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Esa Järvenpää*, Maarit Järvenpää*, and Juho Niemelä*, Department of Mathematics and Statistics, P.O. Box 35, FIN-40014 University of Jyväskylä, Finland.
email: esaj@maths.jyu.fi, amj@maths.jyu.fi and juniemel@maths.jyu.fi

TRANSVERSAL MAPPINGS BETWEEN MANIFOLDS AND NON-TRIVIAL MEASURES ON VISIBLE PARTS

Abstract

This paper has two aims. On the one hand, we generalize the notion of sliced measures by means of transversal mappings and study dimensional properties of these measures. On the other hand, as an application of these results, we explain in what sense typical visible parts of a set with large Hausdorff dimension are smaller than the set itself. This is achieved by establishing a connection between dimensional properties of generalized slices and those of visible parts.

1 Background and Preliminary Discussion.

Given integers k and d such that $0 \leq k \leq d-1$, and an affine k-plane K in $\mathbb{R}^d$ (0-plane is simply a point), we use the notation Proj_K for the projection onto K. The following definition of visibility goes back to Urysohn [U] in the 1920's. Let $E \subset \mathbb{R}^d$ be compact. A point $a \in E$ is visible from K, if a is the only point of E in the closed line segment joining a to $\mathrm{Proj}_K(a)$. The visible part of E from K, denoted by $V_K(E)$, is the set of all points that are visible from K (see Section 2).

Visibility was investigated in connection with set theoretic problems by Nikodym in [N]. The study of dimensional properties of visible parts was

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initiated in [JJMO]. Denoting by $\dim_{\mathrm{H}}$ the Hausdorff dimension, we have for almost all affine k-planes K not intersecting E that

$$\dim_{\mathrm{H}}(V_K(E)) = \dim_{\mathrm{H}} E \tag{1.1}$$

provided that $\dim_{\mathrm{H}} E \le d-1$ [JJMO, Theorem 3.2]. On the other hand, under the assumption $\dim_{\mathrm{H}} E > d-1$, we have

$$\dim_{\mathrm{H}}(V_K(E)) \ge d-1 \tag{1.2}$$

for almost all k-planes K not intersecting E [JJMO, Proposition 3.3]. It is not known whether the opposite inequality holds in (1.2). Some special examples of planar sets with Hausdorff dimension bigger than 1 are investigated in [JJMO]. In particular, it is shown that Hausdorff dimensions of all visible parts of a quasi-circle are equal to 1. In [O] an upper bound bigger than 1 is verified for Hausdorff dimensions of typical visible parts of connected and compact subsets of the plane. For further information on related topics, see [M3].

In this paper we approach the open question concerning the validity of the equality in (1.2) by proving the following weaker result.

Theorem 1.1. *Let k and d be integers such that $0 \le k \le d-1$ and let $E \subset \mathbb{R}^d$ be a compact set with $\dim_{\mathrm{H}} E > d-1$. Assume that μ is a Radon measure on E such that $\mu(E) > 0$ and $\dim_{\mathrm{H}} \mu > d-1$. Then $\mu(V_K(E)) = 0$ for $\Gamma_{d,k}$-almost all $K \in A_{d,k}$ with $K \cap E = \emptyset$.*

Here we use the notation $\dim_{\mathrm{H}} \mu$ for the Hausdorff dimension of any finite measure μ on $\mathbb{R}^d$ (and later in Sections 2 and 3 on a metric space X); that is,

$$\dim_{\mathrm{H}} \mu = \mu\text{-}\operatorname*{ess\,inf}_x \underline{\dim}_{\mathrm{loc}}\, \mu(x)$$

where

$$\underline{\dim}_{\mathrm{loc}}\, \mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}. \tag{1.3}$$

(In (1.3) $B(x,r)$ is the closed ball with center at x and with radius $r>0$.) For more information about dimensions of a measure, see [Fa, Chapter 10]. Moreover, the natural Radon measure on the space of affine k-planes of $\mathbb{R}^d$ is denoted by $\Gamma_{d,k}$ (see Section 2 for definitions). Observe that, if (1.2) holds as an equality, Theorem 1.1 follows from it.

