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TRIANGULAR MAPS NON-DECREASING ON THE FIBERS

Abstract

There is a list of about 50 properties which characterize continuous maps of the interval with zero topological entropy. Most of them were proved by A. N. Sharkovsky [cf., e.g., Sharkovsky et al., Dynamics of One-Dimensional Mappings, Kluwer 1997]. It is also well known that only a few of these properties remain equivalent for continuous maps of the square. Recall, e.g., the famous Kolyada's example of a triangular map of type 2^∞ with positive topological entropy.

In 1989 Sharkovsky formulated the problem to classify these conditions in a special case of triangular maps of the square. The present paper is a step toward the solution. In particular, we give a classification of 23 conditions in the case of triangular maps of the square which are non-decreasing on the fibers. We show that the weakest is "no homoclinic trajectory", the two strongest, mutually incomparable, are "map restricted to the set of chain recurrent points is not Li & Yorke chaotic"and "every ω-limit set contains a unique minimal set".

1 Introduction.

As is well-known, there is a long list of properties characterizing continuous maps of the interval with zero topological entropy. The most representative one can be found in [13] (see also [7], [11], [12]). Below we present a reduced list of 27 properties, regardless of the fact that the last four properties were recently found not to be equivalent to the others (cf. Šindelářová [14]–[16], Alsedà et al. [1]).

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In 1989 A. N. Sharkovsky [12] asked the question which of these properties are equivalent in the case of triangular maps of the square. This problem seems to be very difficult. Until now there are only partial results showing that many of these conditions are not equivalent (cf., e.g., [8], [9], [11], [12]). A systematic approach to the problem, containing also some positive results can be found in [10]. The main problem is caused by the fact that, for general triangular maps of the square, the properties "the map has zero topological entropy" and "the map is of type not greater than 2^∞" are not equivalent [11]. However, they are equivalent in a special case when the triangular maps are non-decreasing on the fibers. Therefore, the present paper is devoted just to these special maps.

In Section 2 we recall some known results. In Section 3 we prove some implications and in Section 4 we present examples of triangular maps showing that some implications are not true. Non-trivial new results are given in Lemmas 3.4, 4.4, and 4.5. The main result, a survey, is given at the end of the paper as a diagram.

In the sequel, $I = [0,1]$ is the unit compact interval, I^2 the unit square, and X a compact metric space with a metric ρ. Let $\mathcal{C}(X,X)$ be the set of continuous mappings of X into itself, $\mathbb{N}$ the set of positive integers, and $\mathbb{N}_0$ the set of non-negative integers. For $\varphi \in \mathcal{C}(X,X)$, let $\varphi^n(x)$ denote the n-th iterate of φ at x, for $n \in \mathbb{N}$ and $x \in X$. The set of cluster points of the sequence $(\varphi^n(x))_{n\in\mathbb{N}}$ is the *ω-limit set $\omega_\varphi(x)$ of x*. Let $\pi : I^2 \to I$ be the projection $(x,y) \mapsto x$.

Let $f : I \to I$, and $g_x : \{x\} \times I \to I$, for $x \in I$. A map $F \in \mathcal{C}(I^2, I^2)$ such that $F(x,y) = (f(x), g_x(y))$, for any x, y in I, is a *triangular map*, f is the *base of* F, and the set $I_x := \{x\} \times I$ is the *fiber over* x. Throughout the paper, F always denotes a triangular map, and f its base.

We proceed with the list of properties of continuous maps of a compact metric space into itself; the symbols used in them are explained below.

(P1) $h(\varphi) = 0$
(P2) $h(\varphi|\ \mathrm{CR}(\varphi)) = 0$
(P3) $h(\varphi|\ \Omega(\varphi)) = 0$
(P4) $h(\varphi|\ \omega(\varphi)) = 0$
(P5) $h(\varphi|\ \mathrm{C}(\varphi)) = 0$
(P6) $h(\varphi|\ \mathrm{Rec}(\varphi)) = 0$
(P7) $h(\varphi|\ \mathrm{UR}(\varphi)) = 0$
(P8) $h(\varphi|\ \mathrm{AP}(\varphi)) = 0$
(P9) $h(\varphi|\ \mathrm{Per}(\varphi)) = 0$
(P10) Every cycle is simple
(P11) Period of any cycle is a power of 2

(P12) There is no minimal set with positive topological entropy
(P13) φ has no homoclinic trajectory
(P14) $\varphi|\ \mathrm{CR}(\varphi)$ is non-chaotic
(P15) $\varphi|\ \Omega(\varphi)$ is non-chaotic
(P16) $\varphi|\ \omega(\varphi)$ is non-chaotic
(P17) $\varphi|\ \mathrm{C}(\varphi)$ is non-chaotic
(P18) $\varphi|\ \mathrm{Rec}(\varphi)$ is non-chaotic
(P19) $\varphi|\ \mathrm{UR}(\varphi)$ is non-chaotic
(P20) $\mathrm{UR}(\varphi) = \mathrm{Rec}(\varphi)$
(P21) Every ω-limit set contains a unique minimal set
(P22) No infinite ω-limit set contains a cycle
(P23) Every ω-limit set either is a cycle or contains no cycle
(P24) $\varphi|\ \mathrm{Per}(\varphi)$ is Lyapunov stable
(P25) $\mathrm{Per}(\varphi)$ is a G_δ-set
(P26) $\mathrm{Rec}(\varphi)$ is an F_σ-set
(P27) Every linearly ordered chain of ω-limit sets is countable

In the sequel, $\mathrm{CR}(\varphi)$ denotes the set of *chain recurrent points* of φ. Thus, $x \in \mathrm{CR}(\varphi)$ if, for any $\varepsilon > 0$, there is a sequence of points $(x_i)_{i=0}^n$ with $x_0 = x$ and $x_n = x$ such that $\rho(x_{i+1}, \varphi(x_i)) < \varepsilon$, for $i = 0, 1, 2, \dots, n-1$. $\Omega(\varphi)$ is the set of *non-wandering points* of φ; i.e., $x \in \Omega(\varphi)$ if, for any neighborhood U of x, there is an $n \in \mathbb{N}$ with $\varphi^n(U) \cap U \neq \emptyset$. By $\omega(\varphi)$ we denote the union of all ω-limit sets of φ, and by $\mathrm{Rec}(\varphi)$ the set of *recurrent points* of φ; i.e., the set of $x \in X$ such that $x \in \omega_\varphi(x)$, while $\mathrm{C}(\varphi) = \mathrm{cl}(\mathrm{Rec}(\varphi))$ is the *center* of φ. $\mathrm{UR}(\varphi)$ denotes the set of *uniformly recurrent points* of φ; i.e., the set of $x \in X$ such that for any neighborhood U of x, there is an $n \in \mathbb{N}$ such that if $\varphi^m(x) \in U$, where $m \geq 0$, then $\varphi^{m+k}(x) \in U$ for some k with $0 < k \leq n$. By $\mathrm{AP}(\varphi)$ we denote the set of *almost periodic points* of φ; i.e., the set of $x \in X$ such that for any neighborhood U of x, there is an $n \in \mathbb{N}$ such that $\varphi^{in}(x) \in U$, for any i. $\mathrm{Per}(\varphi)$ is the set of *periodic points* of φ.

Denote by $h_\rho(\varphi|\ M)$ the *topological entropy of the map φ with respect to the subset M* and by $h_\rho(\varphi)$ the *topological entropy of the map φ*. If no confusion can arise we write h instead of h_ρ.

