Real Analysis Exchange
Vol. 30(2), 2004/2005, pp. 703–718

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ON GREEN'S THEOREM AND CAUCHY'S THEOREM

In memory of Pawel Szeptycki

Abstract

Green's Theorem is proved using only the geometric (or physical) definition of curl, without the use of partial derivatives. The curl free (conservative) case can then be used to prove Cauchy's Theorem.

1 Introduction.

Tradition has it that in order to deduce Cauchy's Integral Theorem from Green's Theorem, one must assume continuity of the partial derivatives in the Cauchy-Riemann equations. A generalization of this hypothesis requires that the pointwise partial derivatives are distribution derivatives in L^2_{loc}. The continuity then follows from elliptic regularity of the Cauchy-Riemann operator (cf. [2]). But this is still an extra assumption as is seen by the example of Cantor's singular function. For Cantor's function, the usual (pointwise) derivative exists and is zero almost everywhere, but the distribution derivative is the corresponding Lebesgue-Stieltjes measure (by the Riesz Representation Theorem, cf. [8]).

In the next section concerning continuous two dimensional vector fields we show that an extra hypothesis is unnecessary if curl is treated as a strictly geometric measure of circulation intensity. It is sufficient that the limit defining curl be uniform on compacta. When curl vanishes identically, this uniformity is equivalent to zero circulation around every square with sides parallel to the coordinate axes. This is analogous to the proof of Cauchy's Theorem for a square, and is accomplished, as there, by bisection (cf. [15]). Thus by verifying that complex contour integrals of holomorphic functions satisfy

Key Words: Green's Theorem, Cauchy's Theorem
Mathematical Reviews subject classification: Primary 53A45, 30E20
Received by the editors November 15, 2004
Communicated by: Stefan Schwabik

a condition implying Green's Theorem, one obtains Cauchy's Theorem from Green's Theorem.

Our hypothesis that the limit defining curl is uniform on compacta enables one to mimic the proof of Green's Theorem given in [3]. But that proof requires a rather intricate geometric construction to treat general boundaries. In contrast, the treatment of complex contour integrals over simple closed rectifiable oriented curves to prove Cauchy's Theorem involves considerably less geometrical insight (cf. [3, Chapter 9] or [10]). In the treatment of Green's Theorem in many references, this issue is avoided by only considering curves which are also smooth or piecewise smooth manifolds (cf. [16]). But as is pointed out in [9], there are quite elementary examples of simple closed rectifiable oriented curves which are not even topological manifolds. Our proof of Green's Theorem is constructed so that the boundary integral is treated in a fashion analogous to the complex contour integral case. The key is the observation that when one has a parameterization, integration over a rectifiable curve which may not be simple is not a problem. The end result is some simplification to an exposition including both the theorems of Green and Cauchy, as compared with, for example, [14]. We also obtain that a continuous radial vector field is conservative even if nowhere pointwise differentiable.

In the final section, we extend Green's Theorem by regularization to vector fields with integrable partial derivatives in the sense of distributions. Some vector fields with stronger singularities are then examined geometrically, and it is shown that Cauchy's Integral Formula of complex analysis is a natural consequence of the geometric approach. We then close with comments on two dimensional divergence, and a geometric characterization of holomorphy.

2 Results for Continuous Vector Fields.

Let Ω be a domain in $\mathbb{R}^2$, and let $\mathbb{F} = (P, Q) : \Omega \to \mathbb{R}^2$ be continuous. For $(x_0, y_0) \in \Omega$ and $0 < \epsilon < \operatorname{dist}((x_0, y_0), \partial\Omega)$, where $\partial\Omega$ is the boundary of Ω, let R_ϵ be the open square $\{(x, y) : |x - x_0| < \epsilon/2 \text{ and } |y - y_0| < \epsilon/2\}$. Let C_ϵ denote the positively oriented (square) curve whose range is ∂R_ϵ. Recall that the circulation of $\mathbb{F}$ around C_ϵ is

$$\int_{C_\epsilon} \mathbb{F} \cdot d\mathbf{r} = \int_{C_\epsilon} (P\,dx + Q\,dy),$$

and that positive orientation means in the counterclockwise sense. The circulation intensity, or curl, of $\mathbb{F}$ at (x_0, y_0) is defined to be

$$\operatorname{curl} \mathbb{F}(x_0, y_0) = \lim_{\epsilon \to 0} \frac{\int_{C_\epsilon} \mathbb{F} \cdot d\mathbf{r}}{\text{Area } (R_\epsilon)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_{C_\epsilon} \mathbb{F} \cdot d\mathbf{r}, \tag{1}$$

provided the limit exists. For the present, we will assume that the limit in (1) is uniform on compact subsets of Ω; i.e., if K is a compact subset of Ω with dist $(K, \partial\Omega) = d$, for each $\eta > 0$ there exists $\epsilon_0 \leq d$ such that whenever $0 < \epsilon < \epsilon_0$,

$$\left|\text{curl } \mathbb{F}(x_0, y_0) - \frac{1}{\epsilon^2}\int_{C_\epsilon} \mathbb{F} \cdot d\mathbf{r}\right| < \eta \text{ for all } (x_0, y_0) \in K.$$