Theorem 1.1 is proved in Section 2 as an application of the results concerning dimensional properties of generalized sliced measures. These measures are defined as a natural extension of the following notion of sliced measures introduced by Mattila in [M2]. Given a k-dimensional linear subspace V of $\mathbb{R}^d$, let $V^\perp$ be the orthogonal complement of V. The slices of a finite Radon measure μ by affine planes $V_a = V + a$, where $a \in V^\perp$, are defined as weak limits of the normalized restriction measures

$$\mathcal{H}^{d-k}(B(a,\delta))^{-1}\mu|_{\mathrm{Proj}_{V^\perp}^{-1}(B(a,\delta))} \tag{1.4}$$

as δ goes to 0. (Here $\mathcal{H}^{d-k}$ is the $(d-k)$-dimensional Hausdorff measure, $B(a,\delta) = \{v \in V^\perp \mid |v-a| \leq \delta\}$, and $\mu|_A$ is the restriction of μ to a set A, in other words, $\mu|_A(B) = \mu(A \cap B)$ for all $B \subset \mathbb{R}^d$.) The dimensional properties of these measures, which turn out to exist for $\mathcal{H}^{d-k}$-almost all $a \in V^\perp$, are investigated in [JM]. In this paper we generalize the notion of (1.4) by replacing the preimage of the projection by a preimage of a transversal mapping between manifolds. Our results in Sections 2 and 3, culminating in Theorem 2.1, have a similar flavor to some of the results in [PS]. [PS, Theorem 7.7] may be regarded as a generalization of the projection results in [M1] for a parametrized family of transversal mappings, whilst Sections 2 and 3 extend the results of [JM] to a similar setting.

Our basic setting is as follows. Let (L, ρ_L), (M, ρ_M), and (N, ρ_N) be l-, m-, and n-dimensional smooth Riemann manifolds, respectively. Note that when considering visible parts of a set $E \subset \mathbb{R}^d$ in Section 2, L will be a submanifold of the set of affine planes in $\mathbb{R}^d$, $M = \mathbb{R}^d$, and N will be a submanifold of $\mathbb{R}^d$. Assume that $l \geq n$, $m > n$, and $\Pi : L \times M \to N$ is a continuous function such that for $i = 0, 1, 2$ there is a constant $c_i > 1$ for which

$$\left|\left| D^i_\lambda \Pi(\lambda, x) \right|\right| \leq c_i \tag{1.5}$$

for all $\lambda \in L$ and $x \in M$. Here the i^{th} derivative with respect to λ is denoted by D^i_λ, the 0^{th} derivative meaning the function itself. Furthermore, suppose that there are finite collections $\{\phi, V\}$ and $\{\varphi, U\}$ of C^2-charts on L and N, respectively, with the following property. There exists $R > 0$ such that for all $\lambda \in L$ and $y \in N$

$$B(\lambda, R) \subset V \text{ and } B(y, R) \subset U \tag{1.6}$$

for some V and U. Assume also that the second derivatives, and thus the Lipschitz constants, of the mappings φ, φ^{-1}, ϕ, and ϕ^{-1} are uniformly bounded from above by a positive constant K.

We will restrict our consideration to the class of transversal mappings whose rôle becomes evident in the proof of Lemma 3.1.

1.1 Transversality.

For $\lambda \in L$, let $\Pi_\lambda = \Pi(\lambda, \cdot)$. Suppose that the following form of transversality is satisfied. There is a constant $C_t > 0$ such that for all $\lambda \in L$ and $x_1 \neq x_2 \in M$ for which $\rho_N(\Pi_\lambda(x_1), \Pi_\lambda(x_2)) \leq R$, the following condition holds. Defining

$$\Phi_{x_1,x_2}(\lambda) = \frac{\varphi \circ \Pi_\lambda(x_1) - \varphi \circ \Pi_\lambda(x_2)}{\rho_M(x_1, x_2)},$$

the property

$$|\Phi_{x_1,x_2}(\lambda)| < C_t \tag{1.7}$$

implies that

$$\det\big(D\Phi_{x_1,x_2}(\lambda)\, D\Phi_{x_1,x_2}(\lambda)^T\big) > C_t^2. \tag{1.8}$$

Here φ is as in (1.6), the derivative with respect to λ is denoted by D, and A^T is the transpose of a matrix A. Moreover, we assume that there exists a constant $\widetilde{L} > 0$ such that

$$\|D^2\Phi_{x_1,x_2}(\lambda)\| \leq \widetilde{L} \tag{1.9}$$

for all x_1, x_2, and λ.

We continue by generalizing (1.4) in a way that is useful for our purposes.

1.2 Sliced Measures Determined by Means of Transversal Mappings.

Let μ be a Radon measure on M and let $\lambda \in L$. Denote by $\mathcal{C}_0^+(M)$ the family of continuous non-negative functions on M with compact support. As indicated below, it follows from the axiomatic theory of derivation in [F, 2.9] that for all $\psi \in \mathcal{C}_0^+(M)$ the limit

$$\lim_{\delta \to 0} \frac{((\Pi_\lambda)_* \nu_\psi)(B(y,\delta))}{\mathcal{H}^n(B(y,\delta))} = \lim_{\delta \to 0} \frac{1}{\mathcal{H}^n(B(y,\delta))} \int_{\Pi_\lambda^{-1}(B(y,\delta))} \psi \, d\mu \tag{1.10}$$