Let $\varphi \in \mathcal{C}(I, I)$ and let $\alpha = \{x_1, x_2, \dots, x_{2^n}\} \subset I$, where $n \in \mathbb{N}_0$, be a cycle of φ with period 2^n such that $x_1 < x_2 < \ldots < x_{2^n}$. Then α is a *simple cycle of φ*, if either $n = 0$ (and $\alpha = \{x\}$ is a fixed point), or $n > 0$ and the sets $\{x_1, x_2, \dots, x_{2^{n-1}}\}$, $\{x_{2^{n-1}+1}, \dots, x_{2^n}\}$ are invariant sets with respect to φ^2, and each of them is a simple cycle of φ^2.

Let α be a cycle of a triangular map F with period 2^k, $k \in \mathbb{N}_0$, such that $\pi(\alpha)$ is a simple cycle of the base f with period $2^n = m$, for some $n \leq k$. Then α is a *simple cycle of F* if, for every $x \in \pi(\alpha)$ and every $z \in \alpha \cap I_x$,

$\{F^{im}(z)|\ i = 1, 2, \ldots, 2^{k-n}\} \subset I_x$ is a simple cycle of $F^m|\ I_x$ (which is a one-dimensional map $I_x \to I_x$).

A subset M of X is a *minimal set* if $M = \omega_\varphi(x)$, for any $x \in M$.

Let $x \in X$ be a fixed point of φ. A sequence $(x_n)_{n=1}^\infty$ of distinct points in X such that $\varphi(x_{n+1}) = x_n$, for every $n \in \mathbb{N}$, $\varphi(x_1) = x$, and $\lim_{n\to\infty} x_n = x$, is a *homoclinic trajectory related to the point* x. A sequence $(y_n)_{n=1}^\infty$ of distinct points in X such that $\varphi(y_{n+1}) = y_n$, for every $n \in \mathbb{N}$, $\varphi(y_1) = y_k$, for some $k \in \mathbb{N}$ (i.e., $\{y_1, \ldots, y_k\}$ is a cycle of period k), and $\lim_{n\to\infty} y_{kn+i} = y_i$ for $i = 1, 2, \ldots, k$, is a *homoclinic trajectory related to the cycle* $\{y_1, \ldots, y_k\}$.

A map φ is *chaotic* (in the sense of Li and Yorke) if there is a φ-*chaotic pair* $\{x, y\} \subset X$; i.e., points $x, y \in X$ such that

$$0 = \liminf_{n\to\infty} \rho\left(\varphi^n(x), \varphi^n(y)\right) < \limsup_{n\to\infty} \rho\left(\varphi^n(x), \varphi^n(y)\right).$$

2 Known Results.

Throughout this section, X denotes a compact metric space, $\mathcal{T}$ the class of triangular maps of the square, and $\mathcal{T}_m$ the class of triangular maps of the square which are non-decreasing on the fibers.

Proposition 2.1. *Let $\varphi \in \mathcal{C}(I, I)$. Then conditions* (P1)–(P23) *are mutually equivalent.*

Remark. The complete list of references can be found in [13]. In some papers and books it was stated that for $\varphi \in C(I, I)$, also properties (P24)–(P27) are equivalent to (P1) (cf., e.g., [7], [11], [12], [13]), but actually this is not the case. In [14] there is proved that (P24) is not equivalent to (P1), in [15] that (P25) is not equivalent to (P1), in [16] that (P26) is not equivalent to (P1), and in [1] it is proved that (P27) is not equivalent to (P1).

Proposition 2.2. *Let $\varphi \in \mathcal{C}(X, X)$, and $F \in \mathcal{T}$.*

(i) If $A \subset B \subset X$ are invariant sets of φ, then $h(\varphi|\ A) \le h(\varphi|\ B)$.

(ii) $h(f) + \sup_{x\in I} h(F|\ I_x) \ge h(F) \ge \max\{h(f), \sup_{x\in I} h(F|\ I_x)\}$.

PROOF. Property (i) follows from the definition of topological entropy, (ii) is proved in [5] (cf. also [11]). ☐

Lemma 2.3. [11] *Let $F \in \mathcal{T}$. For $\varphi \in \mathcal{C}(X, X)$, let $A(\varphi, X)$ denote one of the sets* $\mathrm{Per}(\varphi)$, $\mathrm{Rec}(\varphi)$, $\mathrm{UR}(\varphi)$, $\mathrm{C}(\varphi)$, $\omega(\varphi)$, $\Omega(\varphi)$, $\mathrm{CR}(\varphi)$. *Then $\pi(A(F, I^2)) = A(f, I)$.*

Lemma 2.4. [4] *If $\varphi \in \mathcal{C}(X,X)$, then*

$$\mathrm{Per}(\varphi) \subset \mathrm{AP}(\varphi) \subset \mathrm{UR}(\varphi) \subset \mathrm{Rec}(\varphi) \subset \begin{matrix} \mathrm{C}(\varphi) \\ \omega(\varphi) \end{matrix} \subset \Omega(\varphi) \subset \mathrm{CR}(\varphi).$$

Lemma 2.5. [3] *Let $\varphi \in \mathcal{C}(X,X)$ be surjective. If $h(\varphi) > 0$, then φ is chaotic.*

Lemma 2.6. [16] *There is a map $\chi \in \mathcal{C}(I,I)$ of type 2^∞ such that*

(i) *χ has a unique infinite maximal ω-limit set $\tilde{\omega} = Q \cup P$, where Q is a Cantor set, and $P = \{p_n\}_{n=-\infty}^{\infty}$ an infinite set of points isolated in $\tilde{\omega}$ such that $\chi(p_n) = p_{n+1}$, for any n;*
(ii) $\mathrm{cl}(\mathrm{Per}(\chi)) = \mathrm{Per}(\chi) \cup P \cup Q$*;*
(iii) $P \subset \mathrm{cl}(\mathrm{Per}(\chi)) \setminus \mathrm{Per}(\chi)$*;*
(iv) *any point in* $\mathrm{Per}(\chi)$ *is isolated in $\omega(\chi)$, and repelling;*
(v) *χ is monotone in a neighborhood of any $p \in P$.*

Lemma 2.7. $(13 \not\Rightarrow 1)$ [10] *There is an $F \in \mathcal{T}_m$ with no homoclinic trajectory such that f has positive topological entropy.*

Lemma 2.8. $(21 \Rightarrow 22)$ [10] *Let $F \in \mathcal{T}$. If every ω-limit set of F contains a unique minimal set, then no infinite ω-limit set of F contains a cycle.*

3 Implications.

In this section we prove some implications for functions in $F \in \mathcal{T}_m$. However, if the argument for the class $\mathcal{T}$ is almost the same, we prove the more general statement. For the reader's convenience we include some results that can be found elsewhere, in particular, in [10].

