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DYNAMICAL SYSTEMS GENERATED BY FUNCTIONS WITH CONNECTED G_δ GRAPHS

Abstract

In 2001, Csörnyei, O'Neil and Preiss proved that the composition of any two Darboux Baire-1 functions $[0,1] \to [0,1]$ possesses a fixed point, solving a long-standing open problem. In 2004 Szuca proved that this result can be generalized to any f in the class $\mathcal{J}$ of functions $[0,1] \to [0,1]$ with connected G_δ graph. As a consequence, he proved that for such functions the Sharkovsky theorem is satisfied.

As the main result of this paper we prove that as for continuous maps of the interval, any f in $\mathcal{J}$ has positive topological entropy if and only if it has a periodic point of period different from 2^n, for any $n \in \mathbb{N}$. To do this we show that using Bowen's approach it is possible to define topological entropy for discontinuous maps of a compact metric space with almost all of the standard properties. In particular, the variational principle is true, and consequently, topological entropy is supported by the set of recurrent points. We also develop theory of recurrent, ω-limit, and nonwandering points of functions in $\mathcal{J}$ since, in general, standard results from the topological dynamics, are not true. For example, there is a Darboux Baire-1 function f (hence, $f \in \mathcal{J}$) such that neither the set of recurrent points nor the set of ω-limit points of f are invariant.

1 Introduction and the Main Results.

In this paper we show that some classical results concerning dynamical properties of continuous mappings of the interval are true for more general mappings

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of the interval whose graph is a connected G_δ set; in the sequel we denote the class of these functions by $\mathcal{J}$. The starting point is the result by Csörnyei, O'Neil and Preiss [CNP] from 2001, that the composition of any two Darboux Baire-1 functions has a fixed point and its generalization by Szuca [Szuc] for maps in $\mathcal{J}$.

A nontrivial consequence of this theorem is the Itinerary Lemma, and the fact that the classical Sharkovsky's theorem on the coexistence of periodic orbits for continuous maps of the interval from 1964 (cf., e.g., [BC]), is true for the more general maps in $\mathcal{J}$ [Szu]. The main aim of this paper is to show that for maps in $\mathcal{J}$ another classical result is true – the Misiurewicz's characterization of continuous maps of the interval possessing no periodic orbits of period $\neq 2^n$, for $n = 0, 1, 2, \dots$, as the maps with zero topological entropy (for reference, cf., e.g., [BC]). To do this it is necessary to develop a theory of topological entropy of functions (not necessarily continuous) from a compact metric space into itself; we do this in Section 3. We define topological entropy for maps in $\mathcal{F}$ using Bowen's approach with (n, ε)-separated sets, and show that this notion has the usual properties – cf., e.g., Propositions 3.6 and 3.7 below. The key result is the fact, that topological entropy is supported by the set of recurrent points of the map, similar to the case of continuous maps, cf. Theorem 3.8 below. Its proof, for maps in $\mathcal{F}$ is not simple. The natural way is to use the Poincaré Recurrence Theorem for measurable maps, and the Variational Principle which is proved as Theorem 3.18. The proof is rather complicated, and follows from a sequence of lemmas and propositions. We apply the Misiurewicz's approach for continuous maps (cf. [Sz]), with proper modifications. We also found some useful ideas in [AKLS].

The proof of our main result is in Section 4. The next section, Section 2 contains a sequence of rather elementary results concerning properties of periodic, recurrent and nonwandering points of maps in $\mathcal{J}$. Note that dynamical properties of systems generated by functions in the Baire class 1 were considered also in an older paper [K].

In the sequel we use the standard notions and terminology concerning dynamical systems and real functions, like, e.g., [BC] or [BBT]. We start with the following definition.

Definition 1.1. Let $I = [0, 1]$ and let f be a function $I \to I$. Then $f \in Conn$ if f has a connected graph, $f \in G_\delta$ if the graph of f is a G_δ set, $f \in \mathcal{D}$ if f has Darboux property and $f \in \mathcal{B}_\infty$ if f is a Baire $-$ 1 function. It is well known that

$$\mathcal{DB}_1 := \mathcal{D} \cap \mathcal{B}_1 \subset Conn \cap \mathcal{G}_\delta =: \mathcal{J} \subset \mathcal{D}.$$

We say that an interval U *f-covers* V if $f(U) \supset V$; in this case we write $U \to_f V$.

Lemma 1.2. *(Itinerary Lemma [Szu].) Let $f \in \mathcal{J}$. For every family $\{I_k\}_{1\le k\le n}$ of closed intervals which satisfies $I_1 \to_f I_2 \to_f \ldots \to_f I_n \to_f I_1$ there is an $x \in I_1$ such that $f^n(x) = x$ and $f^i(x) \in I_{i+1}$ for every $i = 1, 2, \ldots, n-1$.*

Recall that, for a continuous map f of the interval, the Itinerary Lemma is well-known. Its proof follows easily by the fact, that if J is a compact interval and $f(J) \supset J$, then f has a fixed point in J. The Itinerary Lemma immediately implies the following result, which is the main tool used to prove Sharkovsky's Theorem 1.4.

Lemma 1.3. *Let $f \in \mathcal{J}$, and let $\pi = \{I_1, \ldots, I_k\}$ be a partition of I into compact subintervals. Let G be a Markov graph of (f, π). Thus, G is an oriented graph whose vertices are intervals from π. And there is an arrow from I_i to I_j in G if and only if I_i f-covers I_j. Now if there is a loop*

$$I_{k_1} \to_f I_{k_2} \to_f \ldots \to_f I_{k_n} \to_f I_{k_1}$$

in G of length n which is not repetition of a single smaller loop, then f has a periodic point of period n.

Theorem 1.4. *(Sharkovsky's Theorem [Szu].) Let $\prec$ be the linear ordering of the set of positive integers given by*

$$3 \prec 5 \prec 7 \prec \ldots \prec 2\cdot 3 \prec 2\cdot 5 \prec \ldots \prec 2^2\cdot 3 \prec 2^2 \cdot 5 \prec \ldots \prec 2^3 \prec 2^2 \prec 2 \prec 1,$$

and let m, n be positive integers such that $n \prec m$. If $f \in \mathcal{J}$ has a periodic orbit of period n, then f also has a periodic orbit of period m.

The following is the main result of our paper.

Theorem 1.5. *Let $f \in \mathcal{J}$. Then f has positive topological entropy if and only if f has a periodic point whose period is not a power of 2.*

Proof. This follows by Theorem 4.7 and 4.8. □

2 Preliminaries.

In this section we provide a list of simple results concerning maps in $\mathcal{J}$, which will be of some use in the sequel, or which exhibit interesting phenomena impossible for continuous maps.

Definition 2.1. We say that the map $f \in \mathcal{J}$ is *turbulent* if there are compact subintervals J, K with at most one point in common such that

$$J \cup K \subseteq f(J) \cap f(K),$$

and is *strictly turbulent* if the subintervals J, K can be chosen disjoint.

