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Tomasz Natkaniec, Institute of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland. email: `mattn@math.univ.gda.pl`

THE DENSITY TOPOLOGY CAN BE NOT EXTRARESOLVABLE

Abstract

We note that in some models of ZFC the density topology can be not extraresolvable.

A topological space X is called *extraresolvable* if there exists a family $\mathcal{D}$ of dense subsets of X such that $|\mathcal{D}| > \Delta(X)$, where $\Delta(X)$ is the dispersion character of X, and $D \cap D'$ is nowhere dense for every distinct $D, D' \in \mathcal{D}$. (See [3].)

Assuming Martin Axiom (MA) A. Bella proved that the real line $\mathbb{R}$ with the (Lebesgue) density topology $\mathcal{T}_d$ is extraresolvable [1]. He also asked whether this fact can be proved in ZFC. In this note we answer this question in the negative.

We use standard terminology. In particular, $|X|$ denotes the cardinality of a set X. For a topological space X, the dispersion character $\Delta(X)$ of X is the smallest cardinality of a non-empty open subset of X. If $\mathcal{J}$ is an ideal of subsets of a set X, then its cofinality $\mathrm{cf}(\mathcal{J})$ is the smallest cardinality of a basis of $\mathcal{J}$, that is, the family $\mathcal{J}_0 \subset \mathcal{J}$ such that each $J \in \mathcal{J}$ is contained in some $J_0 \in \mathcal{J}_0$. $\mathrm{shr}(\mathcal{J})$ is the minimal cardinal κ such that in each set $A \subset X$, $A \notin \mathcal{J}$, there is an $A_0 \subset A$ with $|A_0| \le \kappa$ and $A_0 \notin \mathcal{J}$.

It is well known that a set $A \subset \mathbb{R}$ is nowhere dense in the density topology if and only if it has a measure null. (See [4].) The ideal of null sets in $\mathbb{R}$ is denoted by $\mathcal{N}$.

We start with the following strengthening of [1, Proposition 3].

Lemma 1. *If X is a topological space,* NWD *is the ideal of nowhere dense sets in X, and $|X|^{\mathrm{shr}\,(\mathrm{NWD})} \le \Delta(X)$, then X is not extraresolvable.*

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PROOF. Suppose X is extraresolvable. Let $\{D_\alpha\colon \alpha < \kappa\}$ be a family of dense sets in X such that $\Delta(X) < \kappa$ and $D_\alpha \cap D_\beta \in \mathrm{NWD}$ whenever $\alpha < \beta < \kappa$. For each $\alpha < \kappa$ choose a set $E_\alpha \subset D_\alpha$ such that $|E_\alpha| \le \mathrm{shr}\,(\mathrm{NWD})$ and $E_\alpha \notin \mathrm{NWD}$. Since $|\{E_\alpha\colon \alpha < \kappa\}| \le |X|^{\mathrm{shr}\,(\mathrm{NWD})} \le \Delta(X) < \kappa$, there are $\alpha < \beta < \kappa$ for which $E_\alpha = E_\beta$. Then $E_\alpha \subset D_\alpha \cap D_\beta$, so $D_\alpha \cap D_\beta \notin \mathrm{NWD}$, a contradiction. □

Theorem 2. *There exists a model of ZFC in which the space $X = (\mathbb{R}, \mathcal{T}_d)$ is not extraresolvable.*

PROOF. We will work in ZFC with three additional axioms:

(A1) $\mathfrak{c} = \omega_2$;

(A2) $2^{\omega_1} = \omega_2$;

(A3) $\mathrm{cf}\,(\mathcal{N}) = \omega_1$.

It is known that the axioms (A1), (A2), (A3) are consistent with ZFC. In fact, (A1) and (A3) are consequences of so called *Covering Property Axiom* CPA, introduced by K. Ciesielski and J. Pawlikowski. Moreover, it is known that under CPA 2^{ω_1} can be arbitrarily large. (See [2].)

First note that (A3) implies that $\mathrm{shr}\,(\mathcal{N}) = \omega_1$. In fact, let $\{H_\alpha\colon \alpha < \omega_1\}$ be a basis of the ideal $\mathcal{N}$, that is, such that for each $N \in \mathcal{N}$ there is an $\alpha < \omega_1$ with $N \subset H_\alpha$. Assume $A \notin \mathcal{N}$. For each $\alpha < \omega_1$ choose an $x_\alpha \in A \setminus \bigcup_{\beta<\alpha} H_\alpha$. Then $A_0 = \{x_\alpha\colon \alpha < \omega_1\}$ is a subset of A and $A_0 \notin \mathcal{N}$.

It is clear that $\Delta(X) = \mathfrak{c}$. Thus (A1) implies $\Delta(X) = \omega_2$. Now we have $|X|^{\mathrm{shr}\,(\mathrm{NWD})} = \mathfrak{c}^{\mathrm{shr}\,(\mathcal{N})} = \mathfrak{c}^{\omega_1} = 2^{\omega_1}$. Therefore condition (A2) yields $|X|^{\mathrm{shr}\,(\mathrm{NWD})} = \omega_2 \le \Delta(X)$, and, by Lemma 1, we obtain that X is not extraresolvable. □

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