Lemma 3.1. $(1 \Rightarrow 13)$ *Let $F \in \mathcal{T}$. If $h(F) = 0$, then F has no homoclinic trajectory.*

Proof. Let F have a homoclinic trajectory γ. Without loss of generality we may assume that γ is a homoclinic trajectory related to a fixed point, since otherwise it suffices to replace F by F^k where k is the period of the cycle related to γ. Then $\pi(\gamma)$ is either a homoclinic trajectory or a fixed point. If $\pi(\gamma)$ is a homoclinic trajectory, then $h(F) \geq h(f) > 0$ (cf. Lemma 2.1). If $\pi(\gamma)$ is a fixed point x, then $\gamma \subset I_x$ and $F|\ I_x$ has a homoclinic trajectory. Thus $0 < h(F|\ I_x) \leq h(F)$. □

Lemma 3.2. *For $F \in \mathcal{T}_m$, condition* (P11)*, that period of any cycle is a power of* 2*, is necessary for any of the following conditions:*
(P12) *There is no minimal set with positive topological entropy.*

(P19) $F|\ \mathrm{UR}(F)$ *is non-chaotic.*
(P20) $\mathrm{Rec}(F) = \mathrm{UR}(F)$.
(P22) *No infinite ω-limit set contains a cycle.*

Proof. Let $F \in \mathcal{T}$, let γ be a cycle of F with period p that is not a power of 2, and let $\pi(\gamma) = \alpha = \{a_1, \dots, a_q\}$. Consider two cases.

CASE A. Assume q is not a power of 2. Then there is a minimal set M_f of f with positive topological entropy, and in $M_f \times I$ there is a minimal set M_F of F such that $h(F|\ M_F) \geq h(f|\ M_f) > 0$. This proves that (P12) implies (P11). By Lemma 2.5 there are points $z_1, z_2 \in M_F$ such that

$$\liminf_{n\to\infty} |F^n(z_1) - F^n(z_2)| = 0 \text{ and } \limsup_{n\to\infty} |F^n(z_1) - F^n(z_2)| > 0.$$

Since $M_F \subset \mathrm{UR}(F)$ (cf., e.g., [4]), $F|\ \mathrm{UR}(F)$ is chaotic and consequently, (P19) implies (P11). By Proposition 2.1 there is a recurrent point $\hat{x}$ of f which is not uniformly recurrent, and by Lemma 2.3, $I_{\hat{x}}$ contains a recurrent point of F that is not uniformly recurrent. Hence (P20) implies (P11). Since $h(f) > 0$ Proposition 2.1 implies that f has an infinite ω-limit set $\omega_f(x_0)$ containing a periodic point a with period r. Take $z_0 = (x_0, y)$, for arbitrary $y \in I$. Then $\omega_F(z_0)$ is an infinite set and $\tilde{\omega} = \omega_F(z_0) \cap I_a$ contains a point z. We have $F^r|\ I_a : I_a \to I_a$ and $F^r(\tilde{\omega}) = \tilde{\omega}$. Now assume $F \in \mathcal{T}_m$. Then $F^r|\ I_a$ is non-decreasing and the sequence $(F^{kr}(z))_{k=0}^{\infty}$ has a periodic accumulation point $c \in \tilde{\omega} \subset \omega_F(z_0)$, and this proves that (P22) implies (P11).

CASE B. Assume q is a power of 2, and $G := F^q|\ I_{a_1} : I_{a_1} \to I_{a_1}$ has a cycle whose period is not a power of 2. By Proposition 2.1, G has a minimal set M_G with positive topological entropy, $G|\ \mathrm{UR}(G)$ is chaotic, and G has a recurrent point (a_1, y_0) which is not uniformly recurrent. Obviously, $(a_1, y_0) \in \mathrm{Rec}(F) \setminus \mathrm{UR}(F)$. □

Lemma 3.3. $(1 \Leftrightarrow \dots \Leftrightarrow 12)$ *For $F \in \mathcal{T}_m$, conditions* (P1)–(P12) *are mutually equivalent.*

Proof. Take $F \in \mathcal{T}_m$ and assume $h(F|\ \mathrm{Per}(F)) = 0$. Then, by Proposition 2.2(ii), $h(f|\ \mathrm{Per}(f)) = 0$, and by Proposition 2.1, $h(f) = 0$. Since for every $x \in I$, g_x is non-decreasing, we have $\sup_{x\in I} h(F|\ I_x) = 0$ and by 2.2(ii), $h(F) = h(f)$. That is, $h(F) = 0$. Thus (P9) implies (P1), and by Lemma 2.4, (P1)–(P9) are mutually equivalent.

Let $F \in \mathcal{T}$. If $h(F) = 0$, then by Proposition 2.2(ii), $h(f) = 0$ and by Proposition 2.1, every cycle α of f is a simple cycle with period 2^n, for some $n \in \mathbb{N}$. For any $x_0 \in \alpha$, every cycle of $F^{2^n}|\ I_{x_0}$ is either a fixed point or a cycle with period 2^k, for some $k \in \mathbb{N}$. That is, every cycle of F is simple, and (P1) implies (P10). The implication (P10) $\Rightarrow$(P11) is obvious.

Let $F \in \mathcal{T}_m$ and let the period of any cycle of F be a power of 2. Then period of any cycle of f is a power of 2 and $h(f) = 0$. Thus (P11) implies (P1) since $h(F) = h(f)$.

By Lemma 3.2, (P12) implies (P11). So to finish the proof it suffices to show that (P7) implies (P12). But this follows since every point of a minimal set is uniformly recurrent (cf., e.g., [4]) hence if $h(F|\ \mathrm{UR}(F)) = 0$, then there is no minimal set with positive topological entropy. □

Lemma 3.4. (18 ⇒ 20) *Let $F \in \mathcal{T}_m$. If $F|\ \mathrm{Rec}(F)$ is non-chaotic, then $\mathrm{UR}(F) = \mathrm{Rec}(F)$.*

PROOF. Let $F|\ \mathrm{Rec}(F)$ be non-chaotic. Then by Lemmas 2.3, 3.2 and 3.3, $h(F) = 0$ and hence, $h(f) = 0$. So, any isolated point of any infinite ω-limit set of f is a wandering point (cf., e.g., [6]), and since $\mathrm{Rec}(f) \subset \omega(f)$, we have $\mathrm{Rec}(f) = \mathrm{Per}(f) \cup W$ where W is the union of the perfect ω-limit sets of f. Consequently,

$$\mathrm{Rec}\,(F) \subset (\mathrm{Per}\,(f) \cup W) \times I \tag{1}$$

since $\pi(\mathrm{Rec}(F)) = \mathrm{Rec}(f)$, by Lemma 2.3.

Now assume, contrary to what we wish to show, that $z_0 = (x_0, y_0) \in \mathrm{Rec}(F) \setminus \mathrm{UR}(F)$. Then $x_0 \notin \mathrm{Per}(f)$; this follows easily since F is non-decreasing on the fibers. Hence, by (1), $\omega_f(x_0) = Q$ is a perfect set containing x_0. Thus, $\omega_F(z_0)$ is a subset of $Q \times I$ and contains a minimal set $M \neq \omega_F(z_0)$. Let $M_{x_0} = \{y \in I|\ (x_0, y) \in M\}$. Since $y_0 \notin M_{x_0}$, we have either $\max M_{x_0} < y_0$ or $\min M_{x_0} > y_0$. Assume without loss of generality that $\max M_{x_0} < y_0$. Let $m = \min M_{x_0}$, and $z = (x_0, m)$. Then $z \in M$ and consequently, since M is a minimal set, $z \in \mathrm{UR}(F)$. Since M is invariant and $z_0 \in \mathrm{Rec}(F) \setminus M$, we have $\limsup_{n\to\infty} \rho(F^n(z), F^n(z_0)) > 0$.