Since $\mathbb{F}$ is continuous, this implies that curl $\mathbb{F}$ is continuous.

Theorem 1. (Green's Theorem) *Let C be a simple closed rectifiable oriented curve with interior R and $\overline{R} = R \cup \partial R \subset \Omega$. Then if the limit in (1) is uniform on compact subsets of Ω,*

$$\int_R \text{curl } \mathbb{F} \, dA = \int_C \mathbb{F} \cdot d\mathbf{r}.$$

Before considering the proof of Theorem 1, we proceed to show how it implies Cauchy's Theorem. For this, we need part ii) of the following lemma. The notation $\Omega' \subset\subset \Omega$ is used to denote that Ω' is a set whose closure $\overline{\Omega}'$ is a compact subset of Ω.

Lemma 2. *Let K be a compact subset of Ω which is the closure of its interior, and let Ω' be open with $K \subset \Omega' \subset\subset \Omega$. Then:*

i) the limit in (1) is uniform on K if $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$ are continuous on Ω'; and

ii) if curl $\mathbb{F} = 0$ on Ω', the limit in (1) is uniform on K if and only if for every square R_ϵ as above with center in K and closure in Ω', the circulation around C_ϵ is zero.

PROOF. Criterion i) is classical and is a consequence of the identity,

$$\int_{C_\epsilon} \mathbb{F} \cdot d\mathbf{r} = \int_{R_\epsilon} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA,$$

since then

$$\left|\frac{1}{\epsilon^2}\int_{C_\epsilon} \mathbb{F} \cdot d\mathbf{r} - \left(\frac{\partial Q}{\partial x}(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0)\right)\right|$$
$$\leq \sup_{(x,y) \in R_\epsilon} \left|\left(\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y)\right) - \left(\frac{\partial Q}{\partial x}(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0)\right)\right|,$$

and uniform convergence on compacta follows from uniform continuity there. In case ii) sufficiency is obvious. For necessity, let R_ϵ be any open square with center in K, sides parallel to the coordinate axes, and closure in Ω'. Further, let

$$k = \left| \int_{C_\epsilon} \mathbb{F} \cdot d\mathbf{r} \right|.$$

Then if R_ϵ is subdivided into four congruent squares, at least one, call it $R_{1,\epsilon}$, satisfies

$$\left| \int_{C_{1,\epsilon}} \mathbb{F} \cdot d\mathbf{r} \right| \geq \frac{k}{4}.$$

Continuing by induction, at the n^{th} stage R_ϵ is subdivided into 4^n congruent squares, at least one of which, $R_{n,\epsilon}$, with $R_{n,\epsilon} \subset R_{n-1,\epsilon}$ satisfies,

$$\left| \int_{C_{n,\epsilon}} \mathbb{F} \cdot d\mathbf{r} \right| \geq \frac{k}{4^n}.$$

There is exactly one point $(\widehat{x}, \widehat{y})$ common to all the nested closed squares $\overline{R}_{n,\epsilon}$ and there

$$\begin{aligned} \text{curl } \mathbb{F}(\widehat{x}, \widehat{y}) &= \lim_{n\to\infty} \text{ curl } \mathbb{F}(x_n, y_n) \\ &= \lim_{n\to\infty} \frac{1}{(\epsilon/2^n)^2} \int_{C_{n,\epsilon}} \mathbb{F} \cdot \ d\mathbf{r} \\ &= \lim_{n\to\infty} \frac{4^n}{\epsilon^2} \int_{C_{n,\epsilon}} \mathbb{F} \cdot d\mathbf{r} = 0, \end{aligned}$$

by local uniformity of the limit in (1). So $k = 0$, and we are done. □

To prove Cauchy's Theorem recall that if Ω is a domain in $\mathbb{C}$, a function $f : \Omega \to \mathbb{C}$ is holomorphic on Ω if

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{2}$$

exists at each $z_0 = x_0 + iy_0 \in \Omega$. Herein also $z = x + iy$, and i is the imaginary unit. Then f is continuous on Ω, and we may write (2) as

