for which any independent set of nonzero ideals has cardinality less than or equal to $\mathbf{m}$.

3 Main Results.

Theorem 1. *There exists a connected, uncountable, Hausdorff space X such that every Darboux function $f : X \to \mathbb{R}$ is constant.*

PROOF. The proof is based on the construction of a countable and connected set [2][2].

Let $Z = \{(p, q) \in \mathbb{Q}^2 : q \geq 0\}$. Of course, $\operatorname{card} Z = \aleph_0$. For each $x_0 = (p_0, q_0) \in Z$ we define a neighborhood system $B^Z(x_0)$ of x_0 in the following way. If $x_0 = (p_0, q_0)$ and $q_0 = 0$, then

$$B^Z(x_0) = \{U_n(x_0) = ((p_0 - \frac{1}{n}, p_0 + \frac{1}{n}) \cap \mathbb{Q}) \times \{0\} : n = 1, 2, \dots\}.$$

If $x_0 = (p_0, q_0)$ and $q_0 > 0$, then let $(y_0, 0), (t_0, 0)$ be a vertex of the equilateral triangle with the third vertex at x_0. (We may assume that $y_0 < t_0$.) Let $n^\star$ be a positive integer such that $y_0 + \frac{1}{n^\star} < p_0 < t_0 - \frac{1}{n^\star}$. Put

$$U_n(x_0) = \{x_0\} \cup \{(q, 0) \in \mathbb{Q}\times\{0\} : \mid q - y_0 \mid < \frac{1}{n}\} \cup \{(q, 0) \in \mathbb{Q}\times\{0\} : \mid q - t_0 \mid < \frac{1}{n}\},$$

for $n \geq n^\star$. Let

$$B^Z(x_0) = \{U_n(x_0) : n = n^\star, n^\star + 1, \dots\}.$$

Let $\mathcal{T}_Z$ be the topology generated (in the set Z) by the neighborhood system $\{B^Z(x) : x \in Z\}$. Then $(Z, \mathcal{T}_Z)$ is a countable, Hausdorff and connected space ([2], p. 433).

[2]Since the construction of the required space X is based on the structure of the set Z, we describe briefly a construction of this set.

Now we consider sets $Y_r = Z \times \{r\}$ such that $r \in \mathbb{R}$. Let $\mathcal{T}_r$ be the topology in Y_r defined by $\mathcal{T}_r = \{U \times \{r\} : U \in \mathcal{T}_Z\}$. Then $\{(Y_r, \mathcal{T}_r) : r \in \mathbb{R}\}$ is a family of topological spaces such that $Y_{r_1} \cap Y_{r_2} = \emptyset$, for $r_1 \neq r_2$.

Let $(Y, \mathcal{T}_Y)$ be the sum of topological spaces $\{(Y_r, \mathcal{T}_r) : r \in \mathbb{R}\}$:

$$(Y, \mathcal{T}_Y) = \bigoplus_{r \in \mathbb{R}} (Y_r, \mathcal{T}_r);$$

i.e., $Y = \bigcup_{r \in \mathbb{R}} Y_r$ and $\mathcal{T}_Y = \{U \subset Y : U \cap Y_r \in \mathcal{T}_r\}$.

Let us put $w = (0, -1, 0) \in \mathbb{R}^3$ and let $X = \{w\} \cup Y$. Now we define a neighborhood system in X. If $x \in X \setminus \{w\}$, then let $B^X(x)$ be a neighborhood system of x in the space $(Y, \mathcal{T}_Y)$. If $x = w$, then we put

$$B^X(x) = \{\{w\} \cup \bigcup_{r \in \mathbb{R}} [((p, +\infty) \cap \mathbb{Q}) \times \{0\} \times \{r\}] : p \in \mathbb{Q}\}.$$

Let $\mathcal{T}_X$ be a topology generated by the neighborhood system $\{B^X(x) : x \in X\}$. Of course, X is an uncountable set (more precisely the cardinality of X is equal to continuum).

Now we show that $(X, \mathcal{T}_X)$ is a Hausdorff space. It is easy to see ([2], Theorem 2.2.7) that $(Y, \mathcal{T}_Y)$ is a T_2-space. So, it is sufficient to show that for every $x_1 \in X \setminus \{w\}$ there exist $\mathcal{T}_X$-open sets U_{x_1} and U_w, respectively, such that $x_1 \in U_{x_1}$, $w \in U_w$, and $U_{x_1} \cap U_w = \emptyset$.

