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K. J. Falconer, Mathematical Institute, University of St Andrews, North Haugh, St Andrews, Fife, KY16 9SS, Scotland. email: kjf@st-and.ac.uk

DIMENSIONS OF INTERSECTIONS AND DISTANCE SETS FOR POLYHEDRAL NORMS

Abstract

We obtain an estimate for the typical Hausdorff dimension of the intersection of a set E with homothetic copies of a set F, where E and F are Borel subsets of $\mathbb{R}^n$. We apply this to the 'distance set problem' for a polyhedral norm on $\mathbb{R}^n$, by showing that there are subsets of full dimension with distance set of Lebesgue measure 0.

1 Introduction.

Geometric properties of Hausdorff dimensions and measures have been studied in great detail. For example, the dimensions of projections and products of sets have been related to the dimensions of the sets themselves in various ways. Definitions of Hausdorff dimensions and measures and discussions of such results may be found in various texts, such as [4, 13].

Dimensions of intersections of sets have also been investigated in some detail. In particular, given Borel subsets E, F of $\mathbb{R}^n$, we have that, as σ ranges over an appropriate family G of geometric transformations of $\mathbb{R}^n$, such as the group of isometries or similarities, 'in general'

$$\dim_{\mathrm{H}}(E \cap \sigma(F)) \leq \max\{0, \dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F - n\}$$

in the sense that this fails only for exceptional transformations σ, and 'often'

$$\dim_{\mathrm{H}}(E \cap \sigma(F)) \geq \dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F - n \tag{1}$$

in that this holds for a set of transformations $\sigma \in G$ of positive measure (with respect to the natural invariant measure on G).

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In particular (1) holds for a set of transformations $\sigma \in G$ of positive measure in the following cases, where E and F are Borel sets:

(a) G is the group of similarities and E and F are arbitrary;

(b) G is the group of isometries, E is arbitrary and F is a smooth manifold or rectifiable set;

(c) G is the group of isometries, and E and F are arbitrary with $\dim_{\mathrm{H}} E > \frac{1}{2}(n+1)$ or $\dim_{\mathrm{H}} F > \frac{1}{2}(n+1)$.

Further details and proofs of these results may be found in [6, 11] [4, Chapter 8] [13, Chapter 13].

Note that such properties are fractal analogues of the classical 'codimension' formula for the dimensions of intersection of subspaces or submainfolds of $\mathbb{R}^n$ in general position; of course in the classical situation the dimensions are integers.

In all of the above cases, the transformations σ range over a group G which includes rotations, and indeed the rotational component is crucial in the proofs. Here we consider the group of homotheties, where there is no rotational component. Recall that a *homothety* is a similarity transformation which maps subspaces onto parallel affine spaces, equivalently one which is the composition of a dilation and translation. Thus a typical homothety $\sigma_{\lambda,a} : \mathbb{R}^n \to \mathbb{R}^n$ is of the form $\sigma_{\lambda,a}(x) = \lambda x + a$ where $\lambda \in \mathbb{R}^+$ is the scaling factor and $a \in \mathbb{R}^n$ is the translation vector.

There is a natural invariant measure on the group of homotheties; for our purposes it enough to be aware that this is equivalent to Lebesgue measure on the parameterization by $\mathbb{R}^+ \times \mathbb{R}^n$. In particular, the set of homotheties $\{\sigma_{\lambda,a} : (\lambda, a) \in E\}$ has zero measure if and only if $E \subset \mathbb{R}^+ \times \mathbb{R}^n$ has zero $(n+1)$-dimensional Lebesgue measure.

Our main result is as follows.

Theorem 1. *Let E and F be Borel subsets of $\mathbb{R}^n$, and let G be the group of homotheties on $\mathbb{R}^n$. Suppose that $\dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F > 2n-1$. Then*

$$\dim_{\mathrm{H}}(E \cap \sigma(F)) \geq \dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F - n \tag{2}$$

for a set of homotheties σ of positive measure.

A simple example will show that we cannot dispense with the condition $\dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F > 2n-1$.

In the final section of this paper we apply this theorem to a question that is attracting current interest, namely the 'distance set problem'. Let $\| \ \|$ be a norm on $\mathbb{R}^n$. The 'distance set' $D(A)$ of a set $A \subset \mathbb{R}^n$ with respect to this norm is the set of distances realized between its pairs of points, that is

$D(A) = \{\|x - y\| : x, y \in \mathbb{R}^n\}$. We show that for polyhedral norms, there are 'large' subsets of $\mathbb{R}^n$, that is sets of Hausdorff dimension n, with distance sets that are 'small' in the sense of having Lebesgue measure 0.