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\eta(z), \tag{3}$$

where $\eta(z)$ is continuous on Ω and $\eta(z) \to 0$ as $z \to z_0$. With C_ϵ as previously we may integrate both sides of (3) around C_ϵ, and obtain

$$\int_{C_\epsilon} f(z)dz = \int_{C_\epsilon} (z - z_0)\eta(z)dz,$$

since 1 and z have primitives. But the magnitude of the latter integral is dominated by $2\sqrt{2}\epsilon^2 \max_{\partial R_\epsilon} |\eta(z)|$. Thus

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_{C_\epsilon} f(z)\, dz = 0,$$

and if $f(z) = f(x+iy) = u(x,y) + iv(x,y)$,

$$\int_{C_\epsilon} f(z)\, dz = \int_{C_\epsilon} (u\, dx - v\, dy) + i \int_{C_\epsilon} (v\, dx + u\, dy).$$

So holomorphy of f on Ω implies that both of the real vector fields $(u, -v)$ and (v, u) are continuous with vanishing curl on Ω. To verify that ii) of Lemma 2 applies, fix ϵ and let

$$k = \left| \int_{C_\epsilon} f(z)\, dz \right|.$$

Then bisection as in the proof of ii) of Lemma 2 yields the inequality

$$\frac{k}{4^n} \leq \left| \int_{C_{n,\epsilon}} f(z)\, dz \right| = \left| \int_{C_{n,\epsilon}} (z - z_0)\eta(z)\, dz \right| \leq \frac{4\epsilon}{2^n} \cdot \frac{\sqrt{2}\epsilon}{2^n} \max_{\partial R_{n,\epsilon}} |\eta(z)|,$$

where z_0 is the limit of the centers of the nested squares $\{C_{n,\epsilon}\}$. Thus

$$k \leq 4\sqrt{2}\epsilon^2 \max_{\partial R_{n,\epsilon}} |\eta(z)| \to 0 \text{ as } n \to \infty.$$

Thus $k = 0$, so by Lemma 2 and Theorem 1, we have proved the following.

Theorem 3. *(Cauchy's Theorem) Let C be a simple closed rectifiable oriented curve in Ω with interior R and $\overline{R} \subset \Omega$. Then if f is holomorphic on Ω,*

$$\int_C f(z)\, dz = 0.$$

Proof of Theorem 1. We first use classical geometric ideas (cf. [13]) to prove the theorem for the square $R = \{(x,y) : |x - x_0| < \ell/2 \text{ and } |y - y_0| < \ell/2\}$. For each positive integer m, consider the subdivision of R by the grid of m^2 congruent squares of side length ℓ/m, and pick $\eta > 0$. Since the limit in (1) is uniform on compacta, there exists an integer M such that for each square $R_{j,\epsilon}$ of the grid with $\epsilon = \ell/M$,

$$\left| \epsilon^2 \operatorname{curl} \mathbb{F}(x_j, y_j) - \int_{C_{j,\epsilon}} \mathbb{F} \cdot d\mathbf{r} \right| < \eta\epsilon^2,$$

where (x_j, y_j) is the center of $R_{j,\epsilon}$. Addition over the M^2 squares of the grid gives

$$\Big|\sum_j \text{curl } \mathbb{F}(x_j, y_j)\epsilon^2 - \int_C \mathbb{F}\cdot d\mathbf{r}\Big| < \eta\ell^2,$$

and this inequality holds if M is increased. But the sum is a Riemann sum for $\int_R \text{curl } \mathbb{F}\, dA$, and η is arbitrary. So the theorem is proved for squares with sides parallel to the coordinate axes.

We next use ideas from [3] to prove the theorem for a polygon. Since every polygon can be triangulated (cf. [10]), a standard argument shows that it suffices to prove that the theorem holds whenever R is an open triangle with $\overline{R} \subset \Omega$.

For $\epsilon > 0$ let $S(\epsilon)$ be the collection of open squares in $\mathbb{R}^2$ determined by the lines $x = m\epsilon,\ y = m\epsilon,\ m = 0, \pm1, \pm2, \ldots$. Then R can be covered by a finite union of closures $\overline{R}_1, \overline{R}_2, \ldots, \overline{R}_n$ of squares of $S(\epsilon)$, where $R_1, \ldots, R_k, k < n$, are all the squares in $S(\epsilon)$ whose closures are contained in R, and $R_{k+1}, \ldots, R_n$ are the squares in $S(\epsilon)$ whose closures intersect ∂R. We discard any of $R_{k+1}, \ldots, R_n$ which have empty intersections with R. It is obvious that if $\mathcal{R}_k = \text{Int}\Big(\bigcup_{j=1}^{k} \overline{R}_j\Big)$,