On the other hand, there is a sequence $(n_k)_{k=0}^{\infty}$ of positive integers such that $\lim_{k\to\infty} F^{n_k}(z_0) = z$. Denote by y_k and m_k the second coordinate of $F^{n_k}(z_0)$ or $F^{n_k}(z)$, respectively. Since $F \in \mathcal{T}_m$, we have $m_k \le y_k$, for any k. Thus,

$$m_\infty = \liminf_{k\to\infty} m_k \le \lim_{k\to\infty} y_k = m. \tag{2}$$

Since M is invariant, $(x_0, m_\infty) \in M$. Hence $m_\infty \ge m$. Consequently, by (2), $\lim_{k\to\infty} \rho(F^{n_k}(z_0), F^{n_k}(z)) = 0$ and $F|\ \mathrm{Rec}(F)$ would be chaotic which is impossible. □

4 Examples.

In this section we present some examples of functions from $\mathcal{T}_m$ showing that some implications are not true. The next two examples are known.

Lemma 4.1. [9] *There is an $F \in \mathcal{T}_m$ with the following properties:*

(i) $h(F) = 0$.
(ii) $F|\ \mathrm{UR}(F)$ *is chaotic.*
(iii) *Every ω-limit set contains a unique minimal set.*

Proof. It follows easily by Theorems 3 and 4 in [9]. □

Lemma 4.2. [9] *There is an $F \in \mathcal{T}_m$ with the following properties:*

(i) $h(F) = 0$.
(ii) $F|\ \mathrm{Rec}(F)$ *is chaotic.*
(iii) $F|\ \mathrm{UR}(F)$ *is non-chaotic.*
(iv) $\mathrm{UR}(F) \neq \mathrm{Rec}(F)$.
(v) *F has an ω-limit set containing more than one minimal set.*
(vi) *No infinite ω-limit set contains a cycle.*

Proof. It follows easily by Theorem 10 in [9]. The corresponding map F is even distributionally chaotic on $\mathrm{Rec}(F)$. □

Lemma 4.3. [10] *There is an $F \in \mathcal{T}_m$ with the following properties:*

(i) $h(F) = 0$.
(ii) $F|\ \mathrm{CR}(F)$ *is non-chaotic.*
(iii) $\mathrm{UR}(F) = \mathrm{Rec}(F)$.
(iv) *F has an ω-limit set containing more than one minimal set.*
(v) *F has an infinite ω-limit set containing a cycle.*

Proof. The result is implicitly contained in the proof of Lemma 4.4 of [10]. We recall the construction. Let $f(x) = kx$, where $k \in (0,1)$ is a constant. For $\delta \in (0,1)$, let $\tau_\delta, \tau_\delta^* : I \to I$ be such that

$$\tau_\delta(x) = (1-\delta)x + \delta,$$
$$\tau_\delta^*(x) = \begin{cases} 0 & \text{for } x \in [0,\delta] \\ \frac{1}{1-\delta}x + \frac{\delta}{\delta-1} & \text{for } x \in (\delta, 1]. \end{cases}$$

Thus, $\tau_\delta^* \circ \tau_\delta$ is the identity on I. Put $g_0(y) = y$ and for $n = 0, 1, 2, \dots$,

$$g_{f^n(1)}(y) = \begin{cases} \tau_{1/(k+2)}(y) & \text{for } n_k \le n < \frac{1}{2}(n_k + n_{k+1}) \\ \tau^*_{1/(k+2)}(y) & \text{for } \frac{1}{2}(n_k + n_{k+1}) \le n < n_{k+1} \end{cases}$$

for $x \in (f^{n+1}(1), f^n(1))$, $x = \lambda f^{n+1}(1) + (1-\lambda) f^n(1)$, $\lambda \in (0,1)$,

$$g_x(y) = \lambda g_{f^{n+1}(1)}(y) + (1-\lambda) g_{f^n(1)}(y)$$

where $(n_k)_{k=0}^{\infty}$ is a sequence of non-negative even numbers such that $n_0 = 0$ and $\lim_{k\to\infty}(1 - \frac{1}{k+1})^{(n_{k+1}-n_k)/2} = 0$. Since f and $F|\ I_x$, for every $x \in I$, are non-decreasing, $h(F) = 0$. We have $\omega_F(1,0) = I_0 = \mathrm{Fix}(F) = \mathrm{UR}(F) = \mathrm{Rec}(F) = \mathrm{CR}(F)$, and $\omega_F(1,0)$ is an infinite ω-limit set containing a cycle. The other conditions are obvious. □

In the proofs of the next Lemmas 4.4 and 4.6 we need symbolic dynamics, in particular, the adding machine on the space $\Sigma = \{0,1\}^{\mathbb{N}}$ of sequences of two symbols. We assume that Σ is equipped with a metric ρ of pointwise convergence, e.g., $\rho((x(i))_{i=1}^{\infty}, (y(i))_{i=1}^{\infty}) = 1/k$ if k is the first integer such that $x(k) \neq y(k)$. Let $A : \Sigma \to \Sigma$ be the adding machine which is defined by $A(\underline{x}) = \underline{x} + 1\overline{0}$, where $\underline{x} \in \Sigma$ and $\overline{0}$ is the zero sequence; the adding is mod 2 from the left to the right.

Let $f \in \mathcal{C}(I,I)$ be a map of type 2^{∞} possessing the unique Cantor-type ω-limit set Q such that $f|\ Q$ is one-to-one. In this case the system $(Q, f|\ Q)$ is conjugate (i.e., homeomorphic) to (Σ, A) (cf. [6], for example). In the sequel, we identify both systems and refer to $f|\ Q$ as to the adding machine. Clearly $\omega_{f|Q}(x) = Q$, for $x \in Q$. Hence Q is a minimal set for f. For details concerning the adding machine and its representation as a factor of a continuous map of the interval see, e.g., [6], [8], or [9].

Lemma 4.4. *There is an $F \in \mathcal{T}_m$ with the following properties:*

(i) $h(F) = 0$.
(ii) $F|\ \mathrm{Rec}(F)$ *is chaotic.*
(iii) $\mathrm{UR}(F) \neq \mathrm{Rec}(F)$.
(iv) *Every ω-limit set contains a unique minimal set.*

Proof. The construction of such a mapping is based on the proof of Theorem 1 in [8] which states that there is a triangular map with properties (i) and (iii). But our map is much simpler and has the other properties.

Let $f \in \mathcal{C}(I,I)$ be a map of type 2^{∞} possessing the unique infinite ω-limit set Q such that f is one-to-one on Q (e.g., let f be the Feigenbaum map). Then $f|\ Q$ is the adding machine. To get $F \in \mathcal{T}_m$ it suffices to define its restriction $F|\ Q \times I$, non-decreasing on the fibers, and then extend it properly to the whole of I^2. This will give $h(F) = 0$ since $h(f) = 0$. As already mentioned, we assume without loss of generality that Q is the space (Σ, ρ).

For any $k \geq 1$ and any $a \in \{0,1\}$, let $\varphi(k,a)$ be a non-decreasing continuous map $I \to I$ with the properties:

$$\varphi(k,1) \circ \varphi(k,0) = \mathrm{id} \tag{3}$$

where id is the identity map of I, and

$$\|\varphi(k,a) - \mathrm{id}\| = \delta_k \to 0 \text{ as } k \to \infty. \tag{4}$$

Any $\varphi(k,a)$ is a map of *rank* k. If $x = x(1)x(2)x(3)\cdots \in Q$, then $x(2i)$ is the i-th *control digit* of x. For $(x,y) \in Q \times I$ let $F(x,y) = (f(x), g_x(y))$ where $g_x = \varphi(k, x(2k-1))$ if the first zero control digit is the k-th one, and g_x is the identity if x has no zero control digit.