2 Proof of the Main Theorem.

Our proof of Theorem 1 depends on the following result, due to Marstrand [9] in $\mathbb{R}^2$ and Mattila [10] in $\mathbb{R}^n$, on intersections of sets with planes, see also [13, Chapter 10]. We write $A + x = \{a + x : a \in A\}$ for the translate of a set $A \subset \mathbb{R}^n$ by $x \in \mathbb{R}^n$, with $P^\perp$ for the orthogonal complement of a subspace P of $\mathbb{R}^n$, and $\mathcal{L}$ for 1-dimensional Lebesgue measure.

Proposition 2. *Let E be a Borel subset of $\mathbb{R}^n$ with $\dim_{\mathrm{H}} E \geq 1$.*
(a) For all $(n-1)$-dimensional subspaces P of $\mathbb{R}^n$

$$\dim_{\mathrm{H}}(E \cap (P + t)) \leq \dim_{\mathrm{H}} E - 1 \quad \textit{for } \mathcal{L}\textit{-almost all } t \in P^\perp. \tag{3}$$

(b) For almost all $(n-1)$-dimensional subspaces P of $\mathbb{R}^n$ (with respect to the natural invariant measure)

$$\mathcal{L}\{t \in P^\perp : \dim_{\mathrm{H}}(E \cap (P + t)) = \dim_{\mathrm{H}} E - 1\} > 0. \tag{4}$$

PROOF OF THEOREM 1. We use induction on n, the dimension of the ambient space. For $n = 1$ the homotheties are just the (orientation preserving) similarities, and (2) has been established for the group of similarities in $\mathbb{R}^1$, see, for example, [13, Chapter 13].

Now assume inductively that (2) holds in $\mathbb{R}^n$. Let E and F be Borel subsets of $\mathbb{R}^{n+1}$ with $\dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F > 2n + 1$ (so in particular, $\dim_{\mathrm{H}} E, \dim_{\mathrm{H}} F > n$). By Proposition 2(b), we may choose an n-dimensional subspace P of $\mathbb{R}^{n+1}$ such that (4) holds simultaneously for E and for F (with F replacing E in (4)). By a rotation of space if necessary, we may assume that P is the coordinate subspace of the first n coordinates in $\mathbb{R}^{n+1}$. Thus, writing points in $\mathbb{R}^{n+1}$ as $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, and writing E_t for the section of E given by $E_t = \{(x, t') \in E : t' = t\}$, etc., there are Lebesgue measurable sets $T, U \subset \mathbb{R}$ with $\mathcal{L}(T), \mathcal{L}(U) > 0$ such that

$$\dim_{\mathrm{H}}(E_t) = \dim_{\mathrm{H}} E - 1 \text{ for } t \in T \text{ and } \dim_{\mathrm{H}}(F_u) = \dim_{\mathrm{H}} F - 1 \text{ for } u \in U. \tag{5}$$

We may parameterize the homotheties on $\mathbb{R}^{n+1}$ as follows. For $\lambda \in \mathbb{R}^+, a \in \mathbb{R}^n$ and $h \in \mathbb{R}$, write $\sigma_{\lambda,a,h} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ for the homothety given by

$$\sigma_{\lambda,a,h}(x, t) = \lambda(x, t) + (a, h) = (\lambda x + a, \lambda t + h) \text{ for } (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

Suppose, for a contradiction, that (2) is false for these $E, F \subset \mathbb{R}^{n+1}$. Then for almost all $(\lambda, a, h) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}$, we have that

$$\dim_{\mathrm{H}}(E \cap \sigma_{\lambda,a,h}(F)) < \gamma \equiv \dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F - (n+1).$$

Taking sections parallel to the coordinate subspace $\mathbb{R}^n$, it follows from (3) that, for almost all (λ, a, h),

$$\dim_{\mathrm{H}}(E_t \cap \sigma_{\lambda,a,h}(F)) < \gamma - 1 \tag{6}$$

for $\mathcal{L}$-almost all t. By Fubini's theorem, it follows that for almost all t, (6) is true for almost all (λ, a, h). (Note that functions such as $(\lambda, a, h) \mapsto \dim_{\mathrm{H}}(E \cap \sigma_{\lambda,a,h}(F))$ are Borel measurable.)

The set $\sigma_{\lambda,a,h}(F_{(t-h)/\lambda}) = \lambda F_{(t-h)/\lambda} + (a, h)$ is the intersection of $\sigma_{\lambda,a,h}(F)$ with the n-plane $\mathbb{R}^n \times \{t\}$, so for almost all t, we have that for almost all (λ, a, h),

$$\dim_{\mathrm{H}}(E_t \cap \sigma_{\lambda,a,h}(F_{(t-h)/\lambda})) < \gamma - 1.$$

The coordinate transformation on $\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}$ given by $\lambda := \lambda, a := a, u := (t-h)/\lambda$ preserves sets of measure zero. Thus for almost all $t \in \mathbb{R}$, for almost all $(\lambda, a, u) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}$

$$\dim_{\mathrm{H}}(E_t \cap \sigma_{\lambda,a,t-\lambda u}(F_u)) < \gamma - 1. \tag{7}$$