Lemma 2.2. *Let $f \in \mathcal{J}$ and J be a subinterval of I which contains no periodic point of f. If $x \in J, f^m(x) \in J$ for some $m > 0$ and $y \in J, f^n(y) \in J$ for some $n > 0$, then $y < f^n(y)$ if $x < f^m(x)$ and $y > f^n(y)$ if $x > f^m(x)$.*

Proof. We suppose that $x < f^m(x)$ and put $g := f^m$. Then the interval $[x, g(x)]$ contains no periodic point of g. If $g^k(x) > x$ for some $k \geq 1$, then $g^{k+1}(x) > g(x)$, since g^k does not have a fixed point in the interval $[x, g(x)]$. Evidently $g^k(x) > x$ for every $k \geq 1$ (by induction). In particular: $f^{mn}(x) > x$. If $y > f^n(y)$, then in the same way we would obtain $y > f^{mn}(y)$. But since $f \in \mathcal{J}$, f^{mn} has a fixed point between x and y, by the Itinerary Lemma, which is a contradiction. Similarly we can prove that $y > f^n(y)$ if $x > f^m(x)$. □

Corollary 2.3. *Let $f \in \mathcal{J}$ and J be a subinterval of I which contains no periodic point of f. Then, for any $x \in I$, the points of the trajectory $\{f^n(x)\}_{n \geq 0}$ which lie in J form a strictly monotonic (finite or infinite) sequence.*

Proof. Let
$$\{y_k\}_{k=1}^N := J \cap \{f^n(x)\}_{n=1}^\infty,$$
where $N \in \mathbb{N}$ or $N = \infty$. We may assume that there are positive integers m_i such that $f^{m_i}(y_i) = y_{i+1}$ for $i < N$. By the previous lemma,
$$y_1 < f^{m_1}(y_1) = y_2 < f^{m_2}(y_2) = y_3,$$
or
$$y_1 > f^{m_1}(y_1) = y_2 > f^{m_2}(y_2) = y_3.$$
It follows by induction that the sequence $\{y_n\}_{n=1}^N$ is strictly monotone. □

Lemma 2.4. *Let $f \in \mathcal{J}$ and J be an open subinterval which contains no periodic point of f. Then (i) J contains at most one point of any ω-limit set $\omega_f(x)$, (ii) J contains no recurrent point, (iii) if $x \in J$ is nonwandering, then no other point of its trajectory lies in J.*

Proof. (i) Let x be a point in I and $\omega_f(x)$ be its ω-limit set. Suppose that u, $v \in \omega_f(x) \cap J$, $u \neq v$. Then there exist sequences $\{f^{n_k}(x)\}_{k=1}^\infty$ and $\{f^{m_k}(x)\}_{k=1}^\infty$ such that
$$\{f^{n_k}(x)\}_{k=1}^\infty \to u, \text{ and } \{f^{m_k}(x)\}_{k=1}^\infty \to v.$$
By Corollary 2.3, $\{f^{n_k}(x)\}_{k=1}^\infty \cup \{f^{m_k}(x)\}_{k=1}^\infty$ forms a strictly monotonic sequence which is a contradiction and $u = v$.

(ii) Suppose that $x \in J \cap \mathrm{Rec}(f)$. Let $U \subset J$ be a neighborhood of x. Then there is $n \in \mathbb{N}$ such that $f^n(x) \in U$. If $V \subset U$ is a neighborhood of x without

$f^n(x)$, then there exists $m > n$ such that $f^m(x) \in V$. Thus either the point $f^m(x)$ lies between x and $f^n(x)$ or x is between $f^m(x)$ and $f^n(x)$. But this is a contradiction to Corollary 2.3.

(iii) Let $x \in J \cap \Omega(f)$ and $f^m(x) \in J$ for some $m > 0$. Since $x \neq f^m(x)$, there is an open interval $G \subseteq J$ containing x such that $f^m(G) \subseteq J$ and $G \cap f^m(G) = \emptyset$. Assuming that $x \in \Omega(f)$, we can choose $y \in G$ and $n > m$ such that $f^n(y) \in G$. Then $f^m(y)$ does not lie between y and $f^m(y)$ which is a contradiction. □

The following lemma is stated for $f \in \mathcal{J}$, but its proof is valid for any $f \in \mathcal{F}$.

Lemma 2.5. *Let $f \in \mathcal{J}$. Then $\Omega(f) = \overline{\Omega(f)}$.*

Proof. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\Omega(f)$ and $x_n \to x$. Then x lies in $\Omega(f)$. Indeed, if U is a neighborhood of x, then there is an $m \in \mathbb{N}$ such that $x_m \in U$. Since $x_m \in \Omega(f)$, $f^p(U) \cap U \neq \emptyset$ for some p and thus, $x \in \Omega(f)$. □

Lemma 2.6. *If $f \in \mathcal{J}$, then $\mathrm{Rec}(f) \subset \overline{\mathrm{Per}(f)}$.*

Proof. We can write

$$I \setminus \overline{\mathrm{Per}(f)} = \bigcup_{n=1} J_n,$$

where every J_n is an open interval and $J_n \cap \overline{\mathrm{Per}(f)} = \emptyset$. By Lemma 2.4, $J_n \cap \mathrm{Rec}(f) = \emptyset$ and $\mathrm{Rec}(f)$ must be a subset $\overline{\mathrm{Per}(f)}$. □

Lemma 2.7. *There is an $f \in \mathcal{J}$ possessing a finite ω-limit set which fails to be a cycle. (Recall that for continuous maps this is impossible, cf., e.g. [BC].)*

Proof. Let $f : [0,1] \to [0,1]$ be a function such that f is continuous on the interval $(0,1)$ and satisfies the following properties: $f(0) = 0$, $f(x) \in (x, 1-x)$ for every $x \in (0, \frac{1}{2})$, $f(\frac{1}{2}) = \frac{1}{2}$, $f(x) \in (1-x, x)$ for every $x \in (\frac{1}{2}, 1)$ and $f(1) = 1$. Then f is continuous everywhere except for $x = 0$ and $x = 1$, and clearly, $f \in \mathcal{J}$, but on the other hand, the ω-limit set containing only 0 and 1, which do not form a cycle. □

Lemma 2.8. *There is an $f \in \mathcal{J}$ possessing a recurrent point a whose image $f(a)$ is not recurrent. (Recall that for continuous maps this is impossible, cf., e.g. [BC].)*

Proof. Let $f : [0,1] \to [0,1]$ be such that $f(x) = \frac{1}{2}$ for $x \in [\frac{3}{4}, 1]$, $f(0) = 1$, and $f(2^{-n}) = 2^{-(n+1)}$, for $n = 0, 1, 2, \dots$. Let f be continuous on any interval $I_n = [2^{-(n+1)}, 2^{-n}]$ with $f(I_n) = [0,1]$. Then f is continuous everywhere except for $x = 0$, and clearly, $f \in \mathcal{J}$. On the other hand, $a = 0$ is a recurrent point of f but $f(a) = 1$ fails to be recurrent. □

Lemma 2.9. *Let $f \in \mathcal{J}$, $a \in \mathrm{Rec}(f)$, and $n_0 \in \mathbb{N}$. If $f^n(a) > a$ for every $n \geq n_0$, $n \in \mathbb{N}$, then there is a sequence $\{p_n\}_{n=1}^{\infty} \subset \mathrm{Per}(f)$ such that $p_n > a$ and $\lim_{n\to\infty} p_n = a$.*

PROOF. Let $\delta > 0$ be sufficiently small such that, for some $k \geq n_0$, $f^i(a) \notin (a, a+\delta)$ whenever $i < k$, and $f^k(a) \in (a, a+\delta)$. Let $r \geq n_0$ be the minimal integer such that $f^{k+r}(a) \in (a, f^k(a))$. Such an r exists since $a \in \mathrm{Rec}(f)$ and $f^n(a) > a$, for $n > n_0$. Then, for $b = f^k(a)$, we have $f^r(a) > a$ and $f^r(b) < b$. By the Itinerary Lemma there is a point $p \in (a,b)$, such that $f^r(p) = p$. $\square$

Lemma 2.10. *Let $f \in \mathcal{J}$ and let $a \in \mathrm{Rec}(f)$. If $f^k(a) > a$, for some $k \in \mathbb{N}$, and if there is a sequence $\{p_n\}_{n=1}^{\infty} \subset \mathrm{Fix}(f)$ such that $p_n > a$ and $\lim_{n\to\infty} p_n = a$, then f has a periodic point of period $\neq 2^n$, $n \in \mathbb{N}$.*

PROOF. Put $a_k = f^k(a)$, and let $q \in (a, a_k) \cap \mathrm{Fix}(f)$. Consider two cases.