exists and is finite for $\mathcal{H}^n$-almost all $y \in N$. Here $\nu_\psi(A) = \int_A \psi \, d\mu$ for all Borel sets $A \subset M$ and $f_* m$ is the image of a measure m under a function $f : X \to Y$; that is, $f_* m(A) = m(f^{-1}(A))$ for all $A \subset Y$. It is well-known that in a separable metric space (X, ρ), satisfying a certain geometric condition described in [F, 2.8.9], the family

$$\mathcal{V} = \{(x, B(x,r)) \mid x \in X, r > 0\}$$

is a μ-Vitali relation for any locally finite Borel regular measure μ on X [F, Theorem 2.8.18] and [F, 2.8.16]. As indicated in [F, 2.8.9] this covers as a special case Riemann C^k-manifolds ($k \geq 2$) with the usual metrics. The existence of (1.10) follows now from [F, Theorem 2.9.5]. Using the separability of $\mathcal{C}_0^+(M)$ it may be shown that the exceptional set of points $y \in N$ in (1.10) is independent of the choice of ψ, and therefore we conclude from the Riesz representation theorem [M1, Theorem 1.16] that, given $\lambda \in L$, for $\mathcal{H}^n$-almost all $y \in N$ there is a Radon measure $\mu_{\lambda,y}$ such that

$$\int \psi \, d\mu_{\lambda,y} = \lim_{\delta \to 0} \frac{1}{\mathcal{H}^n(B(y,\delta))} \int_{\Pi_\lambda^{-1}(B(y,\delta))} \psi \, d\mu \tag{1.11}$$

for all $\psi \in \mathcal{C}_0^+(M)$. Clearly

$$\operatorname{spt} \mu_{\lambda,y} \subset \operatorname{spt} \mu \cap \Pi_\lambda^{-1}(\{y\}), \tag{1.12}$$

where $\operatorname{spt} \mu$ is the support of μ.

Remark 1.2. (1) In (1.11) transversality plays no rôle; only the continuity of Π is needed.

(2) By [F, Lemma 2.9.6] the function $y \mapsto \int \psi \, d\mu_{\lambda,y}$ is $\mathcal{H}^n$-measurable for all $\psi \in \mathcal{C}_0^+(M)$.

The following disintegration formula holds for the measures $\mu_{\lambda,y}$: Given a non-negative Borel function f on M with $\int f \, d\mu < \infty$, it follows from [F, Theorem 2.9.7] that for all Borel sets $B \subset N$

$$\int_B \int f \, d\mu_{\lambda,y} \, d\mathcal{H}^n y \leq \int_{\Pi_\lambda^{-1}(B)} f \, d\mu. \tag{1.13}$$

Moreover, the equality holds in (1.13) provided that $(\Pi_\lambda)_*\mu$ is absolutely continuous with respect to $\mathcal{H}^n$ [F, Theorem 2.9.2]. In this case we write $(\Pi_\lambda)_*\mu \ll \mathcal{H}^n$.

2 Sliced Measures and Non-Trivial Measures on Visible Parts.

In this section, we will state our main result concerning dimensional properties of sliced measures defined in (1.11) and apply it to visible parts.

According to [JM, Theorem 3.8], if μ is a Radon measure on $\mathbb{R}^d$ with compact support and with $\dim_{\mathrm{H}} \mu > k$, then for almost all $(d-k)$-dimensional linear subspaces V of $\mathbb{R}^d$ we have

$$\operatorname{ess\,inf}\{\dim_{\mathrm{H}} \mu_{V,a} \mid a \in V^\perp \text{ with } \mu_{V,a}(\mathbb{R}^d) > 0\} = \dim_{\mathrm{H}} \mu - k. \tag{2.1}$$

The following analogue of (2.1) will be proved in Section 3.

Theorem 2.1. *Let μ be a Radon measure on M with compact support. Assuming that $\dim_{\mathrm{H}} \mu > n$, we have*

$$\mathcal{H}^n\text{-ess}\inf\{\dim_{\mathrm{H}} \mu_{\lambda,y} \mid y \in N \text{ with } \mu_{\lambda,y}(M) > 0\} = \dim_{\mathrm{H}} \mu - n$$

for $\mathcal{H}^l$-almost all $\lambda \in L$.

Theorem 2.1 is being used as a tool when verifying Theorem 1.1. We proceed by introducing the notation needed for this purpose.