Then $F|\, Q \times I$ is continuous. Indeed, let $\rho(u,v) < 1/2k$, for some $u, v \in Q$. If there exists $i \le k$ with $u(2i) = 0$ $(= v(2i))$, then $g_u = g_v$. Otherwise $\|g_u - \mathrm{id}\| \le \delta_m$ and $\|g_v - \mathrm{id}\| \le \delta_m$, for some $m > k$. In any case $\|g_u - g_v\| \le 2\delta_m$ which, by (4), implies $\lim_{u \to v} \|g_u - g_v\| = 0$ and hence, the continuity of $g_x(y)$ in $Q \times I$.

Next we prove some identities for F. Let $\underline{0}$ be the zero sequence in Q and let $y_0 \in I$. Denote by y_j the second coordinate of $F^j(\underline{0}, y_0)$, for $j \ge 0$. Let $m \le 2 \cdot 4^k$. Then

$$y_m = \varphi_m \circ \varphi_{m-1} \circ \cdots \circ \varphi_1(y_0) \tag{5}$$

where every φ_i is a map of rank $\le k+1$ since during the first $2 \cdot 4^k = 2^{2k+1}$ iterations, the $(k+1)$-th control digit of $f^i(\underline{0})$ is zero. If $m \ge 2$, then $\varphi_2 \circ \varphi_1$ is the block

$$\beta_1 = \varphi(1,1) \circ \varphi(1,0) = \mathrm{id}$$

(see (3)) and this block β_1 repeats in (5) periodically with period 4 (since $f^i(\underline{0})$ begins with two zeros periodically with period 4). Hence, in (5) there are, from the right to the left, the block β_1, then a block of two maps, then again β_1, etc.

Similarly, if $m \ge 2^3$, then $\varphi_8 \circ \cdots \circ \varphi_1$ is the block β_2 of the form

$$\beta_2 = \varphi^2(2,1) \circ \beta_1 \circ \varphi^2(2,0) \circ \beta_1$$

which, by $\beta_1 = \mathrm{id}$ and (3), again gives $\beta_2 = \mathrm{id}$. Block β_2 repeats in (5) periodically with period $2^4 = 16$. By induction we get that the block β_k of the first $m_k = 2^{2k-1}$ maps in (5) amounts to the identity; so

$$y_{m_k} = y_0, \text{ for } k \ge 1, \tag{6}$$

and for large m the structure of (5) is

$$\cdots \circ \beta_k \circ \beta_k'' \circ \beta_k \circ \beta_k' \circ \beta_k$$

where β_k', β_k'', ... are blocks of the same length m_k as β_k. Since the maps of rank less than $k+1$ are organized into blocks β_i, $i \le k$, which by (6) can be cancelled, and since the number of appearances of $\varphi(k+1,0)$ in (5), for $m = 2m_k$, equals to the number of n's between 0 and $2m_k$ for which the first k control digits in $f^n(\underline{0})$ are ones; i.e., $2m_k/2^k = 2^k$, we get

$$y_{2m_k} = \varphi^{2^k}(k+1,0)(y_0)\,,\ k \geq 1.$$

Now we specify $\varphi(k,0)$ and $\varphi(k,1)$. For any $y \in I$, let

$$\varphi(k,0)(y) = (1-\delta_k)\,y, \tag{7}$$

$$\varphi(k,1)(y) = \min\{1, y/(1-\delta_k)\}\,. \tag{8}$$

So, it is clear that $\omega_F(\alpha,0) = Q \times \{0\} = M$, for any $\alpha \in Q$. That is, M is a minimal set of F. Moreover, we want F to be such that $\omega_F(\underline{0},y) \supset M$, for any $y \in I$. It suffices to let $y_{2m_k} \to 0$ as $k \to \infty$. Thus, we have to specify the parameters δ_k. Take, for example, $\delta_k = 1/k$. Then

$$y_{2m_k} = \varphi^{2^k}(k+1,0)(y_0) = (1-\delta_k)^{2^k} y_0 = (1-1/k)^{2^k} y_0. \tag{9}$$

It remains to extend $F|\ Q \times I$ properly to get a map $F \in \mathcal{T}_m$ which is easy.

For any $\alpha \in Q$ and $y \in I$, the trajectory of (α, y) visits every neighborhood of $I_{\underline{0}}$ and it follows that $\omega_F(\alpha,y)$ contains a point $y_0 \in I_{\underline{0}}$. Consequently, $\omega_F(\alpha,y) \supset \omega_F(\underline{0},y_0)$ and by (9), $\omega_F(\underline{0},y_0) \supset M$. It follows that M is a unique minimal set in $Q \times I$. Therefore every ω-limit set in $Q \times I$ contains a unique minimal set, the set M. For any ω-limit set $\tilde{\omega}$ not in $Q \times I$, $\pi(\tilde{\omega})$ is a cycle of f. Since, for every x, g_x is non-decreasing $\tilde{\omega}$ must be a cycle as well. This proves (iv).

Since $\lim_{k\to\infty} f^{m_k}(\underline{0}) = \underline{0}$, the point $z_1 = (\underline{0},1)$ is by (6) recurrent, and since M is the unique minimal set in $Q \times I$, it follows that z_1 is not uniformly recurrent. This proves (iii).

Points $z_0 = (\underline{0},0)$ and $z_1 = (\underline{0},1)$ are recurrent. By (9),

$$\liminf_{n\to\infty} \rho(F^n(z_0), F^n(z_1)) = 0,$$

and by (6),

$$\limsup_{n\to\infty} \rho\left(F^n(z_0), F^n(z_1)\right) = 1.$$

That is, $F|\ \mathrm{Rec}(F)$ is chaotic. This proves (ii). □

Lemma 4.5. *There is an $F \in \mathcal{T}_m$ with the following properties:*

(i) *$\omega(F)$ is a proper subset of* $\mathrm{C}(F)$.

(ii) *$F|\ \omega(F)$ is non-chaotic.*

(iii) *$F|\ \mathrm{C}(F)$ is chaotic.*

(iv) $\mathrm{UR}(F) = \mathrm{Rec}(F)$.

PROOF. As the base f of F take the function χ from Lemma 2.6. Then, by Lemmas 2.3 and 2.6, $\mathrm{C}(F) \subset \mathrm{C}(f) \times I$ and $\mathrm{C}(f) = \mathrm{Per}(f) \cup Q \cup P$. We are going to find F such that

$$\omega(F) \subset \mathrm{C}(F) \setminus (P \times (0,1]), \tag{10}$$

$$F|\ P \times \{0,1\} \text{ is chaotic.} \tag{11}$$

Moreover, all triangular maps (f, g_x) used in the argument have the property that $g_x(0) = 0$, for any x. This yields

$$F\left(I \times \{0\}\right) \subset I \times \{0\}.$$

We proceed with several steps. In each step we construct a triangular map defined on a subset of I^2, whose base map is always the corresponding restriction of $f = \chi$ from Lemma 2.6.