$$\int_{\mathcal{R}_k} \text{curl}\,\mathbb{F}\, dA = \int_R \chi_{\mathcal{R}_k} \text{curl}\,\mathbb{F}\, dA = \int_{\mathcal{C}_k} \mathbb{F}\cdot d\mathbf{r}, \tag{4}$$

where $\chi_{\mathcal{R}_k}$ is the characteristic function of $\mathcal{R}_k$, and $\mathcal{C}_k$ is the positively oriented curve whose range is $\partial\mathcal{R}_k$. The dominated convergence theorem (the restricted version for Riemann integrals, cf. [11], is sufficient) implies that the left side of (4) converges to $\displaystyle\int_R \text{curl}\,\mathbb{F}\, dA$ as $\epsilon \to 0$.

To find the limit of the right side of (4) first let $B_j = R \bigcap R_j,\ j = k + 1, \ldots, n$, and note that by construction, none of the sets B_j is empty. Then each set B_j is either a triangle or a convex quadrilateral, and

$$\int_C \mathbb{F}\cdot d\mathbf{r} = \int_{\mathcal{C}_k} \mathbb{F}\cdot d\mathbf{r} + \sum_{j=k+1}^{n} \int_{C_j} \mathbb{F}\cdot d\mathbf{r},$$

where C_j is the positively oriented curve with range ∂B_j. It remains to show that the latter sum of integrals tends to zero with ϵ. For this purpose, let N be the greatest integer less than or equal to $L(C)/\epsilon$, where $L(C)$ is the length of C. Starting at one corner of the triangle R, decompose C into N arcs of length ϵ, and possibly one of length less than ϵ. Of such arcs, the linear ones can (by construction) intersect at most three of $\overline{R}_{k+1}, \ldots, \overline{R}_n$, while the at most two arcs with corners can intersect at most four of $\overline{R}_{k+1}, \ldots, \overline{R}_n$. Thus $n - k < 8 + 3(N-1) = 5 + 3N \leq 5 + 3L(C)/\epsilon$. For $j = k+1, \ldots, n$, let

$$\rho_j = \max_{\partial B_j} P - \min_{\partial B_j} P, \;\; \sigma_j = \max_{\partial B_j} Q - \min_{\partial B_j} Q,$$

and choose $\eta > 0$. Now note, as in [3], that for any constant K

$$\int_{C_j} P\,dx = \int_{C_j} (P - K)\,dx,$$

so by choosing $K = \dfrac{1}{2}\left(\max\limits_{\partial B_j} P - \min\limits_{\partial B_j} P\right)$ one obtains $|P(x,y) - K| \leq \frac{1}{2}\rho_j$ for all $(x,y) \in \partial B_j$ and so

$$\left|\int_{C_j} P\,dx\right| \leq \frac{1}{2}\rho_j L(C_j). \tag{5}$$

Similarly,

$$\left|\int_{C_j} Q\,dy\right| \leq \frac{1}{2}\sigma_j L(C_j), \tag{6}$$

and so by continuity of P and Q there exists $\epsilon_0 \leq 1$ such that if $0 < \epsilon < \epsilon_0$,

$$\left|\int_{C_j} (P\,dx + Q\,dy)\right| \leq \frac{1}{2}(\rho_j + \sigma_j)L(C_j) < \eta L(C_j)$$

for all $j = k+1, \ldots, n$. But then

$$\sum_{j=k+1}^{n} L(C_j) \leq L(C) + 4\epsilon(n-k) < 20 + 12L(C),$$

so

$$\left|\sum_{j=k+1}^{n} \int_{C_j} \mathbb{F} \cdot d\mathbf{r}\right| < \eta(20 + 12L(C)).$$

Since η is arbitrary, this concludes the proof for polygons.

We are now ready to prove the theorem in full generality. Let π be a polygonal path approximation to the rectifiable curve C, with $\pi \bigcup \operatorname{Int}\pi \subset \Omega$. As noted in [10], π may enclose more than one open connected polygonal region. If this is the case, parameterization of π will not yield a simple curve.

This is of concern in the non-conservative case due to the reversal of orientation at crossing points. We proceed by first parameterizing π.