Let k and d be integers such that $0 \le k \le d-1$. The Grassmann manifold of linear k-dimensional subspaces of $\mathbb{R}^d$ and the space of affine k-dimensional subspaces of $\mathbb{R}^d$ are denoted by $G_{d,k}$ and $A_{d,k}$, respectively. (A 0-plane is simply a point.) Letting $\gamma_{d,k}$ be the unique orthogonally invariant Radon probability measure on $G_{d,k}$, and defining for all Borel sets $A \subset A_{d,k}$

$$\Gamma_{d,k}(A) = \int \mathcal{H}^{d-k}(\{a \in V^\perp \mid V + a \in A\})\, d\gamma_{d,k}(V),$$

gives a Radon measure $\Gamma_{d,k}$ on $A_{d,k}$. Observe that both $G_{d,k}$ and $A_{d,k}$ are smooth Riemann manifolds [F, 3.2.28], and $\Gamma_{d,k}$ is equivalent to $\mathcal{H}^s$ with $s = \dim A_{d,k}$.

2.1 Visible Parts.

The visible part of a compact set $E \subset \mathbb{R}^d$ from an affine subspace $K \in A_{d,k}$ with $E \cap K = \emptyset$ is

$$V_K(E) = \{x \in E \mid [\mathrm{Proj}_K(x), x] \cap E = \{x\}\}.$$

Here $\mathrm{Proj}_K(x) = \mathrm{Proj}_V(x) + a$ is the closest point to x on the affine plane $K = V + a$, where $V \in G_{d,k}$ and $a \in V^\perp$, and $[x, y]$ is the closed line segment between x and y.

Remark 2.2. (a) The visible part $V_K(E)$ is a Borel set being the graph of a lower semi-continuous function [JJMO, Remark 2.2 (a)].

(b) Let $\Pi : L \times M \to N$ be as in Section 1, and let $\lambda \in L$ such that $(\Pi_\lambda)_*\mu \ll \mathcal{H}^n$. Then for all Borel sets $B \subset M$

$$\mu_{\lambda,y}|_B = (\mu|_B)_{\lambda,y}$$

for $\mathcal{H}^n$-almost all $y \in N$. This can be verified using similar arguments as in [JM, Lemma 3.2].

PROOF OF THEOREM 1.1. Consider a compact set $E \subset \mathbb{R}^d$ with $\dim_{\mathrm{H}} E > d-1$. Let μ be a Radon measure on E such that $\mu(E) > 0$ and $\dim_{\mathrm{H}} \mu > d-1$. Given $\varepsilon > 0$, let $A_{d,k}(E,\varepsilon)$ consist of those $K \in A_{d,k}$ for which $\operatorname{dist}(K,E) > \varepsilon$. Defining for all $K \in A_{d,k}(E,\varepsilon)$

$$N_K = \{x \in \mathbb{R}^d \mid \operatorname{dist}(x,K) = \varepsilon\},$$

the assumptions of Section 1 are satisfied for the natural projection $\Pi_K : \mathbb{R}^d \to N_K$; the transversality of Π_K follows from [JJMO, (3.4)]. Since $\dim_{\mathrm{H}} \mu > d-1$, Proposition 3.8 (2) gives that $(\Pi_K)_*\mu \ll \mathcal{H}^{d-1}$ for $\Gamma_{d,k}$-almost all $K \in A_{d,k}(E,\varepsilon)$. Defining D as the set of such affine k-planes and using Remark 2.2 (b), we get for all $K \in D$ and for all Borel sets $B \subset E$

$$\mu_{K,y}|_B = (\mu|_B)_{K,y}$$

for $\mathcal{H}^{d-1}$-almost all $y \in N_K$. Note that D depends only on μ. Combining this with Theorem 2.1 implies the existence of $\widetilde{D} \subset D$ (depending again only on μ) such that $\Gamma_{d,k}(A_{d,k}(E,\varepsilon) \setminus \widetilde{D}) = 0$ and for all Borel sets $B \subset E$ we have

$$\begin{aligned} &\mathcal{H}^{d-1}\text{-ess}\inf\{\dim_{\mathrm{H}}(\mu|_B)_{K,y} \mid y \in N_K \text{ with } (\mu|_B)_{K,y}(\mathbb{R}^d) > 0\} \\ &\geq \mathcal{H}^{d-1}\text{-ess}\inf\{\dim_{\mathrm{H}} \mu_{K,y} \mid y \in N_K \text{ with } \mu_{K,y}(\mathbb{R}^d) > 0\} \\ &= \dim_{\mathrm{H}} \mu - (d-1) > 0 \end{aligned} \tag{2.2}$$

for all $K \in \widetilde{D}$.