Let[3] $x_1 = (x_1^0, x_2^0, r_0)$, where $x_1^0, x_2^0 \in \mathbb{Q}$, $x_2^0 \geq 0$ and $r_0 \in \mathbb{R}$. Let $U'_{x_1} \in B^Z((x_1^0, x_2^0))$ and let $p' \in \mathbb{Q}$ be a number such that

$$U'_{x_1} \cap ((p', +\infty) \times \{0\}) = \emptyset.$$

Put

$$U_{x_1} = U'_{x_1} \times \{r_0\} \text{ and } U_w = \{w\} \cup \bigcup_{r \in \mathbb{R}} [((p', +\infty) \cap \mathbb{Q}) \times \{0\} \times \{r\}].$$

We have $U_{x_1} \cap U_w = \emptyset$.

Now we show that $(X, \mathcal{T}_X)$ is a connected space. First, let us note that Y_r is a connected set in $(Y, \mathcal{T}_Y)$ (for $r \in \mathbb{R}$) and, at the same time, Y_r is a connected set in $(X, \mathcal{T}_X)$ (for $r \in \mathbb{R}$). Moreover, it is obvious that

$$w \in \overline{Y_r} \text{ (for } r \in \mathbb{R}). \tag{1}$$

This means ([2], Theorem 6.1.9) that $(X, \mathcal{T}_X)$ is a connected space.

[3]For the simplicity of the notation, we suppose association of Cartesian product.

Finally, let $f : X \to \mathbb{R}$ be an arbitrary Darboux function. Let us suppose that $f(w) = \alpha_0$ and let $t \in X \setminus \{w\}$. Then $t \in Y_{r_t}$ ($r_t \in \mathbb{R}$). According to (1), $Y_{r_t} \cup \{w\}$ is a countable and connected set and, consequently, $f(Y_{r_t} \cup \{w\})$ is a singleton, which means that $f(t) = \alpha_0$. □

From the above Theorem we can draw the following conclusion.

Corollary 1. *There exists a connected, uncountable, Hausdorff space X for which there are no essential Darboux rings of real functions defined on X.*

The above statements lead us to pose the following open problems:

1. Is the above Theorem or its Corollary true if we assume that X is a T_i-space, for some $i > 2$?

2. Is the above Theorem or its Corollary true if we additionally assume that X is a locally connected space?

It is well known (e.g., [8], [9], [12]) that there exist essential (as well prime) Darboux rings of real functions of a real variable. The following theorems show that we can formulate general assumptions on a topological space X such that there exist essential (prime) Darboux rings of real functions defined on X.

Theorem 2. *If X is a connected and locally connected topological space such that there exists a nonconstant continuous function*[4] *$f_1 : X \to \mathbb{R}$, then there exists an essential Darboux ring of functions from X to $\mathbb{R}$.*

Proof. Let $f_1 : X \to \mathbb{R}$ be a nonconstant continuous function. Since X is a connected space, $f_1(X)$ is a nondegenerate interval. Let $\beta \in \mathrm{Int}(f_1(X))$. Let us consider a continuous function $f_0 : X \to \mathbb{R}$ such that $f_0 = f_2 \circ f_1$, where $f_2(x) =\mid x - \beta \mid$, for $x \in \mathbb{R}$ ($f_2 : \mathbb{R} \to \mathbb{R}$). Of course, there exists $\delta > 0$ such that $[0, \delta) \subset f_0(X)$. Let $\Re_0$ be an essential Darboux ring of functions $\tau : [0, +\infty) \to \mathbb{R}$ such that $D(\Re_0) = \{0\}$ and $\mathcal{C}([0, +\infty), \mathbb{R}) \subset \Re_0$ ([8], [9], [12]).

Let us consider a family $\Re_1$ of functions from X to $\mathbb{R}$

$$\Re_1 = \{\tau \circ f_0 : \tau \in \Re_0\}.$$

It is easy to see that $\Re_1$ is a ring of Darboux functions $h : X \to \mathbb{R}$. To prove the fact that $\Re_1$ is an essential Darboux ring it is sufficient to show that

$$D(\Re_1) \neq \emptyset. \tag{2}$$

[4]It is well known that there exist regular spaces on which every continuous real-valued function is constant.