For this purpose, pick $\eta > 0$. By continuity of P and Q, each point (x, y) on ∂R is the center of an open disk $D(x, y)$ such that both

$$\rho_{x,y} = \sup P - \inf P \text{ and } \sigma_{x,y} = \sup Q - \inf Q$$

are less than $\eta/2L(C)$, where the suprema and infima are over $D(x, y)$. Since ∂R is compact, there are a finite number $D_1, D_2, \ldots, D_M$ of such disks which cover ∂R, with corresponding quantities ρ_j and σ_j for $j = 1, \ldots, M$. Pick the polygonal path π such that the endpoints of each linear segment lie within a single disk D_j. There may be more than M linear segments in π, $N \geq M$ say. Call the endpoints of the linear segments of π, ordered with the positive orientation of $C, P_1, P_2, \ldots, P_{N+1} = P_1$. For each $j = 1, 2, \ldots, N$ we form a rectifiable oriented closed curve C_j, in general not simple, as follows. Proceed from P_j to P_{j+1} along the linear segment of π joining P_j to P_{j+1} and then proceed back from P_{j+1} to P_j along C via reverse parameterization of C. Then

$$\int_{C_j} \mathbb{F} \cdot d\mathbf{r} = \int_{\pi_j} \mathbb{F} \cdot d\mathbf{r} - \int_{\gamma_j} \mathbb{F} \cdot d\mathbf{r},$$

where π_j and γ_j are the corresponding arcs with orientation induced by π and C, respectively. By refinement, if necessary, all the points of each curve C_j lie in a single disk D_j . The proof of inequalities (5) and (6) did not require that the rectifiable oriented closed curves C_j be simple. Thus

$$\left| \int_{C_j} \mathbb{F} \cdot d\mathbf{r} \right| = \left| \int_{\pi_j} \mathbb{F} \cdot d\mathbf{r} - \int_{\gamma_j} \mathbb{F} \cdot d\mathbf{r} \right| < \eta \frac{L(C_j)}{2L(C)}$$

for each $j = 1, 2, \ldots, N$. Since

$$\sum_{j=1}^{N} L(C_j) = \sum_{j=1}^{N} L(\pi_j) + \sum_{j=1}^{N} L(\gamma_j) = L(\pi) + L(C) \leq 2L(C),$$

addition gives

$$\left| \int_{\pi} \mathbb{F} \cdot d\mathbf{r} - \int_{C} \mathbb{F} \cdot d\mathbf{r} \right| < \eta. \tag{7}$$

Note that (7) remains true if the polygonal approximation π is further refined.

To complete the proof, let $\Lambda(\delta) = \{(x, y) : \operatorname{dist}((x, y), \partial R) < \delta\}$, and assume that δ is small enough that $\Lambda(\delta) \subset \Omega$. By refinement, if necessary, we may assume that the polygonal path π is contained in $\Lambda(\delta)$. Then

$$\int_{R \setminus \Lambda(\delta)} \operatorname{curl} \mathbb{F} \, dA + \int_{\operatorname{Int} \pi \bigcap \Lambda(\delta)} f \, dA = \int_{\pi} \mathbb{F} \cdot d\mathbf{r}, \tag{8}$$

where $f = \pm \operatorname{curl} \mathbb{F}$ with sign that of the orientation previously ascribed to that portion of π bounding a connected component of $\operatorname{Int} \pi$, and $\setminus$ denotes relative complement. The dominated convergence theorem then shows that there exists δ_0 such that if $0 < \delta < \delta_0$, both

$$\left| \int_{R \setminus \Lambda(\delta)} \operatorname{curl} \mathbb{F} \, dA - \int_R \operatorname{curl} \mathbb{F} \, dA \right| < \eta, \tag{9}$$

and

$$\left| \int_{\operatorname{Int} \pi \bigcap \Lambda(\delta)} f \, dA \right| < \eta. \tag{10}$$

The theorem follows from (7), (8), (9), and (10). □

The following corollary which appears to be new, though the analogous result for complex contour integrals has long been known (cf. [3], [10]), is contained in the third part of the proof of Theorem 1. Though our proof is in two dimensions, the n-dimensional proof is the same.

Corollary 4. *Let C be a rectifiable oriented curve, and let $\mathbb{F}$ be a vector field defined and continuous on an open set Ω containing the range of C. Then for each $\eta > 0$ there exists a polygonal path approximation π to C, also contained in Ω, such that*

$$\left| \int_C \mathbb{F} \cdot d\mathbf{r} - \int_\pi \mathbb{F} \cdot d\mathbf{r} \right| < \eta.$$

Theorem 1 implies that geometric shapes other than squares with sides parallel to the coordinate axes can be used to calculate curl.