CASE A. Let $f^{n+k}(a) < a$ for some $n > 0$. Then there is a $y \in (q, a_k)$ such that $f^n(y) = a$. This follows since $f^n[q, a_k] \supset [a, q]$ and $f^n \in \mathcal{D}$. By the Itinerary Lemma (note that $f^k \in \mathcal{D}$) there are points $a < u < p < v < q$ such that $f^k(u) = f^k(v) = y$ (note that $f^k(a) > y$), and $p \in \mathrm{Fix}(f)$. Let $U = [u,p], V = [p,v]$. Then $f^k(U) \supset [p,y]$. Hence

$$f^{n+k}(U) \supset f^n[p,y] \supset [a,q] \supset U \cup V.$$

Similarly, $f^{k+n}(V) \supset U \cup V$. Thus, f^{n+k} is turbulent (cf. Definition 2.1). By the Itinerary Lemma (or by Lemma 1.3), f^{n+k} has a fixed point and so f has a periodic point of period $(n+k)j$, for any $j \in \mathbb{N}$.

CASE B. Let $f^{n+k}(a) \geq a$ for all $n \in \mathbb{N}$. We may assume $a_s \in (q, a_k)$ for some s, otherwise we take smaller q. Since $f^k \in \mathcal{D}$, there is a sequence of points $\{u_i\}$ such that $u_i > a$, $\lim_{i\to\infty} u_i = a$, and $f^k(u_i) = a_s$ (note that every right neighborhood of a contains a fixed point). Choose $u = u_i < p < u_j = v$, where $p \in \mathrm{Fix}(f)$, and let $U = [u,p], V = [p,y]$. Then similar to the previous case we get

$$f^{k+m}(U) \cap f^{k+m}(V) \supset U \cup V$$

for any m such that $a_{m+s} < u$. $\square$

Lemma 2.11. *Let $f \in \mathcal{J}$. If $\mathrm{Per}(f) = \mathrm{Fix}(f)$, then $\mathrm{Rec}(f) = \mathrm{Per}(f)$.*

PROOF. Let $a \in \mathrm{Rec}(f) \setminus \mathrm{Per}(f)$. By Lemma 2.6, there is a sequence $\{p_n\}$ of fixed points, converging to a, say, from the right. Then, by Lemma 2.10, $f^n(a) < a$, for every $n > 0$, and consequently, by the "converse" version of Lemma 2.9, a is the limit point of fixed points of f from the left. By (the "converse" version of) Lemma 2.10, this is impossible. $\square$

Lemma 2.12. *Let $f \in \mathcal{J}$, and let $a \in \operatorname{Rec}(f)$. If there is a neighborhood U of a such that, for some $n \in \mathbb{N}$, any periodic point of f contained in U has period 2^i, with $0 \le i \le n$, then a is periodic.*

PROOF. The proof is similar to that for Lemma 2.11. It is based on Lemma 2.9 and a slight variation of Lemma 2.10, since it could happen that, for some $a \in \operatorname{Rec}(f)$, $a \notin \operatorname{Rec}(f^m)$, for an $m > 0$. But, in any case, there is a k, $0 \le k < m$ such that a is a cluster point of the sequence $\{f^{k+im}(a)\}_{i=1}^{\infty}$. Another possibility is to apply Lemma 2.2. □

Before stating the next lemma recall the following. Let $(X, \mathcal{M}, \mu)$ be a measure space, with μ a probability measure on X, and let $f : X \to X$ be an $\mathcal{M}$-measurable map. If, for any $A \in \mathcal{M}$, $\mu(f^{-1}(A)) = \mu(A)$, then μ is an *invariant measure* for f.

Lemma 2.13. *Let X be a compact metric space, $f : X \to X$ a measurable map with respect to a probability invariant measure μ on X. Then $\mu(\operatorname{Rec}(f)) = 1$.*

PROOF. This result must be known but we are not able to give a reference. By the Poincaré Recurrence Theorem (cf., e.g., [SmS]), for any measurable $A \subset X$, a.e. point $x \in A$ is recurrent with respect to A. This means that $f^i(x) \in A$, for infinitely many integers i. Let $N = X \setminus \operatorname{Rec}(f)$, and let $\mathcal{B} = \{B_n\}$ be a countable base of X. Let N_n be the set of points in B_n which are not recurrent with respect to B_n. To prove the lemma it suffices to show that $N = \bigcup_{n=1}^{\infty} N_n$. Clearly, $N \supset \bigcup N_n$. To prove the converse, let $x \in N$. Then there is an n such that $x \in B_n$ and, for any $k > 0$, $f^k(x) \notin B_n$. Consequently, $x \in N_n$. □

3 Topological Entropy for Discontinuous Functions.

In the literature, topological entropy is defined for *continuous* maps of a compact metric space (or a compact topological space), while metric entropy is defined for measurable functions which may be strongly discontinuous. In this section we show that topological entropy with reasonable properties (cf., e.g., [AKLS]) can be defined for an arbitrary map f from a compact metric space into itself. In the sequel, we assume that (X, ρ) is a compact metric space, and $\mathcal{F}$ is the space of all maps $X \to X$.

Definition 3.1. Let $f \in \mathcal{F}$, $n \in \mathbb{N}$ and $\varepsilon > 0$. A set $M \subset X$ is (n, ε)-*separated* if for every $x, y \in M, x \ne y$ there is $0 \le i < n$ such that $\rho(f^i(x), f^i(y)) > \varepsilon$. A set $E \subset X$ is an (n, ε)-*span* if for every $x \in X$, there is $y \in E$ such that $\rho(f^i(x), f^i(y)) \le \varepsilon$ for every $i \in \{0, 1, \dots, n\}$. Let $S(f, n, \varepsilon)$ denote an (n, ε)-separated set with maximal possible number of points, and $s_n(\varepsilon)$ its cardinality and similarly $r_n(\varepsilon) = \min\{\#F, F \text{ is an } (n, \varepsilon)\text{-span}\}$.

Let

$$\overline{s}(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon) \text{ and } \overline{r}(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log r_n(\varepsilon).$$

Lemma 3.2. *(i)* $r_n(\varepsilon) \le s_n(\varepsilon) \le r_n(\frac{\varepsilon}{2}) < +\infty$, *(ii) if* $\varepsilon_1 < \varepsilon_2$, *then* $\overline{s}(\varepsilon_1) \ge \overline{s}(\varepsilon_2)$, $\overline{r}(\varepsilon_1) \ge \overline{r}(\varepsilon_2)$.

PROOF. (i) If M is an (n,ε)-separated set with maximal cardinality, then M is an (n,ε) spanning set. Therefore $r_n(\varepsilon) \le s_n(\varepsilon)$. To show the other inequality suppose M is an (n,ε)-separated set and K is an $(n,\varepsilon/2)$ spanning set. Define $\vartheta : M \to K$ by choosing, for each $x \in M$, some point $\vartheta(x) = y \in K$ with

$$\rho\left(f^i(x), f^i(y)\right) \le \varepsilon/2,$$

for every $0 \le i < n$. Then ϑ is injective and therefore the cardinality of M is not greater than that of K. Hence $s_n(\varepsilon) \le r_n(\frac{\varepsilon}{2})$. For every $\varepsilon > 0$, an (n,ε)-separated set has finite cardinality, since X is a compact interval and thus (by the first inequality) $r_n(\frac{\varepsilon}{2}) < +\infty$.