The final step is to conclude that if $K \in A_{d,k}(E,\varepsilon)$ with $\mu(V_K(E)) > 0$, then

$$K \notin \widetilde{D}. \tag{2.3}$$

The claim follows then by letting ε tend to 0 along a sequence. To verify (2.3), assume to the contrary that $K \in \widetilde{D}$. Recalling Remark 2.2 (a) and applying (2.2) with $B = V_K(E)$, gives

$$\dim_{\mathrm{H}}(V_K(E) \cap \Pi_K^{-1}(\{y\})) > 0$$

for $\mathcal{H}^{d-1}$-almost all $y \in N_K$ with $(\mu|_{V_K(E)})_{K,y}(\mathbb{R}^d) > 0$. Noting that the set $V_K(E) \cap \Pi_K^{-1}(\{y\})$ contains at most one point for all $y \in N_K$, this gives a contradiction, since

$$\mathcal{H}^{d-1}(\{y \in N_K \mid (\mu|_{V_K(E)})_{K,y}(\mathbb{R}^d) > 0\}) > 0$$

by (1.13). □

3 Proof of Theorem 2.1.

This section is dedicated to the verification of Theorem 2.1. Our methods combine those of [JM] and [PS]. We begin by proving the following generalization of [M1, Lemma 3.11].

Proposition 3.1. *Assume that $B \subset L$ is bounded. Then there are constants $c > 0$ and $\delta_0 > 0$ such that for all $x_1 \neq x_2 \in M$ and $0 < \delta < \delta_0$ we have*

$$\mathcal{H}^l(\{\lambda \in B \mid \rho_N(\Pi_\lambda(x_1), \Pi_\lambda(x_2)) \leq \delta\}) \leq c\delta^n \rho_M(x_1, x_2)^{-n}.$$

Proposition 3.1 is obtained as an outcome of a sequence of lemmas (Lemmas 3.2–3.7) in which the rôle of the transversality condition (1.8) is crucial. The basic idea of the proof is similar to that of [PS, Lemma 7.7]. Note that in the setting of [PS] both L and N are Euclidean spaces whereas M is a metric space.

Let R and C_t be as in (1.6) and (1.7), respectively, and let $R_1 = R/(3c_1)$ where c_1 is as in (1.5). Consider $\lambda_0 \in L$ and $x_1 \neq x_2 \in M$ such that $\rho_N(\Pi_{\lambda_0}(x_1), \Pi_{\lambda_0}(x_2)) \leq R/3$ and $|\Phi_{x_1,x_2}(\lambda_0)| < C_t$. Picking coordinates $(\eta_1, \ldots, \eta_l)$ in $B(\lambda_0, R_1)$ and applying transversality property (1.8) and the Cauchy–Binet theorem, we find an $(n \times n)$-minor $A(\lambda_0)$ of $D\Phi_{x_1,x_2}(\lambda_0)$ with

$$|\det A(\lambda_0)| \geq \tilde{c}\, C_t. \tag{3.1}$$

Here $A(\lambda_0)$ is determined with respect to the coordinates $(\tilde{\eta}_1, \ldots, \tilde{\eta}_n)$ induced by $(\eta_1, \ldots, \eta_n, \ldots, \eta_l)$ and $\tilde{c}$ is a positive constant depending on l and n. Given $\lambda = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1}, \ldots, \lambda_l) \in B(\lambda_0, R_1)$, set

$$H_\lambda = \{\lambda' \in B(\lambda_0, R_1) \mid \lambda' = (\lambda'_1, \ldots, \lambda'_n, \lambda_{n+1}, \ldots, \lambda_l)\}.$$

Defining a function $\psi_\lambda : H_\lambda \to \mathbb{R}^n$ by

$$\psi_\lambda(\lambda') = \Phi_{x_1,x_2}(\lambda')$$

for all $\lambda' \in H_\lambda$, the following lemma holds. (Observe that by (1.5) and the choice of R_1 the function ψ_λ is well defined.)

Lemma 3.2. *There exists $0 < R_0 \leq R_1$ which is independent of λ_0 such that the following properties hold:*

1. *For all $\lambda \in B(\lambda_0, R_0)$ the absolute values of the singular values of $D_\lambda \psi_\lambda$ are bounded below and above by positive constants that do not depend on λ_0 and λ.*

2. *For all* $\lambda \in B(\lambda_0, R_0)$ *the function* $\psi_\lambda : B(\lambda, \widetilde{R}/3) \cap H_\lambda \to \psi_\lambda(B(\lambda, \widetilde{R}/3) \cap H_\lambda)$ *is a diffeomorphism, and*

$$\psi_\lambda(B(\lambda, \rho) \cap H_\lambda) \supset B(\psi_\lambda(\lambda), d\rho)$$

for all $0 < \rho \leq \widetilde{R}$. *Here* $\widetilde{R}$ *and* d *are independent of* λ_0 *and* λ.

Proof. (1) From (1.9) we see that the absolute values of the singular values of $(D\psi_{\lambda_0})(\lambda_0)$ are bounded above by a constant $C(\widetilde{L})$. Furthermore, since $(D\psi_{\lambda_0})(\lambda_0) = A(\lambda_0)$, inequality (3.1) implies that the absolute values of the singular values of $(D\psi_{\lambda_0})(\lambda_0)$ are bounded below by the constant $\tilde{c}\, C_t/C(\widetilde{L})^{n-1}$. The claim follows since the function $\lambda \mapsto (D\Phi_{x_1,x_2})(\lambda)$ is uniformly continuous by (1.9).