Since f_0 is a nonconstant, continuous function, $X \neq f_0^{-1}(0) \neq \emptyset$ and X is a connected set, we can choose an element $x_0 \in \mathrm{Fr}(f_0^{-1}(0))$.

Let $\tau_0 \in \Re_0$ be a function such that $0 \in D(\tau_0)$. Then there exists a sequence $\{\alpha_n\} \subset (0, \delta)$ such that $\alpha_n \searrow 0$ and $\lim_{n \to +\infty} \tau_0(\alpha_n) = \alpha \neq \tau_0(0)$. Let $\{U_t : t \in T\}$ be a local base of X at x_0 consisting of connected sets. Let us consider a set Σ consisting of all ordered pairs (t, n), where $t \in T$, n is a positive integer number and

$$U_t \cap f_0^{-1}(\alpha_n) \neq \emptyset$$

(since $x_0 \in Fr(f_0^{-1}(0))$ and U_t is a non-singleton connected set, then $f_0(U_t) \setminus \{0\}$ is a nonempty connected set contained in $(0, +\infty)$, for $t \in T$. Let us define

$$(t_1, n_1) \ll (t_2, n_2) \Longleftrightarrow (U_{t_2} \subset U_{t_1} \wedge n_1 \leq n_2) .$$

One can verify that Σ is a directed set.

If $\sigma = (t, n) \in \Sigma$, then let x_σ be a fixed element of the intersection $U_t \cap f_0^{-1}(\alpha_n)$. Then we have defined a net $\{x_\sigma\}_{\sigma \in \Sigma}$. We show that

$$x_0 \in \lim_{\sigma \in \Sigma} x_\sigma \text{ and } \lim_{\sigma \in \Sigma} (\tau_0 \circ f_0)(x_\sigma) = \alpha. \tag{3}$$

If W is an open neighborhood of x_0, then let $t_0 \in T$ be an index such that $U_{t_0} \subset W$ and let n_0 be a positive integer such that $U_{t_0} \cap f_0^{-1}(\alpha_{n_0}) \neq \emptyset$. Put $\sigma_0 = (t_0, n_0)$. It is easy to see that $x_\sigma \in W$, for $\sigma \gg \sigma_0$.

Let $\varepsilon > 0$ and let n^0 be a positive integer such that $\tau_0(\alpha_n) \in (\alpha - \varepsilon, \alpha + \varepsilon)$, for $n \geq n^0$. Let $t_1 \in T$ be an index such that $U_{t_1} \cap f_0^{-1}(\alpha_{n_1}) \neq \emptyset$, for some $n_1 \geq n^0$. Put $\sigma_1 = (t_1, n_1)$. Let $\sigma \gg \sigma_1$. Then $(\tau_0 \circ f_0)(x_\sigma) = \tau_0(\alpha_{n_\sigma})$ and $n_\sigma \geq n^0$. This means that $(\tau_0 \circ f_0)(x_\sigma) \in (\alpha - \varepsilon, \alpha + \varepsilon)$. According to the arbitrariness of ε we can deduce that $\lim_{\sigma \in \Sigma} (\tau_0 \circ f_0)(x_\sigma) = \alpha$. The proof of (3) is finished, and so, the proof of (2) is finished, too. □

Theorem 3. *Let X be a non-singleton, connected and locally connected, perfectly normal topological space. Then for every point $x_0 \in X$ there exists a prime Darboux ring*[5] *$\Re$ of real functions defined on X such that $D(\Re) = \{x_0\}$.*

Proof. Let x_0 be a fixed point of X and let $U \neq X$ be an open neighborhood of x_0. Let us put $F = X \setminus U$. According to the Vedenissoff theorem ([2], Theorem 1.5.19) there exists a continuous function $f_0 : X \to \mathbb{R}$ such that $f_0(X) = [0, 1]$, $f_0^{-1}(0) = \{x_0\}$ and $f_0^{-1}(1) = F$.

[5] And, at the same time, there exists an essential Darboux ring.