Corollary 5. *Let Ω be a domain in $\mathbb{R}^2$ and let $\mathbb{F} : \Omega \to \mathbb{R}^2$ be continuous and such that the hypothesis of Green's Theorem holds. For $0 < \epsilon \le \epsilon_0$, let C_ϵ be a family of simple closed rectifiable oriented curves in Ω. Denote by R_ϵ the interior of C_ϵ, and assume that for each $\epsilon \in (0, \epsilon_0]$*

i) $\overline{R}_\epsilon \subset \{(x, y) : |(x, y) - (x_0, y_0)| \le \epsilon\} \subset \Omega$,

ii) $(x_0, y_0) \in R_\epsilon$, *and*

iii) there exists a constant $\alpha > 0$ such that Area $(R_\epsilon) \ge \alpha \pi \epsilon^2$.

Then

$$\operatorname{curl} \mathbb{F}(x_0, y_0) = \lim_{\epsilon \to 0} \frac{\int_{C_\epsilon} \mathbb{F} \cdot d\mathbf{r}}{\operatorname{Area}(R_\epsilon)}.$$

Proof. By Theorem 1,

$$\int_{R_\epsilon} \operatorname{curl} \mathbb{F} \, dA = \int_{C_\epsilon} \mathbb{F} \cdot d\mathbf{r},$$

and so continuity of curl $\mathbb{F}$ at (x_0, y_0) implies the result. $\square$

Example 6. (Continuous Central Fields) With the usual polar coordinates $r = (x^2 + y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$, let $f(r)$ be such that $f : [0, \infty) \to \mathbb{R}$ is continuously differentiable with $f'(0+) = 0$. Further, let $\mathbb{F} = (P, Q) = f'(r)(\partial r/\partial x, \, \partial r/\partial y)$, defined by the limit, zero, at the origin. Thus $\mathbb{F}$ is the gradient of a radial function. Then $\mathbb{F}$ is continuous on $\mathbb{R}^2$ and curl $\mathbb{F} \equiv 0$ even though $\mathbb{F}$ need not be pointwise differentiable. That curl $\mathbb{F}$ vanishes, and that criterion ii) of Lemma 2 is everywhere satisfied can be established by a variant of regularization (cf. [2], [17]).

First extend f by reflection to a continuously differentiable function on $\mathbb{R}$ (cf. [6, Proposition VII.19.1]). Let d be a positive number and multiply f by a function $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi \equiv 1$ on $[-d, d]$. Now regularize φf. Then the regularization, f_ε, $\varepsilon > 0$, has the property that both f_ε and f'_ε converge uniformly to f and f' on $[0, d]$. In particular $f'_\varepsilon(0) \to 0$ as $\varepsilon \to 0$. The gradient of $f_\varepsilon(r) - f'_\varepsilon(0)\, r$, call it $\mathbb{F}_\varepsilon$ is then infinitely continuously differentiable on $\mathbb{R}^2 \setminus \{0\}$, and continuous on $\mathbb{R}^2$. Standard deformation of contours techniques (which are only necessary near the origin) verify criterion ii) of Lemma 2 for $\mathbb{F}_\varepsilon$. So if C is a square, Green's Theorem gives

$$\int_C \mathbb{F}_\varepsilon \cdot d\mathbf{r} = 0.$$

Now pick d large enough that the range of C is interior to $r < d$. Then $\mathbb{F}_\varepsilon$ converges uniformly to $\mathbb{F}$ on C, so

$$\int_C \mathbb{F} \cdot d\mathbf{r} = 0,$$

and we are done.

3 Extensions.

As before, let Ω be a domain in $\mathbb{R}^2$, and let $\mathbb{F} = (P, Q) : \Omega \to \mathbb{R}^2$. We will weaken the continuity assumption made on $\mathbb{F}$ in the previous section. First note that if P and Q are in $L^1_{\text{loc}}(\Omega)$, P and Q have distribution derivatives on

Ω. So we may define curl $\mathbb{F}$ in the sense of distributions by,

$$< \operatorname{curl} \mathbb{F}, \varphi >=< \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}, \varphi >= \int_\Omega \left(P\frac{\partial \varphi}{\partial y} - Q\frac{\partial \varphi}{\partial x} \right) dA$$

for all $\varphi \in C_0^\infty(\Omega)$, where $< \cdot , \cdot >$ denotes the duality between $\mathcal{D}'(\Omega)$ and $C_0^\infty(\Omega)$. This is consistent with uniform convergence on compacta, used in the previous section, since such convergence implies convergence in $L^1_{\text{loc}}(\Omega)$. When the distribution curl $\mathbb{F}$ is in $L^1_{\text{loc}}(\Omega)$, one can hope to obtain a version of Green's Theorem.