(2) Defining a function $\widetilde{\psi}_\lambda : H_\lambda \to \mathbb{R}^n$ by

$$\widetilde{\psi}_\lambda(\lambda') = (D\psi_\lambda)(\lambda)^{-1}\psi_\lambda(\lambda')$$

and using the uniform continuity of the function $\lambda \mapsto (D\Phi_{x_1,x_2})(\lambda)$, we see that the derivative $D\widetilde{\psi}_\lambda(\lambda') = D\psi_\lambda(\lambda)^{-1}D\psi_\lambda(\lambda')$ is close to the identity in some ball $B(\lambda, \widetilde{R}) \cap H_\lambda$. (Here $\widetilde{R}$ does not depend on λ_0 and λ.) Applying [PS, Lemma 7.6] gives that $\widetilde{\psi}_\lambda : B(\lambda, \widetilde{R}/3) \cap H_\lambda \to \widetilde{\psi}_\lambda\big(B(\lambda, \widetilde{R}/3) \cap H_\lambda\big)$ is a diffeomorphism, and $\widetilde{\psi}_\lambda\big(B(\lambda, \rho) \cap H_\lambda\big) \supset B(\widetilde{\psi}_\lambda(\lambda), \rho/2)$ for all $0 < \rho \leq \widetilde{R}$. This in turn completes the proof of (2). Note that the constant d depends on the lower bound of the absolute values of the eigenvalues of $D\psi_\lambda(\lambda)$, and therefore, by (1), it is independent of λ_0 and λ. □

According to the next lemma, there is a zero of the function Φ_{x_1,x_2} close to each parameter λ for which $|\Phi_{x_1,x_2}(\lambda)|$ is small enough.

Lemma 3.3. *Suppose that* $x_1 \neq x_2 \in M$ *and* $\lambda \in L$ *such that* $\rho_N(\Pi_\lambda(x_1), \Pi_\lambda(x_2)) \leq R/3$. *Then, given any* $0 < \delta < \min\{C_t, \widetilde{R}d/4\}$, *the condition* $|\Phi_{x_1,x_2}(\lambda)| < \delta$ *implies the existence of* $\tilde{\lambda} \in L$ *such that* $\Phi_{x_1,x_2}(\tilde{\lambda}) = 0$ *and* $\rho_L(\lambda, \tilde{\lambda}) \leq \delta/d \leq \widetilde{R}/4$.

Proof. Lemma 3.2 gives the inclusion

$$\psi_\lambda\big(B(\lambda, \delta/d) \cap H_\lambda\big) \supset B(\Phi_{x_1,x_2}(\lambda), \delta)$$

which implies the claim. □

Given any $x_1 \neq x_2 \in M$ and $\tilde{\lambda} \in L$ with $\Phi_{x_1,x_2}(\tilde{\lambda}) = 0$, define an $(l-n)$-dimensional submanifold $L_{x_1,x_2}(\tilde{\lambda})$ of L by

$$L_{x_1,x_2}(\tilde{\lambda}) = \Phi^{-1}_{x_1,x_2}(0) \cap B(\tilde{\lambda}, \widetilde{R}).$$

Lemma 3.4. *Let $x_1 \neq x_2 \in M$ and $\tilde\lambda \in L$ such that $\Phi_{x_1,x_2}(\tilde\lambda) = 0$. Then for all $0 < \delta < \widetilde{R}d/4$ we have*

$$\{\lambda \in B(\tilde\lambda, \widetilde{R}/2) \mid |\Phi_{x_1,x_2}(\lambda)| \leq \delta\} \subset \bigcup_{\lambda' \in L_{x_1,x_2}(\tilde\lambda)} B(\lambda', \delta/d).$$

Proof. We may assume that $\widetilde{R} \leq 2R_0$. Then $\Phi_{x_1,x_2}(\lambda)$ is defined for all $\lambda \in B(\tilde\lambda, \widetilde{R}/2)$. Letting $\lambda \in B(\tilde\lambda, \widetilde{R}/2)$ such that $|\Phi_{x_1,x_2}(\lambda)| < \delta$ and using Lemma 3.3, we find $\lambda' \in B(\lambda, \delta/d) \cap H_\lambda$ such that $\Phi_{x_1,x_2}(\lambda') = 0$. Since $\lambda' \in B(\tilde\lambda, \widetilde{R})$, the claim follows. □