(ii) Let $\varepsilon_1 < \varepsilon_2$. Then every (n,ε_2)-separated set is also an (n,ε_1)-separated set and hence, by the definition of $\overline{s}(\varepsilon)$, $\overline{s}(\varepsilon_1) \ge \overline{s}(\varepsilon_2)$. If K is an (n,ε_1) spanning set with minimal cardinality, then K is also an (n,ε_2) spanning set and hence $r_n(\varepsilon_2) \le r_n(\varepsilon_1)$. Therefore $\overline{r}(\varepsilon_1) \ge \overline{r}(\varepsilon_2)$. □

The following definition was introduced by R. Bowen for *continuous* maps.

Definition 3.3. The *Topological entropy* of an $f \in \mathcal{F}$ is the number

$$h(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon).$$

If $A \subset X$, then the *topological entropy* $h(f|_A)$ of $f|_A$ is given by the same formula, except that $s_n(\varepsilon)$ means the maximal cardinality of the sets $S(f,n,\varepsilon) \subset A$.

Proposition 3.4. *For any* $f \in \mathcal{F}$,

$$\lim_{\varepsilon\to 0} \overline{s}(\varepsilon) = \lim_{\varepsilon\to 0} \overline{r}(\varepsilon) = h(f).$$

PROOF. It follows by Definition 3.3 and Lemma 3.2. □

Proposition 3.5. *Let* $f \in \mathcal{F}$, *and let* $A, B \subset X$ *be arbitrary sets. Then* $h(f|_{A\cup B}) = \max\{h(f|_A), h(f|_B)\}$.

PROOF. The proof follows easily by Definition 3.3. □

Proposition 3.6. *Let $f \in \mathcal{F}$. Then $h(f^k) = k \cdot h(f)$, for every positive integer k.*

PROOF. First we show
$$k \cdot h(f) \geq h(f^k). \tag{3.1}$$
Let $\#S(f,n,\varepsilon) = s_n(\varepsilon)$ and $\#S(f^k,n,\varepsilon) = s_n^k(\varepsilon)$. Since any set which is (n,ε)-separated by the function f^k, is also (nk,ε)-separated by the function f, we have $s_n^k(\varepsilon) \leq s_{nk}(\varepsilon)$. It follows that
$$\frac{1}{n}\log s_n^k(\varepsilon) \leq \frac{k}{nk}\log s_{nk}(\varepsilon),$$
and consequently,
$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log s_n^k(\varepsilon) \leq \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{k}{nk}\log s_{nk}(\varepsilon),$$
which yields (3.1). To prove the converse inequality, let M_i, $0 \leq i < k$, be a maximal subset of $S(f,nk,\varepsilon)$ with the following property. For every distinct x, y in M_i there is a $0 \leq j < n$ such that $\rho(f^{i+jk}(x), f^{i+jk}(y)) > \varepsilon$; i.e., $\rho(f^{jk}(f^i(x)), f^{jk}(f^i(y))) > \varepsilon$. By definition of M_i, the points $f^i(x)$ and $f^i(y)$ are (n,ε)-separated by the function f^k. Hence, $\#M_i \leq s_n^k(\varepsilon)$. Since the union of the sets M_i is $S(f,nk,\varepsilon)$, we get
$$\begin{aligned} s_{nk}(\varepsilon) &\leq k \cdot s_n^k(\varepsilon), \\ \frac{k}{nk}\log s_{nk}(\varepsilon) &\leq \frac{1}{n}\log\left(k \cdot s_n^k(\varepsilon)\right) = \frac{1}{n}\log s_n^k(\varepsilon) + \frac{1}{n}\log k, \\ k \cdot h(f) &\leq h(f^k) + \limsup_{n\to\infty}\frac{\log k}{n}, \end{aligned}$$
and hence,
$$k \cdot h(f) \leq h(f^k). \tag{3.2}$$
By (3.1) and (3.2), $kh(f) = h(f^k)$. $\square$

Proposition 3.7. *Assume $f, g \in \mathcal{F}$ are topologically conjugate via a homeomorphism φ of I (so that $\varphi \circ f = g \circ \varphi$). Then $h(f) = h(g)$.*

PROOF. Let $n \in \mathbb{N}, \varepsilon > 0$. We have
$$f = \varphi^{-1} \circ g \circ \varphi. \tag{3.3}$$
Let $\#S(f,n,\varepsilon) = s_n^f(\varepsilon)$ and $\#S(g,n,\varepsilon) = s_n^g(\varepsilon)$. Take arbitrary points $x, y \in S(f,n,\varepsilon), x \neq y$. By definition of (n,ε)-separated set there is $0 \leq i < n$ such that $\rho(f^i(x), f^i(y)) > \varepsilon$ and (3.3) implies
$$\rho\left(\varphi^{-1}\left(g^i \circ \varphi(x)\right), \varphi^{-1}\left(g^i \circ \varphi(y)\right)\right) > \varepsilon. \tag{3.4}$$

Since φ is homeomorphism, φ^{-1} is continuous. Hence there is a $\delta > 0$ such that if $\rho(u,v) \leq \delta$, then $\rho\big(\varphi^{-1}(u), \varphi^{-1}(v)\big) \leq \varepsilon$. By (3.4) we have $\rho\big(g^i \circ \varphi(x), g^i \circ \varphi(y)\big) > \delta$, so $\varphi(x)$, $\varphi(y) \in S(g,n,\delta)$ and hence,

$$\#S(g,n,\delta) \geq \#\varphi\left(S(f,n,\varepsilon)\right).$$

Consequently, $s_n^g(\delta) \geq s_n^f(\varepsilon)$, and $h(g) \geq h(f)$. By the symmetry, $h(g) \leq h(f)$ and thus, $h(g) = h(f)$. □

The remainder of this section is devoted to the proof of the following Theorem 3.8 which is essential in proving our main result. Theorem 3.8 is a consequence of Lemma 2.13 and Theorem 3.18 below.

Theorem 3.8. *For $f \in \mathcal{F}$, $h(f) = h(f|_{\mathrm{Rec}(f)})$.*

This result is well-known if f is a continuous map of the interval; the proof consisting of a sequence of definitions, lemmas and propositions can be found, e.g., in [Sz]. It can be modified for mappings from $\mathcal{F}$ since the major part of the original argument is based on the fact that a continuous map of the interval has the Darboux property and is measurable. We start with the definition of the metric entropy, and an alternative definition of topological entropy.

Definition 3.9. Let $(X, \mathcal{M}, \mu)$ be a probability measure space (i.e., $\mu(X) = 1$), such that μ is an invariant measure of a map $f : X \to X$. Let $\xi = \{A_i, i = 1, \ldots, m\}$ be a decomposition of X, where $A_i \in \mathcal{M}$. If

$$R_{n-1}(\xi) := \bigvee_{k=0}^{n-1} f^{-k}(\xi)$$

is the set containing all intersections of the form $A_{i_1} \cap f^{-1}(A_{i_2}) \cap \ldots \cap f^{-(n-1)}(A_{i_n})$, then

$$h_\mu(f,\xi) = -\lim_{n\to\infty} \frac{1}{n} \sum_{B \in R_{n-1}(\xi)} \mu(B) \cdot \log \mu(B).$$

The *Metric entropy*, with respect to the measure μ, is the number

$$h_\mu(f) = \sup_{\xi} h_\mu(f,\xi).$$

Let X be a compact topological space and α, β two covers of set X (not necessarily by open sets). Then $\alpha \vee \beta := \{A_i \cap B_j, A_i \in \alpha, B_j \in \beta\}$, $f^{-1}(\alpha) := \{f^{-1}(A_i), A_i \in \alpha\}$, and similarly for $f^{-k}(\alpha)$.