STAGE 1. *Construction of a map $F_1 : (P \cup Q) \times I \to (P \cup Q) \times I$ satisfying (11) with F replaced by F_1.*

Let

$$\varphi(k,0)(y) = \max\left\{0, y - \frac{1}{2^k+1}\right\}, y \in I, k \in \mathbb{N}_0, \tag{12}$$

$$\varphi(k,1)(y) = \min\left\{4y, y + \frac{1}{2^k+1}, 1\right\}, y \in I, k \in \mathbb{N}_0. \tag{13}$$

It is easy to see that, for any k,

$$\varphi(k,1) \circ \varphi(k,0)(1) = 1. \tag{14}$$

For $x \in P \cup Q$ let

$$g_x = \begin{cases} \varphi(k,0) & \text{if } x = p_n \text{ and } n \in [2(2^k-1), 3(2^k-1)], \\ \varphi(k,1) & \text{if } x = p_n \text{ and } n \in [3 \cdot 2^k - 2, 2^{k+2} - 3], \\ \mathrm{id} & \text{otherwise.} \end{cases} \tag{15}$$

Let $F_1^j(p_0, 1) = (p_j, y_j)$, for $j \in \mathbb{N}_0$. Then, by (12)–(15),

$$y_{2(2^k-1)} = 1 \quad \text{and} \quad y_{3\cdot 2^k - 2} = \varphi^{2^k}(k,0)(1) = 1 - \frac{2^k}{2^k+1}. \tag{16}$$

By Lemma 2.6(i), $\omega_f(p_n) = Q$ for any n, hence $\lim_{n\to\infty} \mathrm{dist}(p_n, Q) = 0$. Since $\lim_{k\to\infty} \varphi(k,0) = \lim_{k\to\infty} \varphi(k,1) = \mathrm{id}$, the map F_1 is continuous on $(P \cup Q) \times I$, and (11) for F replaced by F_1 follows from (16).

STAGE 2. *Extension of F_1 to an auxiliary triangular map $F_2 : I^2 \to I^2$ such that*

$$P \times (0,1] \cap \omega(F_2) = \emptyset. \tag{17}$$

By Lemma 2.6(i),(v), for any non-negative integer j there is an open interval V_j such that $V_j \cap (P \cup Q) = \{p_j\}$ and $f|\, V_j$ is strictly monotone. It is well-known and easy to verify that p_j divides V_j into two subintervals such that one of them is non-wandering (and the other one is wandering, cf., e.g., [6]); denote this non-wandering interval by W_j. Since $f|\, V_j$ is strictly monotone for every j, there is a strictly monotone sequence $(a_0^i)_{i=0}^\infty \subset W_0$ such that $\lim_{i\to\infty} a_0^i = p_0$, and

$$a_j^i = f^j(a_0^i) \in W_j, \text{ for } j \le 2^{i+3}, \quad i, j \in \mathbb{N}_0. \tag{18}$$

Since $f|\, V_j$ is strictly monotone for every j, and since W_j are non-wandering intervals, for any i there is a minimal $k = k(i) > 2^{i+3}$ such that $f^k(a_0^i) \notin W_k$. For $0 \le j < k$ denote by W_j^i the compact interval with endpoints p_j and a_j^i. Thus, by (18),

$$f(W_j^i) = W_{j+1}^i, \text{ whenever } j+1 < k(i), \ i, j \in \mathbb{N}_0. \tag{19}$$

By Lemma 2.6(iv), we may assume that no a_j^i is periodic.

For $x \in W_j^i$, $j < k(i)$, let

$$g_{a_j^i} = \begin{cases} g_{p_j} \circ g_{p_j} & \text{if } j = 3(2^i - 1), \\ g_{p_j} & \text{otherwise,} \end{cases} \tag{20}$$

and if $\lambda \in (0,1)$, let

$$g_x = \lambda g_{a_j^i} + (1-\lambda)\, g_{a_j^{i+1}} \text{ if } x = \lambda a_j^i + (1-\lambda)\, a_j^{i+1}. \tag{21}$$

Thus, we have defined g_x on the union W of all W_j^i. Since $Q \cup P$ is compact, $W \cup Q \cup P = W \cup Q$ is also compact. Moreover, g_x depends continuously on $x \in W \cup Q$. (Note that $\|g_{p_j} \circ g_{p_j} - g_{p_j}\| \to 0$ for $j \to \infty$, cf. (20).) Therefore F_1 (obtained in Stage 1) can be extended onto the whole of I^2, to get an $F_2 \in \mathcal{T}_m$ such that, for $(x, y) \in W_j^i \times I$, $F_2(x, y) = (f(x), g_x(y))$ where g_x is given by (20) and (21).

Let $J_i \subset W_0$ be the compact interval with endpoints a_0^i and a_0^{i+1}. Then

$$F_2^j(I_x) = \left(f^j(x), 0\right), \text{ if } x \in J_i, \ j \ge 3 \cdot 2^{i+1} - 2. \tag{22}$$

Indeed, for $(x, y) \in I^2$ put $F^j(x, y) = (x_j, y_j)$. If $x = a_0^i$ and $y = 1$, then

$$y_{3\cdot 2^i - 2} = g_{p^i_{3\cdot(2^i-1)}} \circ g_{p^i_{3\cdot(2^i-1)}} \left(y_{3\cdot(2^i-1)}\right) = g_{p_{3\cdot(2^i-1)}} \left(1 - \frac{2^i}{2^i+1}\right) = 0,$$

by (16), (20), and (13). Since $F_2 \in \mathcal{T}_m$ and $g_x(0) = 0$ for any x,

$$F_2^j\left(I_{a_0^i}\right) = \left(a_j^i, 0\right), \text{ for every } j \geq 3 \cdot 2^i - 2. \tag{23}$$

Now let $x = \lambda a_0^i + (1-\lambda)a_0^{i+1}$, with $\lambda \in I$. Then, by (23) and (21), for $j = 3 \cdot 2^{i+1} - 2 < k(i)$,

$$F_2^j(I_x) \subset \lambda F_2^j(I_{a_0^i}) + (1-\lambda)F_2^j(I_{a_0^{i+1}}) \subset \{(x_j, 0)\},$$

and (22) follows.

It remains to show (17). Assume $(p_0, z) \in \omega_{F_2}(x, y) = \tilde{\omega}$. Then $F_2^m(x, y) = (x_m, y_m) \in W_0^0 \times I$, for some $m > 0$. Thus, by (22), $F_2^n(I_x) = (x_n, 0)$, for all sufficiently large n. Consequently, $z = 0$. Indeed, since $F_2(\tilde{\omega}) = \tilde{\omega}$ and $\pi(\tilde{\omega}) = P \cup Q$ (cf. Lemma 2.3), it follows that the only F_2^n-preimage of (p_n, z) in $\tilde{\omega}$ is $(p_0, 0)$ whence $z = 0$. For $n < 0$ the argument is similar.

STAGE 3. *Construction of a map $F_3 : ((I \setminus G) \cup T) \times I \to ((I \setminus G) \cup T) \times I$ such that*

$$F_3|\ (I \setminus G) \times I = F_2|\ (I \setminus G) \times I,$$

and

$$\{p_0\} \times \{0, 1\} \subset \mathrm{C}(F_3), \tag{24}$$

where $T \subset \mathrm{Per}(f)$, $\mathrm{cl}(T) = T \cup P \cup Q$, and G is a neighborhood of T, disjoint from $P \cup Q$.

By Lemma 2.6(ii) there is a sequence $(t_0^r)_{r=0}^{\infty}$ of periodic points in W_0^0, with the periods $(s_r)_{r=0}^{\infty}$ such that s_0 is arbitrary, $s_r \geq 4s_{r-1}$, for $r > 0$, and $\lim_{r\to\infty} t_0^r = p_0$. Denote $f^i(t_0^r) = t_i^r$, for $i < s_r$, and let T be the set of t_i^r, for all r and i. Let G be an open neighborhood of T. The set G is specified in Stage 5.