Let $\Re_0$ be an essential Darboux ring of functions $\tau : [0,1] \to \mathbb{R}$ such that $D(\Re_0) = \{0\}$ and $\mathcal{C}([0,1],\mathbb{R}) \subset \Re_0$ ([8], [9], [12]). Let us consider a family $\Re_1$ of functions from X to $\mathbb{R}$:

$$\Re_1 = \{\tau \circ f_0 : \tau \in \Re_0\}.$$

It is easy to see that $\Re_1$ is a ring of Darboux functions. Let $\Re$ be a subset of $\Re_1$ consisting of all functions $h \in \Re_1$ for which $x_0 \in Z(h)$. Of course, $\Re$ is a ring of Darboux functions and $D(h) \subset Z(h)$, for any $h \in \Re$.

We show that

$$D(\Re) = \{x_0\}. \tag{4}$$

In fact, if $x \neq x_0$, then $f_0(x)$ is a continuity point for every function belonging to $\Re_0$. Consequently, x is a continuity point of $\tau \circ f_0$, for any $\tau \in \Re_0$. So, $D(\Re) \subset \{x_0\}$. Consider the point x_0. Since $D(\Re_0) = \{0\}$, there exists a function $\tau_0 \in \Re_0$ such that $\{0\} = D(\tau_0)$. Similar to the proof of Theorem 2 one can prove that $x_0 \in D(\tau_0 \circ f_0)$ and, consequently, $x_0 \in D(\Re)$. From these considerations we conclude that equality (4) is true. □

Now we consider some cardinal function connected with the theory of rings of functions. First, we introduce some new notions. Let X be a topological space for which there exists at least one essential Darboux ring of real functions defined on X.

Definition 1. A family $\{\Re_t : t \in T\}$ of essential Darboux rings of real functions defined on X is said to be $\mathcal{D}$-independent if $\{D(\Re_t) : t \in T\}$ is a locally discrete family of sets.

Definition 2. The smallest cardinal number $\mathbf{m}$ such that every $\mathcal{D}$-independent family of essential Darboux rings of real functions defined on X has cardinality less than or equal to $\mathbf{m}$ is called the $\mathcal{D}$-number of X and is denoted by $\mathcal{D}(X)$.

Theorem 4. *Let X be a non-singleton, connected and locally connected, perfectly normal topological space. Then $\mathcal{D}(X) = S_{lf}(X)$.*

Proof. Note that, according to Theorem 3, there exists at least one essential Darboux ring of functions defined on X. First, we show that

$$\mathcal{D}(X) \leq S_{lf}(X). \tag{5}$$

Let $\{\Re_t : t \in T\}$ be an arbitrary $\mathcal{D}$-independent family of essential Darboux rings of real functions defined on X. Thus $\text{card}(T) \leq \mathcal{D}(X)$. Then we can form a transfinite sequence $\{\Re_\alpha\}_{\alpha<\beta}$ of type β such that the cardinality of β is less than or equal to $\mathcal{D}(X)$. Thus we have a transfinite sequence $\{V_\alpha\}_{\alpha<\beta}$ such that $\{V_\alpha : \alpha < \beta\}$ is locally finite family of open sets, $D(\Re_\alpha) \subset V_\alpha$ $(\alpha < \beta)$ and

$$\overline{V_{\alpha_0}} \cap \bigcup_{\alpha\neq\alpha_0} D(\Re_\alpha) = \emptyset, \text{for each } \alpha_0 < \beta.$$

We construct a transfinite sequence $\{U_\alpha\}_{\alpha<\beta}$ of non-empty open and pairwise disjoint subsets of X such that $\{U_\alpha : \alpha < \beta\}$ is a locally finite family. Let us consider a ring $\Re_0$. Then there exists an open set $U_0 = V_0$ such that $D(\Re_0) \subset U_0 \subset \overline{U_0}$ and $\overline{U_0} \cap \bigcup_{0<\alpha<\beta} D(\Re_\alpha) = \emptyset$. Of course, $\bigcup_{0<\alpha<\beta} D(\Re_\alpha) \subset X \setminus \overline{U_0}$ and $X \setminus \overline{U_0}$ is an open set.