Definition 3.10. Let α be a cover of X with a finite subcover. The *entropy* of α is defined to be

$$H_0(\alpha) = \log N(\alpha),$$

where $N(\alpha)$ is the minimal number of sets in any finite subcover.

Lemma 3.11. *Let $f \in \mathcal{F}$. Then for every cover α of X with a finite subcover*

$$\limsup_{n\to\infty} \frac{1}{n} H_0\left(\bigvee_{k=0}^{n-1} f^{-k}(\alpha)\right) =: h(f,\alpha) \le H_0(\alpha).$$

PROOF. Obviously $N(\bigvee_{k=0}^{n-1} f^{-k}(\alpha)) \le N(\alpha)^n$, for every $n \in \mathbb{N}$. Hence

$$\frac{1}{n} H_0(\bigvee_{k=0}^{n-1} f^{-k}(\alpha)) = \frac{1}{n}\log N(\bigvee_{k=0}^{n-1} f^{-k}(\alpha)) \le \frac{1}{n}\log N(\alpha)^n = \log N(\alpha),$$

and consequently, $h(f,\alpha) \le \log N(\alpha) = H_0(\alpha)$. □

A cover β is said to be a *refinement* of a cover α, in symbols $\alpha \prec \beta$, if every set of β is a subset of a set in α.

Lemma 3.12. *Let $f \in \mathcal{F}$, and let α be a cover of X with finite subcover. If $\alpha \prec \beta$, then $h(f,\alpha) \le h(f,\beta)$.*

PROOF. Let $\alpha \prec \beta$. Then clearly

$$\alpha \vee f^{-1}(\alpha) \vee f^{-2}(\alpha) \vee \ldots \vee f^{-n}(\alpha) \prec \beta \vee f^{-1}(\beta) \vee f^{-2}(\beta) \vee \ldots \vee f^{-n}(\beta)$$

so that

$$H_0(\bigvee_{k=0}^{n-1} f^{-k}(\alpha)) \le H_0(\bigvee_{k=0}^{n-1} f^{-k}(\beta)) \text{ and } h(f,\alpha) \le h(f,\beta).$$ □

For $f \in \mathcal{F}$, we set

$$d(f) = \sup_{\alpha} h(f,\alpha), \tag{3.5}$$

where the supremum is taken over all open covers α of X.

Proposition 3.13. *Let $f \in \mathcal{F}$, and let α_n be a cover of X consisting of open balls with diameter $< \frac{1}{n}$. Then $d(f) = \lim_{n\to\infty} h(f,\alpha_n)$.*

PROOF. It follows from (3.5) by Lemmas 3.11 and 3.12. □

Proposition 3.14. *Let $f \in \mathcal{F}$. Then $h(f) = d(f)$.*

PROOF. Let $\alpha_\varepsilon = \{C_i\}$ be a cover, where each C_i is a open ball with diameter smaller than ε. Let $\tilde{\alpha}^n_\varepsilon$ be a minimal subcover of the cover

$$\alpha^n_\varepsilon = \bigvee_{k=0}^{n-1} f^{-k}(\alpha_\varepsilon).$$

Choose one point of every element $\tilde{\alpha}^n_\varepsilon$ and let these points form a set F. Then F is an (n, ε)-span and hence,

$$N(\alpha^n_\varepsilon) = N(\tilde{\alpha}^n_\varepsilon) = \#F \geq r_n(\varepsilon),$$

which is equivalent, for $n \to \infty$,

$$d(f) \geq h(f, \alpha_\varepsilon) \geq \overline{r}(\varepsilon).$$

Letting $\varepsilon \to 0$ we get

$$d(f) \geq h(f). \tag{3.6}$$

Let $\alpha = \{A_i\}$ be an open cover with $\operatorname{diam}(\alpha) < \varepsilon$ and $\delta > 0$ be a number such that for every set $A \subset X$ with $\operatorname{diam}(A) \leq \delta$ there is $A_i \in \alpha$ so that $A \subset A_i$. Since X is compact, such a positive δ (the *Lebesque number of covering* α) always exists. Let F_n be a minimal $(n, \varepsilon/2)$-spanning set and

$$B_x = \bigcap_{k=0}^{n-1} \{y \in X, \rho\left(f^k(x), f^k(y)\right) < \frac{1}{2}\delta\},$$

for every $x \in F_n$. Then $\operatorname{diam}(B_x) < \delta$ and each set B_x is contained in an element of cover α^n. If $x \in \cap_{k=0}^{n-1} f^{-k}(A_{i_k})$, then

$$f^k(B_x) \subset \{y, \rho\left(f^k(x), y\right) < \frac{1}{2}\delta\} \text{ and } f^k(B_x) \subset A_{i_k}.$$

Hence $B_x \subset \bigcap_{k=0}^{n-1} f^{-k}(A_{i_k})$. The cover $\beta_n = \{B_x : x \in F_n\}$ is a refinement of α^n and $N(\beta_n) = \#F_n = r_n(\frac{1}{2}\delta)$. Thus $N(\alpha^n) \leq N(\beta_n) = r_n(\frac{1}{2}\delta)$. Hence

$$h(f, \alpha) = \lim_{n\to\infty} \frac{1}{n} \log N(\alpha^n) \leq \limsup_{n\to\infty} \frac{1}{n} \log r_n(\frac{\delta}{2}) = \overline{r}(\frac{\delta}{2}).$$

Letting $\varepsilon \to \infty$ we get, by Proposition 3.4,

$$d(f) \leq h(f). \tag{3.7}$$

By (3.6) and (3.7), $h(f) = d(f)$. $\square$

Let $\alpha = \{A_i\}$ be an open cover. If $\xi = \{C_i\}$ is a decomposition of X such that C_i is μ-measurable, let

$$p(\alpha,\xi) := \max_{A\in\alpha} \#\{C \in \xi, A \cap C \neq \emptyset\}.$$

Lemma 3.15. *Let $f \in \mathcal{F}$, and let μ be an f-invariant measure. Then, for any μ-measurable cover α of X with a finite subcover, and any μ-measurable decomposition ξ of X, $h_\mu(f,\xi) \leq h(f,\alpha) + \log p(\alpha,\xi)$.*

PROOF. Let

$$H(\xi) = -\sum_{C_i\in\xi} \mu(C_i)\log\mu(C_i) \text{ and } \xi^n = \bigvee_{i=0}^{n-1} f^{-i}(\xi).$$

It is easy to see that

$$p_i > 0,\ \sum_{i=1}^m p_i = 1 \implies -\sum_{i=1}^m p_i \log p_i \leq \log m. \tag{3.8}$$

Since ξ^n is a decomposition of X, (3.8) implies the first inequality in

$$H(\xi^n) \leq \log N(\xi^n) \leq \log\left(p(\alpha,\xi)\, N(\alpha^n)\right),$$

while the second one follows easily from the definition of $p(\alpha,\xi)$. To finish the argument let $n \to \infty$. $\square$

Lemma 3.16. *Let $f \in \mathcal{F}$, and let μ be an f-invariant probability Borel measure on X. Then, for every μ-measurable decomposition ξ of X there is a decomposition $\overline{\xi}$ and an open cover α such that (i) $h_\mu(f,\overline{\xi}) \geq h_\mu(f,\xi) - 1$, (ii) $p(\alpha,\overline{\xi}) \leq 2$.*