For any r there is a maximal integer $n = n_r$ such that $t_0^r \in W_0^n$. Then, by (19), $t_i^r \in W_i^n$, for $i \leq n$. If $n = 0$ let $g_x = \mathrm{id}$, for any $x \in \{t_0^r, t_1^r, \ldots, t_{s_{r-1}}^r\}$. Otherwise there is a maximal integer $m = m_r$ of the form $2^{k+1} - 3$ such that $m \leq n$. Then let $g_{t_i^r} = g_{p_i}$ for $i \leq m$, and $g_{t_i^r} = \mathrm{id}$ otherwise. By (16), $(t_0^r, 1) \in \mathrm{Per}(F_3)$. Since $g_x(0) = 0$ for any x, we have also $(t_0^r, 0) \in \mathrm{Per}(F_3)$. Since $\lim_{r\to\infty} t_0^r = p_0$, (24) follows. It remains to show that F_3 is continuous, but this easily follows by Lemma 2.6(ii),(iv) since $\lim_{i\to\infty} \|g_{p_i} - \mathrm{id}\| = 0$.

STAGE 4. *Any extension $F_4 \in \mathcal{T}_m$ of F_3 satisfies conditions (iii) and (iv).*

Indeed, (iii) follows by (24) and (16) since $\mathrm{C}(F_4) \supset \mathrm{C}(F_3)$. To prove (iv) note that, by Lemma 2.3, $\mathrm{Rec}(F_4) \subset (\mathrm{Per}(f) \cup Q) \times I$. Since $F_4 \in \mathcal{T}_m$ any recurrent point of F_3 in $\mathrm{Per}(f) \times I$ is periodic and hence, uniformly recurrent. Thus, $\mathrm{Rec}(F_4) \setminus \mathrm{UR}(F_4) \subset Q \times I$. But $Q \subset \mathrm{UR}(f)$ and $g_x = \mathrm{id}$ for $x \in Q$, hence $Q \times I \subset \mathrm{UR}(F_4)$.

STAGE 5. *Construction of extension $F \in \mathcal{T}_m$ of F_3 satisfying conditions (i) and (ii).*

The map F_3 from Stage 3 has properties (i) and (ii) since, by (17),

$$P \times (0,1] \cap \omega(F_3) = \emptyset. \tag{25}$$

Therefore to get F it suffices to extend F_3 not violating this condition. For any $t \in T$, let G_t be an open interval with the closure disjoint from $P \cup Q$ such that $G_t \cap \mathrm{Per}(f) = \{t\}$, and containing no point a_j^i. Moreover, let $G_t \cap G_d = \emptyset$ for $t \neq d$ in T. Such a family exists by Lemma 2.6. By the continuity of f, for any $t \in T$ there is an open interval H_t such that

$$f^{s+i}(H_t) \subset G_{f^i(t)}, \quad \text{for } 0 \leq i < s, \tag{26}$$

where s is the period of t. Let H be the union of all H_t, and let $F \in \mathcal{T}_m$ be a continuous extension of $F_3|\ ((I \setminus H) \cup T) \times I$, such that $g_x(0) = 0$ for any x. We show that

$$\{p_0\} \times (0,1] \cap \omega(F) = \emptyset. \tag{27}$$

Let $(p_0, z) \in \omega_F(x,y)$, let $F^n(x,y) = (x_n, y_n)$, and let $\lim_{k\to\infty}(x_{n_k}, y_{n_k}) = (p_0, z)$. We may assume that $x_{n_k} \in W_0$ (cf. Stage 2). If $x_{n_k} \in H_t$ for no k and t, then $(x_{n_k}, y_{n_k}) = F_4^{n_k}(x,y) \to (p_0, 0)$ by (22). On the other hand, if $x_{n_k} \in H_t$ for some t, then, by (26), $x_n \in f^s(H_t) \setminus H_t$ for some $n > n_k$ (since $x_{n_k} \neq t$ and t is repelling by Lemma 2.6). Consequently, by (26), $y_m = 0$ for some $m > n$. This proves (27). Now (25) follows by (27) since any ω-limit set is invariant (cf. end of Stage 2). □

Lemma 4.6. *There are $F_1, F_2, F_3 \in \mathcal{T}_m$ with zero topological entropy such that:*

(i) $F_1|\ \mathrm{C}(F_1)$ *is non-chaotic and* $F_1|\ \omega(F_1)$ *is chaotic.*
(ii) $F_2|\ \omega(F_2)$ *is non-chaotic and* $F_2|\ \Omega(F_2)$ *is chaotic.*
(iii) $F_3|\ \Omega(F_3)$ *is non-chaotic and* $F_3|\ \mathrm{CR}(F_3)$ *is chaotic.*

PROOF. There is a map $f_1 \in \mathcal{C}(I,I)$ with zero topological entropy, such that $\omega(f_1) = T \cup P \cup Q$, where $T = \mathrm{Per}(f_1)$, Q is a (minimal) Cantor set, $P = (p_n)_{n=-\infty}^{\infty}$. Moreover, p_n is isolated in $T \cup P \cup Q$ and $f_1(p_n) = p_{n+1}$, for any n, and there is an $x_0 \in I$ with $\omega_{f_1}(x_0) = P \cup Q$. Consequently, $\mathrm{C}(f_1) = T \cup Q$. Such a function can be found, e.g., in [6]. To get F_1 let $g_x = \mathrm{id}$, for $x \in T \cup Q$. By Lemma 2.3, $\mathrm{C}(F_1) \subset \mathrm{C}(f_1) \times I = (T \cup Q) \times I$. Therefore, F_1 is non-chaotic on $\mathrm{C}(F_1)$, regardless of how g_x is defined for $x \notin \mathrm{C}(f_1)$.

To finish the construction it suffices to get $\{p_0\} \times I \subset \omega(F_1)$ with the points $z_0 = (p_0, 0)$, $z_1 = (p_0, 1)$ forming an F_1-chaotic pair. To do this we use similar construction as in the proof of Lemma 4.4. Identify the integers $0, 1, 2, \ldots$

with the iterates of $\overline{0}$ in the adding machine. For $n < 0$ let $g_{p_n} = \mathrm{id}$, and for $n \geq 0$ let $g_{p_n} = \varphi(k, n(2k-1))$ if the first zero control digit is the k-th one. The functions $\varphi(k,0)$ and $\varphi(k,1)$ are given by (7), and (8), respectively. As in the proof of Lemma 4.5, there is a sequence $(W_n^0)_{n=-\infty}^{\infty}$ of compact, non-wandering, mutually disjoint one-sided neighborhoods of the points p_n such that $W_n^0 \cap (T \cup P \cup Q) = \{p_n\}$. For $x \in W_n^0$, let $g_x = g_{p_n}$, and extend F_1 to a map in $\mathcal{T}_m$.