Suppose that $\gamma < \beta$ is an ordinal number and we have defined open sets $U_\alpha \subset V_\alpha$, for $\alpha < \gamma$ in such a way that

$$D(\Re_\alpha) \subset U_\alpha \subset \overline{U_\alpha}\ (\alpha < \gamma),\ U_{\alpha_1} \cap U_{\alpha_2} = \emptyset\ (\alpha_1 \neq \alpha_2)$$

and ($\{V_\alpha\}_{\alpha<\beta}$ is a locally finite family. Thus $\{U_\alpha\}_{\alpha<\gamma}$ is also a locally finite family)

$$\bigcup_{\gamma\leq\mu<\beta} D(\Re_\mu) \subset X \setminus \overline{\bigcup_{\alpha<\gamma} U_\alpha}.$$

Consider a ring $\Re_\gamma$. Since $\{\Re_\alpha\}_{\alpha<\beta}$ is a $\mathcal{D}$-independent family and $D(\Re_\gamma) \subset X \setminus \overline{\bigcup_{\alpha<\gamma} U_\alpha}$, there exists an open set $U_\gamma \subset \left(X \setminus \overline{\bigcup_{\alpha<\gamma} U_\alpha}\right) \cap V_\gamma$ such that $D(\Re_\gamma) \subset U_\gamma \subset \overline{U_\gamma}$. Thus $U_\alpha \cap U_\gamma = \emptyset$, for $\alpha < \gamma$. Of course, $\bigcup_{\alpha\neq\gamma} D(\Re_\alpha) \subset X \setminus \overline{U_\gamma}$ and

$$\bigcup_{\gamma<\mu<\beta} D(\Re_\mu) \subset \left(X \setminus \overline{\bigcup_{\alpha<\gamma} U_\alpha}\right) \cap \left(X \setminus \overline{U_\gamma}\right) = X \setminus \overline{\bigcup_{\alpha\leq\gamma} U_\alpha}.$$

The construction of a family $\{U_\alpha\}_{\alpha<\beta}$ is finished. It is not hard to verify that $\{U_\alpha\}_{\alpha<\beta}$ is a locally finite family of pairwise disjoint, non-empty, open sets. The existence of this family proves that $\mathrm{card}(T) \leq S_{lf}(X)$. Thus we deduce (5).

In the next step of the proof, we show that

$$S_{lf}(X) \leq \mathcal{D}(X). \tag{6}$$

Let $\{U_s : s \in S\}$ be an arbitrary locally finite family of nonempty, pairwise disjoint open sets. We show that there is a $\mathcal{D}$-independent family $\{\Re_s : s \in S\}$ of essential Darboux rings of functions.

Let x_s be a fixed point of U_s, for $s \in S$. According to Theorem 3 for each $s \in S$, there exists an essential Darboux ring $\Re_s$ of real functions defined on X such that $D(\Re_s) = \{x_s\}$. Since X is a regular space, for every $s \in S$ there exists an open set V_s such that

$$x_s \in V_s \subset \overline{V_s} \subset U_s.$$

Let s_0 be a fixed index. Then $D(\Re_{s_0}) = \{x_{s_0}\} \subset \overline{V_{s_0}} \subset U_{s_0}$. This means that

$$\bigcup_{s \in S \setminus \{s_0\}} D(\Re_s) \subset X \setminus \overline{V_{s_0}}.$$

It is easy to see that $\{V_s\}_{s \in S}$ is a locally finite family of open sets such that $D(\Re_s) \subset V_s$ $(s \in S)$. Thus we can deduce that $\{\Re_s : s \in S\}$ is a $\mathcal{D}$-independent family of essential Darboux rings of real functions defined on X. Consequently, we can infer that $\text{card}(S) \leq \mathcal{D}(X)$ and so, the inequality (6) is true. □

For the final theorem we adopt the following symbol.

$P(X)$ is the family of all prime Darboux rings $\Re$ of real functions defined on X for which there exists $x_\Re \in X$ such that $D(\Re) = \{x_\Re\}$ and

$$\mathcal{C}_{x_\Re}(X, \mathbb{R}) = \{\varphi \in \mathcal{C}(X, \mathbb{R}) : \varphi(x_\Re) = 0\} \subset \Re.$$

Theorem 5. *Let X be a connected Tichonoff topological space such that $P(X) \neq \emptyset$. Then*

$$\dim(\Re) = S(X),$$

for an arbitrary ring $\Re \in P(X)$.