PROOF. Let $\xi = \{C_1, C_2, \ldots, C_s\}$ be a decomposition of X. It is well-known that any probability Borel measure on a metric space has the property, that any measurable set is μ-approximable by open subsets. Take compact sets $K_i \subset C_i$ with $\mu(C_i \setminus K_i)$ small enough such that the decomposition $\overline{\xi} = \{K_1, K_2, \ldots, K_s, X \setminus \bigcup_{i=1}^s K_i\}$ has the property (i). Put $U_{s+1} = X \setminus \bigcup_{i=1}^s K_i$ and $U_i = K_i \cup U_{s+1}$. Then $\alpha = \{U_1, U_2, \ldots, U_s, U_{s+1}\}$ is an open cover, and $p(\alpha,\overline{\xi}) \leq 2$. $\square$

Theorem 3.17. *Let $f \in \mathcal{F}$ and let μ be an f-invariant probability Borel measure on X. Then $h_\mu(f) \leq h(f)$.*

PROOF. Let ξ be a decomposition. Then, by Lemmas 3.15, 3.16 and Theorem 3.14,

$$h_\mu(f^n,\xi) \le h_\mu(f^n,\overline{\xi}) + 1 \le h(f^n,\alpha) + \log p(\alpha,\overline{\xi}) + 1 \le h(f^n) + \log 2 + 1$$

and

$$h_\mu(f^n) \le h(f^n) + \log 2 + 1.$$

By Proposition 3.6, since $h_\mu(f^n) = nh_\mu(f)$, $nh_\mu(f) \le nh(f) + \log 2 + 1$. Let $n \to \infty$ to get $h_\mu(f) \le h(f)$. □

Theorem 3.18. *Let $f \in \mathcal{F}$. Then $h(f) = \sup_\mu h_\mu(f)$, where the* sup *is taken over all probability f-invariant Borel measures μ on X.*

PROOF. For an invariant measure μ we put $S^k_\mu(A) = \mu(f^{-k}(A))$. Let $\varepsilon > 0$ be given and let E_n, for $n = 1, 2, \dots$, be an (n,ε)-separated set of cardinality $s_n(\varepsilon)$. Let σ_n be the atomic measure on the points of E_n; i.e.,

$$\sigma_n(\{x\}) = \frac{1}{s_n(\varepsilon)}, \text{ where } x \in E_n.$$

Define μ_n by $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} S^k_{\sigma_n}$. Since X is compact and the measures μ_n are probability measures, there is a subsequence $\{n_j\}$ of positive integers such that $\mu_{n_j} \to \mu$. Clearly, μ is an f-invariant probability measure on X. Moreover, μ is a Borel measure. This follows by the fact, that the support of every μ_{n_j} is a (finite) compact set B_j, and hence the support B of μ is compact as well. By Definition 3.3, we may assume that $\lim_{j\to\infty} \frac{1}{n}\log s_{n_j}(\varepsilon) = \overline{s}(\varepsilon)$. We show that $h_\mu(f) \ge \overline{s}(\varepsilon)$. To do this we first define a μ-measurable partition α of X consisting of sets with diameter less than ε such that the μ measure of the boundary $\operatorname{Fr}(A)$ of any set $A \in \alpha^n$ is zero, for any n. So let $E = \{x_1, \dots, x_m\}$ be an $\frac{1}{4}\varepsilon$ span. For $\frac{1}{4}\varepsilon < r < \frac{1}{2}\varepsilon$, let $A_i(r) = \{y \in X; \rho(x_i, y) \le r\}$. Since μ is finite, for any i there are only countably many r such that the μ-measure of the boundary of $A_i(r)$ is positive. Hence there is an r_0 such that the boundary of $A_i(r_0)$ is a set zero measure, for any i.

Let $\alpha = \{A_1, \dots, A_m\}$, where

$$\begin{aligned} A_1 &= \{y : \rho(x_1, y) \le r_0\}, \\ A_2 &= \{y : \rho(x_2, y) \le r_0\} \setminus A_1, \\ &\cdots \\ A_m &= \{y : \rho(x_m, y) \le r_0\} \setminus \bigcup_{i=0}^{m-1} A_i. \end{aligned}$$

Then α is the partition. It is easy to see that, for every set A_i, $\operatorname{diam}(A_i) \leq 2r_0 < \varepsilon$ and $\mu(\operatorname{Fr}(A_i)) = 0$, for every $1 \leq i \leq m$ (and so, $\mu(\bigcup_{i=0}^{m} \operatorname{Fr}(A_i)) = 0$, $\mu(\bigcup_{A\in\alpha^n} \operatorname{Fr}(A)) = 0$). Fix positive integers n, q with $n \geq 2q$. Define $s(j)$, for $0 \leq j < q$, by $s(j) = [(n-j)/q] - 1$, where $[b]$ denotes integer part of $b > 0$, and put $\alpha^n = \bigvee_{i=0}^{n-1} f^{-i}(\alpha)$. Then

$$\alpha^n = \bigvee_{k=0}^{s(j)} f^{-kq-j}(\alpha^q) \vee \bigvee_{k\in M} f^{-k}(\alpha),$$

where $M = \{0, 1, \ldots, j-1\} \cup \{qs(j) + j + q, \ldots, n-1\} \subset \{0, 1, \ldots, q-1\} \cup \{n-q, \ldots, n-1\}$. Is clear that M has cardinality at most $2q$. Let

$$H_\nu(\xi) = -\sum_{i=1}^{m} \nu(\xi_i) \log \nu(\xi_i),$$

for some partition $\xi = \{\xi\}_{i=1}^m$ and some measure ν. (We define $\nu(\xi_i)\log\nu(\xi_i) = 0$ for $\nu(\xi_i) = 0$.) It is easy to see that

$$H_{\sigma_n}(\xi \vee \eta) \leq H_{\sigma_n}(\xi) + H_{\sigma_n}(\eta). \tag{3.9}$$

By (3.8) and (3.9),

$$\begin{aligned}
H_{\sigma_n}(\alpha^n) &= H_{\sigma_n}(\bigvee_{k=0}^{s(j)} f^{-kq-j}(\alpha^q) \vee \bigvee_{k\in M} f^{-k}(\alpha)) \\
&\leq \sum_{k=0}^{s(j)} H_{\sigma_n}(f^{-kq-j}(\alpha^q)) + \sum_{k\in M} H_{\sigma_n}(\alpha) \\
&= \sum_{k=0}^{s(j)} H_{\sigma_n}(f^{-kq-j}(\alpha^q)) + \#M \cdot \log \#\alpha \\
&\leq \sum_{k=0}^{s(j)} H_{\sigma_n}(f^{-kq-j}(\alpha^q)) + 2q \log m.
\end{aligned}$$

For each $0 \leq j \leq q-1$, $s(j)q + j = [((n-j)/q) - 1]q + j \leq n - q$. The numbers $\{j + kq |\ 0 \leq j \leq q-1,\ 0 \leq k \leq s(j)\}$ are mutually distinct and not greater than $n - q$. Hence

$$qH_{\sigma_n}(\alpha^n) \leq \sum_{j=0}^{q-1} \sum_{k=0}^{s(j)} H_{\sigma_n}(f^{-kq-j}(\alpha^q)) + 2q^2 \log m,$$

and hence,

$$qH_{\sigma_n}(\alpha^n) \leq \sum_{k=0}^{n} H_{\sigma_n}(f^{-k}(\alpha^q)) + 2q^2 \log m. \tag{3.10}$$

Choose an (n, ε)-separated set E_n so no member of α^n can contain more than one member of E_n. Since $\sigma_n(\{x\}) = 1/s_n(\varepsilon)$, for $x \in E_n$,

$$H_{\sigma_n}(\alpha^n) = -\sum_{x \in E_n} \frac{1}{s_n(\varepsilon)} \log \frac{1}{s_n(\varepsilon)} = \log s_n(\varepsilon).$$