Writing E_t° and F_t° for the orthogonal projections of E_t and F_t respectively onto $\mathbb{R}^n$ in the decomposition $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, the projection of $\sigma_{\lambda,a,t-\lambda u}(F_u)$ onto $\mathbb{R}^n$ is just $\sigma_{\lambda,a}(F_u^\circ)$, where $\sigma_{\lambda,a} : \mathbb{R}^n \to \mathbb{R}^n$ is the homothety of $\mathbb{R}^n$ given by $\sigma_{\lambda,a}(x) = \lambda x + a$. Thus $E_t \cap \sigma_{\lambda,a,t-\lambda u}(F_u)$ is congruent to $E_t^\circ \cap \sigma_{\lambda,a}(F_u^\circ)$. We conclude from (7) that for almost all $t \in T$ and almost all $u \in U$,

$$\dim_{\mathrm{H}}(E_t^\circ \cap \sigma_{\lambda,a}(F_u^\circ)) < \gamma - 1 \text{ for almost all } (\lambda, a) \in \mathbb{R}^+ \times \mathbb{R}^n. \tag{8}$$

But for all $t \in T$ and $u \in U$ we have $\dim_{\mathrm{H}} E_t^\circ = \dim_{\mathrm{H}} E_t = \dim_{\mathrm{H}} E - 1$ and $\dim_{\mathrm{H}} F_u^\circ = \dim_{\mathrm{H}} F_u = \dim_{\mathrm{H}} F - 1$. Applying the inductive assumption (2) to $E_t^\circ, F_u^\circ \subset \mathbb{R}^n$, we conclude that for almost all $t \in T$ and $u \in U$,

$$\dim_{\mathrm{H}}(E_t^\circ \cap \sigma_{\lambda,a}(F_u^\circ)) \geq (\dim_{\mathrm{H}} E - 1) + (\dim_{\mathrm{H}} F - 1) - n = \gamma - 1$$

for a set of $(\lambda, a) \in \mathbb{R}^+ \times \mathbb{R}^n$ of positive measure, which contradicts (8), to complete the inductive step. □

The following example shows that the condition $\dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F > 2n-1$ is necessary for the intersection property (2) to hold for the group of homotheties.

Example
Given $r, s > 0$ with $r + s < 1$, we may easily find Borel sets $E_1, F_1 \subset \mathbb{R}$ with $\dim_{\mathrm{H}} E_1 = r$ and $\dim_{\mathrm{H}} E_2 = s$ such that $E_1 \cap (\lambda F_1 + t) = \emptyset$ for almost all $(\lambda, t) \in \mathbb{R}^+ \times \mathbb{R}$. (For example, this will happen if we choose either E_1 or F_1 to have equal Hausdorff and box-counting dimensions.) Then $(E_1 \times \mathbb{R}^{n-1}) \cap \sigma(F_1 \times \mathbb{R}^{n-1}) = \emptyset$ for almost all homotheties σ, with $\dim_{\mathrm{H}}(E_1 \times \mathbb{R}^{n-1}) = r + (n-1)$ and $\dim_{\mathrm{H}}(F_1 \times \mathbb{R}^{n-1}) = s + (n-1)$; see [4] for details of the intersection and product properties used here.

3 Distance Sets.

In this section we apply Theorem 1 to a problem on distance sets. Let $\| \, \|$ be a norm on n-dimensional Euclidean space $\mathbb{R}^n$. The *distance set* $D(A)$ of $A \subset \mathbb{R}^n$ is the set of distances realized between points of A, that is $D(A) = \{\|x - y\| : x, y \in A\}$. We first give the following corollary which shows that a set A can have full Hausdorff dimension, yet whose projections onto many 1-dimensional subspaces have 'small' distance sets.

Corollary 3. *Let $\theta_1, \ldots, \theta_k$ be a finite collection of vectors in $\mathbb{R}^n$. There exists a Borel set $A \subset \mathbb{R}^n$ with $\dim_{\mathrm{H}} A = n$ such that its orthogonal projections onto the lines in the directions $\theta_1, \ldots, \theta_k$ all have distance sets with Lebesgue measure 0, equivalently*

$$\mathcal{L}\{x \cdot \theta_i - y \cdot \theta_i : x, y \in A\} = 0 \quad (i = 1, 2, \ldots, k).$$

PROOF. Let $F \subset \mathbb{R}$ be a Borel set with $\dim_{\mathrm{H}} F = 1$ but with $\mathcal{L}\{|u-v| : u, v \in \mathbb{R}\} = 0$. (There are various possible constructions of such F, for example the porous sets formed as the intersections of rapidly increasing numbers of spaced intervals of rapidly decreasing lengths can have this property, see [13, Section 4.12]). Let $E = F \times [0,1]^{n-1} \subset \mathbb{R}^n$, so that $\dim_{\mathrm{H}} E = n$, see [4, Chapter 7], and for each i let E_i be a congruent copy of E under a rotation that maps the first coordinate axis to the line through the origin in direction θ_i. Thus for each i, $\dim_{\mathrm{H}} E_i = n$, and $\mathcal{L}\{(x-y) \cdot \theta_i : x, y \in E_i\} = 0$.