Let $\delta_0 = \min\{C_t, \widetilde{R}d/4, R/3\}$. Consider $x_1 \neq x_2 \in M$. For the proof of Lemma 3.1 we may assume that $B \subset V$ for some V defined in (1.6) and $K\delta \leq \delta_0 \rho_M(x_1, x_2)$. Recalling that the Lipschitz constants of φ, φ^{-1}, ϕ, and ϕ^{-1} are uniformly bounded above by K, and applying Lemma 3.3, one finds a constant N (depending only on K, $\widetilde{R}$, l, and the diameter of B) and $\tilde\lambda_1, \ldots, \tilde\lambda_N \in L$ with $\Phi_{x_1,x_2}(\tilde\lambda_i) = 0$ such that

$$\begin{aligned}&\{\lambda \in B \mid \rho_N(\Pi_\lambda(x_1), \Pi_\lambda(x_2)) \leq \delta\}\\ &\subset \bigcup_{i=1}^{N} \{\lambda \in B(\tilde\lambda_i, \widetilde{R}/2) \mid |\Phi_{x_1,x_2}(\lambda)| \leq K\delta/\rho_M(x_1, x_2)\}.\end{aligned}$$

Defining for all $\tilde\lambda \in L$ with $\Phi_{x_1,x_2}(\tilde\lambda) = 0$ and $\delta > 0$

$$N\big(L_{x_1,x_2}(\tilde\lambda), \delta/d\big) = \bigcup_{\lambda' \in L_{x_1,x_2}(\tilde\lambda)} B(\lambda', \delta/d),$$

we see from Lemma 3.4 that Lemma 3.1 is an immediate consequence of the following result.

Lemma 3.5. *Given $x_1 \neq x_2 \in M$ and $\tilde\lambda \in L$ with $\Phi_{x_1,x_2}(\tilde\lambda) = 0$ and $0 < \delta < \delta_0$, we have*

$$\mathcal{H}^l\big(N(L_{x_1,x_2}(\tilde\lambda), \delta/d)\big) \leq c\delta^n.$$

Here the constant c is independent of $\tilde\lambda$ and δ.

Lemmas 3.6 and 3.7, in turn, lead to Lemma 3.5.

Lemma 3.6. *Let $x_1 \neq x_2 \in M$ and $\tilde\lambda \in L$ such that $\Phi_{x_1,x_2}(\tilde\lambda) = 0$. Then there exists $\alpha > 0$ which is independent of $\tilde\lambda$ such that for all $\lambda \in B(\tilde\lambda, \widetilde{R})$, $h \in T_\lambda H_\lambda$ ($T_\lambda H_\lambda$ is the tangent space of H_λ at λ), and $y \in T_\lambda L_{x_1,x_2}(\tilde\lambda)$ with $\|h\| = \|y\| = 1$, we have $\sphericalangle(h, y) \geq \alpha$. (Here $\sphericalangle(h, y)$ is the angle between h and y.)*

Proof. Let $p = h - y \in T_\lambda L$. Denoting by c the lower bound given in Lemma 3.2 (a) and using (1.9), we get

$$c \le |D_\lambda \psi_\lambda(\lambda)h| = |(D_\lambda \Phi_{x_1,x_2})(\lambda)h| = |(D_\lambda \Phi_{x_1,x_2})(\lambda)p| \le c(\widetilde{L})\|p\|$$

since $(D_\lambda \Phi_{x_1,x_2})(\lambda)y = 0$. This completes the proof. □

For $x_1 \ne x_2 \in M$ and $\tilde\lambda = (\tilde\lambda_1, \dots, \tilde\lambda_l) \in L$ with $\Phi_{x_1,x_2}(\tilde\lambda) = 0$, define

$$V_{\tilde\lambda} = \{\lambda' \in L \mid \lambda' = (\tilde\lambda_1, \dots, \tilde\lambda_n, \lambda'_{n+1}, \dots, \lambda'_l)\}.$$

From Lemma 3.6 we immediately get the following result which, in turn, completes the proof of Lemma 3.1 by verifying Lemma 3.5.

Lemma 3.7. *Given $x_1 \ne x_2 \in M$, $\tilde\lambda \in L$ with $\Phi_{x_1,x_2}(\tilde\lambda) = 0$ and $0 < \delta < \delta_0$, there is an integer $I \le c(l,n)(\widetilde{R}/\delta)^{l-n}$ and a covering of $V_{\tilde\lambda} \cap B(\tilde\lambda, \widetilde{R})$ with balls $B_1, \dots, B_I$ of radius δ which induces a covering of $N\big(L_{x_1,x_2}(\tilde\lambda), \delta/d\big)$ with balls $\widetilde{B}_1, \dots, \widetilde{B}_I$ of radius $c(\alpha)\delta$.*

One more tool is needed for the verification of Theorem 2.1. The rôle of Lemma 3.1 is crucial in the following proof which combines the methods from [FM, Lemma 4.1] and [M1, Theorem 9.7]. For reader's convenience we will give a brief outline of the proof in our setting.