Next, $H_{\sigma_n}(f^{-k}(\alpha^q)) = H_{S^k_{\sigma_n}}(\alpha^q)$ and by (3.10) we get

$$q \log s_n(\varepsilon) \leq \sum_{k=0}^{n} H_{S^k_{\sigma_n}}(\alpha^q) + 2q^2 \log m,$$

or equivalently,

$$q\frac{1}{n+1} \log s_n(\varepsilon) \leq \frac{1}{n+1} \sum_{k=0}^{n} H_{S^k_{\sigma_n}}(\alpha^q) + \frac{2q^2}{n+1} \log m. \tag{3.11}$$

Since $1/(n+1) > 0$ and $\sum_{k=0}^{n} 1/(n+1) = 1$, we have

$$H_{\frac{1}{n+1}\sum_{k=0}^{n} S^k_{\sigma_n}}(\alpha^q) \geq \sum_{k=0}^{n} \frac{1}{n+1} H_{S^k_{\sigma_n}}(\alpha^q),$$

which, applied to (3.11), yields

$$\begin{aligned} q\frac{1}{n+1} \log s_n(\varepsilon) &\leq H_{\frac{1}{n+1}\sum_{k=0}^{n} S^k_{\sigma_n}}(\alpha^q) + \frac{2q^2}{n+1} \log m \\ &= H_{\mu_{n+1}}(\alpha^q) + \frac{2q^2}{n+1} \log m. \end{aligned} \tag{3.12}$$

The members of α^q have boundaries of μ-measure zero, so $\lim_{j\to\infty} H_{\mu_{n_j}}(\alpha^q) = H_\mu(\alpha^q)$. Therefore replacing n by n_j in (3.12) and letting $j \to \infty$ we have $q\overline{s}(\varepsilon) \leq H_\mu(\alpha^q)$. We can divide by q and let $q \to \infty$ to get

$$\overline{s}(\varepsilon) \leq \lim_{q\to\infty} \frac{1}{q} H_\mu(\alpha^q) = h_\mu(f, \alpha) \leq h_\mu(f).$$

Since $\mu = \mu(\varepsilon)$,

$$h(f) = \sup_{\varepsilon>0} \overline{s}(\varepsilon) \leq \sup_{\varepsilon>0} h_{\mu(\varepsilon)}(f) \leq \sup_{\mu} h_\mu(f).$$

By Theorems 3.14 and 3.17, $h_\mu(f) \leq h(f)$ and hence $\sup_\mu h_\mu(f) \leq h(f)$. Consequently, $h(f) = \sup_\mu h_\mu(f)$. $\square$

4 Proof of the Main Result.

Lemma 4.1. *Suppose for some $n \in \mathbb{N}$, any periodic point of an $f \in \mathcal{J}$ have period 2^i, $0 \le i \le n$. Then $h(f) = 0$.*

PROOF. Put $g = f^{2^m}$. Then, by Proposition 3.6, Theorem 3.8 and Lemma 2.11,

$$h(f) = \frac{1}{2^m} h(g) = \frac{1}{2^m} h(g|_{\mathrm{Rec}(g)}) = \frac{1}{2^m} h(g|_{\mathrm{Fix}(g)}) = 0. \quad \square$$

Proposition 4.2. *If $f \in \mathcal{J}$ is turbulent, then $h(f)$ is positive.*

PROOF. Since f is turbulent, there are closed intervals J, K for which $J \cup K \subseteq f(J) \cap f(K)$. Let $J = [a, b]$ and $K = [c, d], a < b \le c < d$. If $b \ne c$ (f is strictly turbulent), we take $\varepsilon = c - b$ and an arbitrary $n \in \mathbb{N}$. For every $x = x_0 x_1 x_2 \dots x_{n-1} \in \{0,1\}^n$ there is $y_x \in J \cup K$ such that $f^i(y_x) \in J$ if $x_i = 0$ and $f^i(y_x) \in K$ if $x_i = 1$. The set $M = \{y_x, x \in \{0,1\}^n\}$ containing 2^n elements is (n, ε)-separated. Thus $s_n(\varepsilon) \ge 2^n$ and $h(f) \ge \log 2 > 0$.

If $b = c$, then there is a compact interval $J_0 \subset J$ such that $f(J_0) = J$ or $f(J_0) = K$, and $b \notin J_0$. By the choice of J_0,

$$J_0 \cup K \subset J \cup K \subseteq f^2(J_0) \cap f^2(K),$$

and hence, f^2 is strictly turbulent. By the first part, $h(f^2) > 0$, and by Proposition 3.6, $h(f) > 0$. $\square$

Lemma 4.3. *If $f \in \mathcal{J}$ has periodic point of period $2^k q$, where $q > 1$ is odd and $k \ge 0$, then $f^{2^{k+2}}$ is turbulent.*

PROOF. If f has a point of period $2^k q$, then, by Theorem 1.4, f has a point of period $3 \cdot 2^{k+1}$. Thus $f^{2^{k+1}}$ has a cycle of period 3. Put $g = f^{2^{k+1}}$. There are points $a < b < c$ such that either

$$g(a) = b, \; g(b) = c, \; g(c) = a \tag{4.1}$$

or

$$g(a) = c, \; g(b) = a, \; g(c) = b.$$

Without loss of generality assume (4.1). Let $J = [a, b]$ and $K = [b, c]$. Then obviously $J \cup K \subseteq g^2(J) \cap g^2(K)$ and hence, $g^2 = f^{2^{k+2}}$ is turbulent. $\square$

The following result is crucial in proving that a map of type 2^∞ in $\mathcal{J}$ has zero topological entropy. For continuous maps of the interval the result was proved in [S], but the argument is not transferable to $\mathcal{J}$. However we are able to provide a new, self-contained argument.

Lemma 4.4. *Let $f \in \mathcal{J}$ have only periodic points of periods 2^n, for positive integers n, and let α be a periodic orbit of f with period 2^k, $k > 1$. Let α_0 and α_1 be the left and right half of α, each possessing 2^{k-1} points. Let p be a fixed point of f separating α_0 and α_1, and finally, let J_i be the convex hull of α_i, $i = 0, 1$. Since α is a simple orbit (cf. [BC] and [Szu]),*

$$f(\alpha_i) = \alpha_{1-i}, \text{ and } f^2(\alpha_i) = \alpha_i, \text{ for } i = 0, 1. \tag{4.2}$$

Put

$$U_0 = \bigcup_{n=0}^{\infty} f^{2n}(J_0), \; U_1 = \bigcup_{n=0}^{\infty} f^{2n}(J_1), \tag{4.3}$$

$V_0 = \overline{U_0}$, $V_1 = \overline{U_1}$. Then V_0 and V_1 are compact periodic intervals forming a periodic orbit of f; i.e., $f(V_i) = V_{1-i}$, $i = 0, 1$ such that $V_0 \cap V_1 \subset \{p\}$.