It is well-known that the trajectory of x_0 must be eventually in the union of the sets W_n^0 (cf. [6]). Thus, we may assume $x_0 \in W_0^0$ and $f^{2^k}(x_0) \in W_0^0$ for any $k \geq 0$ [6]. To complete the argument note that, by (6),

$$F_1^{2^{2k-1}}(x,y) = (f_1^{2^{2k-1}}(x), y) \text{ if } x \in W_0^0,\ k > 0.$$

This proves that $\{p_0\} \times I \subset \omega(F_1)$, and $\limsup_{i\to\infty} \rho(F^i(z_0), F_1^i(z_1)) = 1$. Finally, by (9), $\liminf_{i\to\infty} \rho(F^i(z_0), F_1^i(z_1)) = 0$.

Function F_2 is defined similarly. By [2], there is a map f_2 having the same properties as f_1, except that P is disjoint from $\omega(f_2)$, and $(p_n)_{n=0}^{\infty} \subset \Omega(f_2)$. Such a function can be obtained, e.g., by an arbitrarily small perturbation of f_1, making it constant in a one-sided non-wandering neighborhood of one of the points p_n, $n < 0$. It is easy to verify that F_2 has the desired properties.

Finally, construction of F_3 is simple. Let f_3 be a map in $\mathcal{C}(I,I)$ with zero topological entropy such that $\Omega(f_3) = \omega(f_3) = T \cup Q$, where $T = \mathrm{Per}(f_3)$, and Q is the unique infinite ω-limit set. Moreover, let f_3 have a wandering interval J such that $\omega_{f_3}(x) = Q$ whenever $x \in J$. It is well-known (and easy to see) that then the trajectory of J is in $\mathrm{CR}(f_3)$. Such a function f_3 can be obtained either by a simple modification of f_1 or f_2, or by blowing up the orbit, e.g., of the critical point of the Feigenbaum's map (cf. [6]). Now let p_0 be an interior point of J. For $n \geq 0$ define F_3 on $\{p_n\} \times I$ similarly as F_1, and let g_x be the identity for $x \in T \cup Q$. Then extend F_3 onto the whole of I^2. □

5 Survey.

The 23 properties of maps in $\mathcal{T}_m$ which are considered in this paper are related as follows.

The Main Theorem 5.1. *Consider properties* (P1)–(P23) *of triangular maps non-decreasing on the fibers listed in Section 1. The relations between them are displayed by the graph on Figure 1 where a missing arrow means that there is no implication, except for implications that follow by transitivity.*

$$
\begin{tikzcd}
 & 14 \arrow[d] & \\
 & 15 \arrow[dl] \arrow[dr] & \\
16 \arrow[dr] & & 17 \arrow[dl] \\
 & 18 \arrow[dl] \arrow[d] & 21 \arrow[d] \\
19 \arrow[dr] & 20 \arrow[d] & 22,\ 23 \arrow[dl] \\
 & 1\text{--}12 \arrow[d] & \\
 & 13 &
\end{tikzcd}
$$

Figure 1:

Proof. There are two groups of mutually equivalent properties, (P1)–(P12) by Lemma 3.3, and (P22) $\Leftrightarrow$ (P23) by [4]. Keeping (P1) and (P22) as representatives of these equivalence classes we can list the remaining relations as follows (brackets contain references, either to a lemma, or to other implications).

P1 $\Rightarrow$ P13 (3.1)
P1 $\not\Rightarrow$ P14 (4.1); P15 (4.1); P16 (4.1); P17 (4.1); P18 (4.1); P19 (4.1); P20 (4.2); P21 (4.3); P22 (4.3)
P13 $\not\Rightarrow$ P1 (2.7); P14 (4.1); P15 (4.1); P16 (4.1); P17 (4.1); P18 (4.1); P19 (4.1); P20 (4.2); P21 (4.3); P22 (4.3)
P14 $\Rightarrow$ P1 ($14 \Rightarrow 19 \Rightarrow 1$); P13 ($14 \Rightarrow 19 \Rightarrow 13$); P15 (2.4); P16 (2.4); P17 (2.4); P18 (2.4); P19 (2.4); P20 ($14 \Rightarrow 18 \Rightarrow 20$)
P14 $\not\Rightarrow$ P21 (4.3); P22 (4.3)
P15 $\Rightarrow$ P1 ($15 \Rightarrow 19 \Rightarrow 1$); P13 ($15 \Rightarrow 19 \Rightarrow 13$); P16 (2.4); P17 (2.4); P18 (2.4); P19 (2.4); P20 ($14 \Rightarrow 18 \Rightarrow 20$)
P15 $\not\Rightarrow$ P14 (4.6); P21 (4.3); P22 (4.3)
P16 $\Rightarrow$ P1 ($16 \Rightarrow 19 \Rightarrow 1$); P13 ($14 \Rightarrow 19 \Rightarrow 13$); P18 (2.4); P19 (2.4); P20 ($14 \Rightarrow 18 \Rightarrow 20$)
P16 $\not\Rightarrow$ P14 (4.6); P15 (4.6); P17 (4.5); P21 (4.3); P22 (4.3)
P17 $\Rightarrow$ P1 ($17 \Rightarrow 19 \Rightarrow 1$); P13 ($17 \Rightarrow 19 \Rightarrow 13$); P18 (2.4); P19 (2.4); P20 ($14 \Rightarrow 18 \Rightarrow 20$)

P17 $\not\Rightarrow$ P14 (4.6); P15 (4.6); P16 (4.6); P21 (4.3); P22 (4.3)
P18 $\Rightarrow$ P1 ($18 \Rightarrow 19 \Rightarrow 1$); P13 ($18 \Rightarrow 19 \Rightarrow 13$); P19 (2.4); P20 (3.4)
P18 $\not\Rightarrow$ P14 (4.6); P15 (4.6); P16 (4.6); P17 ($16 \Rightarrow 18$, $16 \not\Rightarrow 17$); P21 (4.3); P22 (4.3)
P19 $\Rightarrow$ P1 (3.2, 3.3); P13 ($19 \Rightarrow 1 \Rightarrow 13$)
P19 $\not\Rightarrow$ P14 (4.2); P15 (4.2); P16 (4.2); P17 (4.2); P18 (4.2); P20 (4.2); P21 (4.3); P22 (4.3)
P20 $\Rightarrow$ P1 (3.2, 3.3); P13 ($20 \Rightarrow 1 \Rightarrow 13$)
P20 $\not\Rightarrow$ P14 ($18 \Rightarrow 20$, $18 \not\Rightarrow 14$); P15 ($18 \Rightarrow 20$, $18 \not\Rightarrow 15$); P16 ($18 \Rightarrow 20$, $18 \not\Rightarrow 16$); P17 (4.5); P18; P19; P21 (4.3); P22 (4.3)

The facts, that (P20) does not imply neither (P18) nor (P19) were recently proved by J. Chudziak, Ľ. Snoha, V. Špitalský, and independently by G. L. Forti, L. Paganoni, J. Smítal.

P21 $\Rightarrow$ P1 ($21 \Rightarrow 22 \Rightarrow 1$); P13 ($21 \Rightarrow 22 \Rightarrow 13$); P22 (2.8)
P21 $\not\Rightarrow$ P14 (4.1); P15 (4.1); P16 (4.1); P17 (4.1); P18 (4.1); P19 (4.1); P20 (4.4)
P22 $\Rightarrow$ P1 (3.2, 3.3); P13 ($22 \Rightarrow 1 \Rightarrow 13$)
P22 $\not\Rightarrow$ P14 (4.1, 2.8); P15 (4.1, 2.8); P16 (4.1, 2.8); P17 (4.1, 2.8); P18 (4.1, 2.8); P19 (4.1, 2.8); P20 (4.2, 2.8); P21 (4.2) $\square$

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