PROOF. First, we show that

$$S(X) \leq \dim(\Re). \tag{7}$$

Let $D(\Re) = \{x_\Re\}$. Let $\{U_s : s \in S\}$ be an arbitrary family of nonempty, pairwise disjoint open sets. Of course, $U_s \setminus \{x_\Re\} \neq \emptyset$. We can choose a point $x_s \neq x_\Re$ such that $x_s \in U_s$, for $s \in S$. Since X is a Tichonoff space, for any $s \in S$, there exists a continuous function $h_s : X \to [0,1]$ such that $h_s(x_s) = 1$ and $h_s(X \setminus U_s') = \{0\}$, where U_s' is an open set such that $x_s \in U_s' \subset U_s \setminus \{x_\Re\}$. Of course $h_s \in \Re$. Let us consider a family of ideals $\{(h_s)_\Re : s \in S\}$. We show that

$$\{(h_s)_\Re : s \in S\} \text{ is a family of independent ideals.} \tag{8}$$

Let $s_0 \in S$ and let $\xi \in \Re$ be a function such that

$$\xi \in (h_{s_0})_{\Re} \cap \sum_{s \in S \setminus \{s_0\}} (h_s)_{\Re}.$$

Note that

$$h_{s_0} \cdot h_s = \mathrm{const}_0, \text{ for any } s \in S \setminus \{s_0\}. \tag{9}$$

In fact, if $s \in S \setminus \{s_0\}$, then

$$Z(h_{s_0} \cdot h_s) \supset (X \setminus U_{s_0}) \cup (X \setminus U_s) = X$$

which proves (9). Moreover, since $\xi \in (h_{s_0})_{\Re}$, $\xi = \varphi \cdot h_{s_0}$ for some $\varphi \in \Re$. Since $\xi \in \sum_{s \in S \setminus \{s_0\}} (h_s)_{\Re}$, there exist $s_1, s_2, \ldots, s_n \in S \setminus \{s_0\}$ such that

$$\xi = \varphi_{s_1} \cdot h_{s_1} + \varphi_{s_2} \cdot h_{s_2} + \cdots + \varphi_{s_n} \cdot h_{s_n}.$$

Then, according to (9), we calculate

$$\xi \cdot \xi = (\varphi_{s_1} \cdot h_{s_1} + \varphi_{s_2} \cdot h_{s_2} + \cdots + \varphi_{s_n} \cdot h_{s_n}) \cdot \varphi \cdot h_{s_0} =$$

$$= \varphi_{s_1} \cdot \varphi \cdot \mathrm{const}_0 + \varphi_{s_2} \cdot \varphi \cdot \mathrm{const}_0 + \cdots + \varphi_{s_n} \cdot \varphi \cdot \mathrm{const}_0 = \mathrm{const}_0 .$$

So, we have $\xi = \mathrm{const}_0$. This means that (8) is true and, consequently, we have proved (7).

We now prove that

$$\dim(\Re) \leq S(X). \tag{10}$$

Let $\{\Im_s : s \in S\}$ be an independent family of ideals of $\Re$. For each $s \in S$, we can establish a function $f_s \in \Im_s \setminus \{\mathrm{const}_0\}$. Then $X \setminus Z(f_s) \subset C(f_s)$ (for $s \in S$), which means that $\mathrm{Int}(X \setminus Z(f_s)) \neq \emptyset$ (for $s \in S$).

Note that, for $s_1 \neq s_2$, we have $f_{s_1} \cdot f_{s_2} = \mathrm{const}_0$ and, consequently,

$$\mathrm{Int}(X \setminus Z(f_{s_1})) \cap \mathrm{Int}(X \setminus Z(f_{s_2})) \subset X \setminus (Z(f_{s_1}) \cup Z(f_{s_2})) = \emptyset.$$

This means that $\{\mathrm{Int}(X \setminus Z(f_s)) : s \in S\}$ is a family of nonempty, open and pairwise disjoint sets. This gives (10). From (7) and (10) we conclude $S(X) = \dim(\Re)$. □

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