Proof. By (4.2), $f^2(J_i) \supset J_i$, which yields $f^2(U_i) = U_i$. Similarly, by (4.2), $f(J_i) \supset J_{1-i}$ and hence, $f(U_i) = U_{1-i}$ since $f^2 \in \mathcal{D}$. Now, $p \notin U_0 \cup U_1$. To show this assume, e.g., that $p \in U_0$. If $J_0 = [a, b]$, let $V = [b, p]$. Obviously, $b < p$ since $b \in \alpha$ has period > 1. By (4.3) there is an n such that $p \in f^{2n}(J_0)$. Hence

$$f^{2n}(J_0) \supset J_0 \cup V. \tag{4.4}$$

Since b is a periodic point in the orbit α_0 of f^2, there is an $m \leq 2^{k-1}$ such that $f^{2m}(b) = a$, and consequently,

$$f^{2m}(V) \supset [a, p] = J_0 \cup V. \tag{4.5}$$

By (4.4) and (4.5), f^{2mn} is turbulent and hence f has a periodic point of period $\neq 2^i$, $i \in \mathbb{N}$. This is a contradiction. To finish the argument note that, for *any* Darboux function f, the closure of an f-invariant interval is f-invariant. $\square$

Lemma 4.5. *Let $f \in \mathcal{J}$, and let $U, V \subset I$ be maximal compact intervals with the property, that, for some $k > 0$,*

$$f^k(U) = U, \; f^k(V) = V, \text{ but } f^j(U) \neq U, f^j(V) \neq V, \text{ for } 0 \leq j < k.$$

Then either $U = V$, or $U \cap V$ contains at most one point.

Proof. The argument is easy and follows from the fact that if U and V have non-empty intersection, then $U \cup V$ is a periodic interval of period k, or $k/2$ if the intervals belong to the same periodic orbit, k is even, and $f^{k/2}(U) = V$. $\square$

Lemma 4.6. *Let $f \in \mathcal{J}$ be a function. If there is $\varepsilon > 0$ and $m \in \mathbb{N}$ such that for every $k \geq m$ there is a compact interval $I_k \subset I$ with the following properties:*

(i) $\varepsilon < \operatorname{diam}(I_k) \leq 2\varepsilon$,

(ii) $I_k \supset I_{k+1}$,

(iii) $f^{2^k}(I_k) = I_k$ *and* $f^n(I_k) \neq I_k$ *for every* $0 < n < 2^k$,

(iv) $f^{2^k}(I_{k+1}) = J_{k+1} \subset I_k$,

where I_{k+1} *and* J_{k+1} *are nonoverlapping intervals, and* $S(n,\varepsilon) \subset I_m$ *is the maximal* (n,ε)*-separated set with respect to the function* $g = f^{2^m}$. *Then*

$$\limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon) = 0,$$

where $s_n(\varepsilon)$ *is cardinality of* $S(n,\varepsilon)$.

PROOF. We show by mathematical induction that $s_n(\varepsilon) \leq n+1$ for every $n \in \mathbb{N}$. Evidently if $n = 1$, the maximal $(1,\varepsilon)$-separated set $S(1,\varepsilon) \subset I_m$ contains 2 points. Suppose that $s_n(\varepsilon) \leq n+1$ and show that $s_{n+1}(\varepsilon) \leq n+2$. Let $J = \bigcap_{k=m}^{\infty} I_k$, and $S(n,\varepsilon) = \{x_0, x_1, \ldots, x_p\}$, where $p \leq n$. Let $g = f^{2^m}$. Assume that there are $u, v \in S(n+1,\varepsilon)$ such that $u, v \notin S(n,\varepsilon)$. Thus there exist $x_i, x_j \in S(n,\varepsilon)$ such that

$$\rho\left(g^n(u), g^n(x_i)\right) > \varepsilon \ \wedge \ \rho\left(g^l(u), g^l(x_i)\right) \leq \varepsilon, \tag{4.6}$$

$$\rho\left(g^n(v), g^n(x_j)\right) > \varepsilon \ \wedge \ \rho\left(g^l(v), g^l(x_j)\right) \leq \varepsilon, \tag{4.7}$$

for every $l < n$. Thus a pair of points $g^n(u)$ or $g^n(x_i)$ and $g^n(v)$ or $g^n(x_j)$ lie in J. (If not, some of them are not separable.)

CASE A. If $g^n(x_i) \in J$ and $g^n(x_j) \in J$, then $x_i, x_j \in J_k$ for some k and there is no $s < n$ such that $\rho\left(g^s(x_i), g^s(x_j)\right) > \varepsilon$. This is a contradiction.

CASE B. If $g^n(u),\ g^n(x_j) \in J$, then $x_j, u \in J_k$ for some k and

$$\rho\left(g^s(u), g^s(x_j)\right) \leq \varepsilon, \text{ for every } 0 \leq s < n.$$

By (4.6) we get

$$\rho\left(g^s(x_i), g^s(x_j)\right) \leq \varepsilon, \text{ for every } 0 \leq s < n,$$

and this is a contradiction. Similarly if $g^n(v),\ g^n(x_i) \in J$.

CASE C. If $g^n(u), g^n(v) \in J$, then $u, v \in J_k$ for some k. By (4.7) and by the fact, that $\rho(g^n(u), g^n(v))$ has to be greater than ε, we have

$$\rho\left(g^s(u), g^s(x_j)\right) \leq \varepsilon, \text{ for every } 0 \leq s \leq n,$$

which is a contradiction.

Since for all $n \in \mathbb{N}$, $S(n,\varepsilon)$ can contain at most $n+1$ points, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon) \le \limsup_{n\to\infty} \frac{1}{n} \log(n+1) = 0. \qquad \square$$

Theorem 4.7. *Let $f \in \mathcal{J}$. If the period of any periodic point of f is a power of 2, then $h(f) = 0$.*

PROOF. By Lemma 4.1 we may assume that, for any $n \in \mathbb{N}$, f has a periodic orbit of period 2^n. Let $\mathcal{M}_1$ be the system of maximal compact intervals $U \subset I$ such that $f^2(U) = U$, but $f(U) \ne U$. By Lemmas 4.4 and 4.5, $\mathcal{M}_1 \ne \emptyset$, and consists of nonoverlapping intervals. Letting $M_1 = \bigcup \mathcal{M}_1$ we can see, by Lemmas 4.4 and 2.11, that $\mathrm{Rec}(f) \setminus M_1 \subset \mathrm{Fix}(f)$. Consequently, by Theorem 3.8, $h(f|_{I\setminus M_1}) = 0$. By Proposition 3.5, $h(f) = h(f|_{M_1})$. Thus, it suffices to show that

$$h(f|_{M_1}) = 0. \tag{4.8}$$

To do this, fix an $\varepsilon > 0$. If every interval in $\mathcal{M}_1$ has length less than ε, then (4.8) is true. (Actually, we have $h(f^2|_{M_1}) = 0$.) If not, let $\mathcal{M}_k$ be the system of maximal compact intervals which are fixed by f^{2^k} but not by f^i, $0 < i < 2^k$. Let $M_k = \bigcup \mathcal{M}_k$. Then arguing similarly as before, we can see that (4.8) is satisfied if $h(f^{2^k}|_{M_k}) = 0$. Now if, for some k the intervals in $\mathcal{M}_k$ have diameters less than ε, we are done. Otherwise, there is an $m > 0$ such that in every $\mathcal{M}_k$, for $k \ge k_0$, there is the finite system $J_1^k, J_2^k, \dots J_m^k$ of intervals with diameter greater than ε. Moreover, for any i, $1 \le i \le m$, the intervals $\{J_i^k\}_{k=k_0}^{\infty}$ form a nested system. By Proposition 3.5, to prove (4.8) it suffices to show that

$$h(f^{2^k}|_{J_i^{k_0}}) = 0, \ 1 \le i \le m.$$

But this follows by Lemma 4.6. $\square$

Theorem 4.8. *If $f \in \mathcal{J}$ has a periodic point of period $2^k q$, where $q > 1$ is odd and $k \ge 1$, then $h(f) > 0$.*

PROOF. By Lemma 4.3, $f^{2^{k+2}}$ is turbulent, by Propositions 4.2 and 3.6, $h\left(f^{2^{k+2}}\right) > 0$, and hence, $h(f) > 0$. $\square$

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