Applying Theorem 1 to the sets $E_1, E_2 \ldots E_k$ in turn, we may find homotheties $\sigma_1, \sigma_2, \ldots, \sigma_k$ of $\mathbb{R}^n$ (where we might as well take σ_1 to be the identity) such that $\dim_{\mathrm{H}}(\sigma_1(E_1) \cap \sigma_2(E_2) \cap \ldots \cap \sigma_k(E_k)) = n$, so the set $A = \sigma_1(E_1) \cap \ldots \cap \sigma_k(E_k)$ has the desired properties, noting that for all i,

$$\{(x-y)\cdot\theta_i : x, y \in A\} \subset \{(x-y)\cdot\theta_i : x, y \in \sigma_i(E_i)\} = \{\lambda_i(x-y)\cdot\theta_i : x, y \in E_i\},$$

where $\sigma_i(x) = \lambda_i x + a_i$. $\square$

We apply this corollary to the much studied 'distance set problem'. It has been known for some time [3] that, with distances defined by the usual Euclidean norm, $\mathcal{L}(D(A)) > 0$ if $A \subset \mathbb{R}^n$ is a Borel set with $\dim_{\mathrm{H}} A > (n+1)/2$, and more recently [1, 2, 14] that this conclusion holds if $\dim_{\mathrm{H}} A > n(n+2)/2(n+1)$. It seems widely believed that, with $D(A)$ defined in terms of the usual norm, $\mathcal{L}(D(A)) > 0$ if $\dim_{\mathrm{H}} A > n/2$, but a proof of this still seems some way off.

The same question has been considered recently where distance is defined by non-Euclidean norms on $\mathbb{R}^n$. (Note that the definition of Hausdorff dimension of a subset of a finite dimensional normed space is independent of the norm.) Isoevich and Laba [5] showed that if the norm is such that the unit ball $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is a (symmetric) strictly convex set with non-vanishing curvature, then $\mathcal{L}(D(A)) > 0$ if $\dim_{\mathrm{H}} A \geq (n+1)/2$. On the other hand, Konyagin and Laba [8] worked with regular arrangements of points to show that for a norm on $\mathbb{R}^2$, if the unit ball with respect to the norm is a symmetric convex polygon then 'typically' (i.e., for almost all polygons in the sense of Lebesgue measure on the directions of the sides) there exists a set $A \subset \mathbb{R}^n$ with $\dim_{\mathrm{H}} A = 2$ and $\mathcal{L}(D(A)) = 0$. They also showed [8] that this is the case if the sides of the polygon have algebraic slopes with respect to some coordinate system.

It follows from Corollary 3 that this 'bad' situation extends to norms on $\mathbb{R}^2$ defined by *all* (symmetric) polygons, and indeed to norms on $\mathbb{R}^n$ defined by polytopes for $n \geq 2$.

Corollary 4. *Let $\| \ \|$ be a norm on $\mathbb{R}^n$ ($n \geq 2$), such that the unit ball with respect to the norm, $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, is a symmetric polytope (with finitely many faces). Then there exists a compact set $A \subset \mathbb{R}^n$ with $\dim_{\mathrm{H}} A = n$ such that $\mathcal{L}(D(A)) = 0$, where the distances are defined by the norm $\| \ \|$.*

Proof. We may find vectors $\theta_1, \dots, \theta_k$ such that the unit ball is given by

$$B = \bigcap_{i=1}^{k} \{x : |x \cdot \theta_i| \leq 1\},$$

so for all $x \in \mathbb{R}^n$

$$\|x\| = \max_{i=1,\dots,k} |x \cdot \theta_i|.$$

Taking A to be the set of Hausdorff dimension n constructed in Corollary 3, we see that

$$\{\|x-y\| : x, y \in A\} \subset \bigcup_{i=1}^{k} \{|(x-y) \cdot \theta_i| : x, y \in A\},$$

so

$$\mathcal{L}(D(A)) = \mathcal{L}\{\|x-y\| : x,y \in A\} \leq \sum_{i=1}^{k} \mathcal{L}\{|(x-y)\cdot\theta_i| : x,y \in A\} = 0.$$

Since every Borel subset of $\mathbb{R}^n$ contains a compact subset of the same Hausdorff dimension, we may reduce A to a compact set of dimension n with the required properties. $\square$

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