Given locally finite measures ν_1 and ν_2 on a metric space (X, ρ), denote by $\underline{\mathrm{D}}(\nu_1, \nu_2, x)$ the lower derivative of ν_1 with respect to ν_2 at a point $x \in X$; that is,

$$\underline{\mathrm{D}}(\nu_1, \nu_2, x) = \liminf_{r \to 0} \frac{\nu_1(B(x,r))}{\nu_2(B(x,r))}.$$

Moreover, for $s \ge 0$, the s-energy $I_s(\nu)$ of a measure ν is defined as

$$I_s(\nu) = \iint \rho(x,y)^{-s}\, d\nu(x)\, d\nu(y).$$

Proposition 3.8. *Let μ be a Radon measure on M with compact support. Then the following properties hold:*

1. *We have $(\Pi_\lambda)_*\mu \ll \mathcal{H}^n$ if and only if $\underline{\mathrm{D}}((\Pi_\lambda)_*\mu, \mathcal{H}^n, y) < \infty$ for $(\Pi_\lambda)_*\mu$-almost all $y \in N$.*

2. *Both $\dim_{\mathrm{H}} \mu > n$ and $I_n(\mu) < \infty$ imply that $(\Pi_\lambda)_*\mu \ll \mathcal{H}^n$ for $\mathcal{H}^l$-almost all $\lambda \in L$.*

PROOF. (1) Supposing that $(\Pi_\lambda)_*\mu \ll \mathcal{H}^n$, the finiteness of the lower derivative follows directly from (1.11). On the other hand, assume that $\underline{D}((\Pi_\lambda)_*\mu, \mathcal{H}^n, y) < \infty$ for $(\Pi_\lambda)_*\mu$-almost all $y \in N$. Given $A \subset N$ with $\mathcal{H}^n(A) = 0$, it is sufficient to prove that

$$(\Pi_\lambda)_*\mu\{y \in A \mid \underline{D}((\Pi_\lambda)_*\mu, \mathcal{H}^n, y) < c\} = 0$$

for all $0 < c < \infty$. This follows from [F, Lemma 2.9.3] by choosing $\alpha = \phi = (\Pi_\lambda)_*\mu$ and $\beta = \mathcal{H}^n$ therein. (Recall that the family $\{(y, B(y,r)) \mid y \in N, r > 0\}$ is both a $(\Pi_\lambda)_*\mu$-Vitali relation and a $\mathcal{H}^n$-Vitali relation, and therefore this choice is possible.)

(2) Let $\dim_{\mathrm{H}} \mu > s > n$. Defining for all $i = 1, 2, \ldots$ a restriction measure $\mu_i = \mu|_{M_i}$ where

$$M_i = \{x \in M \mid \mu(B(x,r)) \le ir^s \text{ for all } r > 0\},$$

we have $I_n(\mu_i) < \infty$. Moreover, it follows from Fatou's lemma, Fubini's theorem, and Lemma 3.1 that

$$\int_B \int \underline{D}((\Pi_\lambda)_*\mu_i, \mathcal{H}^n, y)\, d(\Pi_\lambda)_*\mu_i(y)\, d\mathcal{H}^l(\lambda) \le cI_n(\mu_i)$$

for all bounded sets $B \subset L$. From (1) we get $(\Pi_\lambda)_*\mu_i \ll \mathcal{H}^n$ for $\mathcal{H}^l$-almost all $\lambda \in L$. This settles the claim since $x \in \cup_i M_i$ for μ-almost all $x \in M$. □

Now we are ready to prove Theorem 2.1.

PROOF OF THEOREM 2.1. The methods from the proof of [JM, Theorem 3.8] can be extended in a straightforward manner to our setting. Indeed, similarly as in the proof of [JM, Theorem 3.8], we see from the disintegration formula (1.13) that

$$\mathcal{H}^n\text{-ess}\inf\{\dim_{\mathrm{H}} \mu_{\lambda,y} \mid y \in N \text{ with } \mu_{\lambda,y}(M) > 0\} \le \dim_{\mathrm{H}} \mu - n$$

for $\mathcal{H}^l$-almost all $\lambda \in L$. The opposite inequality reduces to proving that for $\mathcal{H}^l$-almost all $\lambda \in L$ the set

$$\begin{aligned} E_\lambda = \{y \in N \mid & \text{there is a Borel set } A \subset \pi_\lambda^{-1}(\{y\}) \\ & \text{with } \dim_{\mathrm{H}} A < s \text{ and } \mu_{\lambda,y}(A) > \varepsilon\} \end{aligned}$$

has $\mathcal{H}^n$ measure zero for fixed $s < \dim_{\mathrm{H}} \mu - n$ and $\varepsilon > 0$. This follows from (1.13), Proposition 3.8, and Remark 2.2 (b). (For details see the proof of [JM, Theorem 3.8].) □

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