Theorem 7. *Let C be a simple closed rectifiable oriented curve with interior R, and $\overline{R} \subset \Omega$. Assume that P, Q and curl $\mathbb{F}$ are in $L^1_{\text{loc}}(\Omega)$. If also $\mathbb{F}$ is continuous on a neighborhood of C, Green's Theorem holds, i.e.,*

$$\int_R \operatorname{curl} \mathbb{F}\, dA = \int_C \mathbb{F} \cdot d\mathbf{r}.$$

Proof. Regularize $\mathbb{F}$, i.e., regularize each of the functions P and Q (cf. [2], [17]). Since the regularization $\mathbb{F}_\epsilon = (P_\epsilon, Q_\epsilon)$ with $\epsilon > 0$ small enough is in $C^\infty(\Omega_\epsilon)$, with $\overline{R} \subset \Omega_\epsilon = \{(x,y) \in \Omega : \operatorname{dist}((x,y), \partial\Omega) > \epsilon\}$, Theorem 1 and part i) of Lemma 2 give

$$\int_R \operatorname{curl} \mathbb{F}_\epsilon\, dA = \int_R \left(\frac{\partial Q_\epsilon}{\partial x} - \frac{\partial P_\epsilon}{\partial y} \right) dA = \int_C \mathbb{F}_\epsilon \cdot d\mathbf{r}. \tag{11}$$

That

$$\lim_{\epsilon \to 0} \int_R \operatorname{curl} \mathbb{F}_\epsilon\, dA = \int_R \operatorname{curl} \mathbb{F}\, dA$$

is a well known property of regularization (cf. [7], [17]). For the right hand side of (11), note that if C is parameterized by arc length, the parameterization is Lipschitzian; hence absolutely continuous. Then since as in the proof of part i) of Lemma 2, or [7], P_ϵ and Q_ϵ converge uniformly to P and Q respectively on the compact subset ∂R of Ω,

$$\lim_{\epsilon \to 0} \int_C P_\epsilon\, dx + Q_\epsilon\, dy = \int_C P\, dx + Q\, dy = \int_C \mathbb{F} \cdot d\mathbf{r}. \qquad \Box$$

To relax the hypothesis of continuity of $\mathbb{F}$ on a neighborhood of ∂R requires assumptions guaranteeing integrable traces (cf. [6]) of P and Q on ∂R. First recall that a simple closed curve need not be a topological manifold (cf. [9]).

If we require that ∂R be a topological manifold, then ∂R satisfies the segment property (cf. [9]), and if also each of P and Q are in $W^{1,1}(R)$, then each of P and Q are limits in $W^{1,1}(R)$ of functions in $C_0^\infty(\mathbb{R}^2)$ (cf. [6]). If we further require R to be a Lipschitz Graph domain, i.e., that ∂R be a Lipschitzian manifold, or geometrically, satisfies the uniform cone property (cf. [1], [5], [9]), a satisfactory trace on ∂R is obtainable. This is somewhat more direct than studying the "fine properties" of functions in $W^{1,1}$ as in [4], [7].

Theorem 8. *Let C be a simple closed rectifiable oriented curve with interior R. Further, let ∂R be Lipschitz and $\mathbb{F} = (P, Q)$ with P and Q in $W^{1,1}(R)$. Then*

$$\int_R \operatorname{curl} \mathbb{F}\, dA = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C \mathbb{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy,$$

where the trace of $\mathbb{F} = (P, Q)$ on ∂R is understood.

PROOF. Regularization as in the proof of Theorem 7 gives convergence of the area integral directly, and that of the line integral by the estimate,

$$\int_C |h|\, dS \leq K ||h||_{W^{1,1}(R)} \tag{12}$$

for all $h \,\epsilon\, W^{1,1}\,(R)$, where K is a constant independent of h, and dS is the element of arc length along C. The estimate (12) is a special case of a well known inequality for BV functions [17, Theorem 5.10.7], and a direct proof in $W^{1,1}$ can be obtained by a trivial modification of the proof of Théorème 1.2 in [12]. □

The following examples illustrate the strong role of geometry in vector calculus.

Example 9. (Central Fields) We return to the situation in Example 6, but now only assume that f is continuously differentiable on $(0, \infty)$. Then $\mathbb{F} = (P, Q) = f'(r) \left(\frac{\partial r}{\partial x}, \ \frac{\partial r}{\partial y} \right)$ may have a singularity at the origin. As in Example 6, curl $\mathbb{F} \equiv 0$, even though $\mathbb{F}$ need not be pointwise differentiable, and no restriction has been placed on the possible singularity of $\mathbb{F}$ at the origin. But the limit in (1) will not be uniform in neighborhoods of $(0, 0)$ if $\mathbb{F}$ is strongly singular there. Standard methods give independence of path provided the path avoids the origin. Note that one may also define curl for $\mathbb{F}$ in the sense of distributions on $\mathbb{R}^2$ by

$$< \operatorname{curl} \mathbb{F}, \varphi >= -\lim_{\epsilon \to 0} \int_{r > \epsilon} \left(Q \frac{\partial \varphi}{\partial x} - P \frac{\partial \varphi}{\partial y} \right) dA$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. If, for simplicity, f' is also continuously differentiable on $(0, \infty)$, integration by parts gives curl $\mathbb{F} = 0$ in $\mathcal{D}'(\mathbb{R}^2)$ no matter how strong the singularity of $\mathbb{F}$ may be at the origin.

Example 10. (A rotational field) On $\mathbb{R}^2$ let $\mathbb{F} = \operatorname{grad}\theta$ where $\theta = \tan^{-1}(y/x)$ is the usual polar angle. It is elementary that curl $\mathbb{F}$ vanishes on $\mathbb{R}^2 \backslash \{0\}$, while use of circles centered at the origin from Corollary 5 (together with an elementary deformation of contours argument) gives curl $\mathbb{F}(0,0) = 2\pi$. Defining curl $\mathbb{F}$ in the sense of distributions as in the previous example yields curl $\mathbb{F} = 2\pi\delta$, where δ is the Dirac measure at the origin.

The reasoning in Example 10 leads directly to Cauchy's Integral Formula of complex analysis.

Theorem 11. *Let Ω be a domain in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be holomorphic. Let C be a simple closed rectifiable oriented curve in Ω with interior R and $\overline{R} \subset \Omega$. Then if $z_0 \in R$,*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$

Proof. Since $f(z)/(z - z_0)$ is holomorphic on $R\backslash\{z_0\}$, Theorem 3 and deformation of contours shows that we may take C to be the circle of radius ϵ centered at z_0. Parameterize C by $z - z_0 = \epsilon e^{i\theta}$, $0 \leq \theta \leq 2\pi$, and write $f(z) = f(x+iy) = u(x,y) + iv(x,y) = U(x - x_0, y - y_0) + iV(x - x_0, y - y_0)$. Then

$$\int_C \frac{f(z)}{z - z_0}\, dz = \int_0^{2\pi} \left(U(\epsilon\cos\theta, \epsilon\sin\theta) + iV(\epsilon\cos\theta, \epsilon\sin\theta)\right) i\, d\theta, \tag{13}$$

and by continuity of U and V, the (constant) limit of the right side of (13) as $\epsilon \to 0$ is $2\pi i(U(0,0) + iV(0,0)) = 2\pi i f(z_0)$. □

As a final comment, note that the two dimensional divergence, or Gauss, theorem can be obtained geometrically by applying the previous methods to $\mathbb{G} = (-Q, P)$. In other words, if $\mathbb{F} = (P, Q)$ and $\mathbb{G} = (-Q, P)$, define div $\mathbb{F}$ by the equation

$$\operatorname{div}\mathbb{F} = \operatorname{curl}\mathbb{G}.$$

The roles of our examples then reverse. The rotational field of Example 10 gives div $\mathbb{F} \equiv 0$ on $\mathbb{R}^2$ both pointwise and in the sense of distributions. The particular central field defined as in Example 9 with $f(r) = \ln r$ yields div $\mathbb{F} = 2\pi\delta$, with δ the Dirac measure at the origin, in $\mathcal{D}'(\mathbb{R}^2)$. This leads to the following geometric characterization of holomorphy - a geometric version of the Cauchy-Riemann equations.

Theorem 12. *Let $\mathbb{F} = (P, Q) : \Omega \to \mathbb{R}^2$ be continuous. Then $P + iQ : \Omega \to \mathbb{C}$ is holomorphic if and only if the limits (1) for curl (Q, P) and curl $(P, -Q) = -$ div (Q, P) both vanish on Ω and are uniform on compact subsets of Ω.*

Proof. For necessity note that if $f(z) = P + iQ$ is holomorphic,

$$\int_C f(z)dz = \int_C (Pdx - Qdy) + i \int_C (Qdx + Pdy) = 0 \tag{14}$$

by Cauchy's Theorem, and the conclusion follows from part ii) of Lemma 2. On the other hand, if the limits defining curl (Q, P) and curl $(P, -Q)$ vanish and are uniform on compacta, Lemma 2 shows that (14) holds for squares with sides parallel to the coordinate axes. Green's Theorem then shows that (14) holds whenever C is a triangle in Ω. Sufficiency thus follows from Morera's Theorem (cf. [10,15]). $\square$

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