Real Analysis Exchange
Vol. 30(1), 2004/2005, pp. 261–266

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AN EXAMPLE OF AN ADDITIVE ALMOST CONTINUOUS SIERPIŃSKI-ZYGMUND FUNCTION

Dedicated to the memory of

Jerry Gibson

Abstract

Assuming that the union of fewer than $\mathfrak{c}$-many meager sets does not cover the real line, we construct an example of an additive almost continuous Sierpiński-Zygmund function which has a perfect road at each point but which does not have the Cantor intermediate value property.

Our terminology is standard. In particular, symbols $\mathbb{Q}$ and $\mathbb{R}$ stand for the sets of all rationals and reals, respectively. We consider only real-valued functions of one real variable. No distinction is made between a function and its graph. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$. If A is a planar set, we denote its x-projection by $\mathrm{dom}\,(A)$. $\mathcal{M}$ denotes the ideal of meager subsets of the real line and $\mathrm{cov}\,(\mathcal{M})$ is the minimal cardinality of a family of meager sets which cover $\mathbb{R}$. (Note that if $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$, $A \subset \mathbb{R}$ is residual in some open interval and B is the union of fewer than $\mathfrak{c}$ meager sets, then $A \setminus B$ is of size $\mathfrak{c}$.)

If $A \subset \mathbb{R}$ (or $A \subset \mathbb{R}^2$), then $\mathrm{LIN}\,(A)$ denotes the linear subspace of $\mathbb{R}$ ($\mathbb{R}^2$, respectively) over $\mathbb{Q}$ generated by A. (Note that if $A \subset \mathbb{R}^2$, then $\mathrm{dom}\,(\mathrm{LIN}\,(A))$ is a linear subspace of $\mathbb{R}$.) In particular, if $q \in \mathbb{Q}$ and $\langle x, y\rangle \in \mathbb{R}^2$, then $q\langle x, y\rangle = \langle qx, qy\rangle$ and if $q \in \mathbb{Q}$ and $A \subset \mathbb{R}^2$, then $qA = \{qa \colon a \in A\}$.

Key Words: almost continuous; additive function; Sierpiński-Zygmund function; perfect road; Cantor Intermediate Value Property
Mathematical Reviews subject classification: Primary 26A15; Secondary: 03E50
Received by the editors June 17, 2004
Communicated by: Krzysztof Chris Ciesielski

A function $f\colon \mathbb{R} \to \mathbb{R}$ is Sierpiński-Zygmund type (SZ function) if the restriction $f|A$ is discontinuous for each $A \subset \mathbb{R}$ of size $\mathfrak{c}$. Recall that f is an SZ function iff for every G_δ set $G \subset \mathbb{R}$ and for each continuous function $g\colon G \to \mathbb{R}$, f agrees with g on the set of size less than $\mathfrak{c}$ [SZ]. A function $f\colon \mathbb{R} \to \mathbb{R}$ is almost continuous (in the sense of Stallings, $f \in \mathrm{AC}$ shortly) if each open subset of the plane containing f contains also a continuous function $g\colon \mathbb{R} \to \mathbb{R}$. A blocking set $K \subset \mathbb{R}^2$ is a closed subset of $\mathbb{R}^2$ that meets the graph of every continuous function $g\colon \mathbb{R} \to \mathbb{R}$ and is disjoint with at least one function. Recall that if $f\colon \mathbb{R} \to \mathbb{R}$ intersects every blocking set, then it is almost continuous. Recall also that for each blocking set $K \subset \mathbb{R}^2$ there exists a continuous function g defined on a G_δ set $G \subset \mathbb{R}$ such that G is residual in some non-degenerate open interval $I \subset \mathbb{R}$ and $g \subset K$. (See [KK, Lemma 1] and the proof of [BCN, Theorem 1].)

A function $f\colon \mathbb{R} \to \mathbb{R}$ has a *perfect road* at $x \in \mathbb{R}$ if there exists a perfect set P with bilateral limit point x such that $f|P$ is continuous at x. PR is the class of all functions which have a perfect road at each point $x \in \mathbb{R}$.

f has the *Cantor intermediate value property* ($f \in \mathrm{CIVP}$) if for each $x, y \in \mathbb{R}$ and every perfect set C between $f(x)$ and $f(y)$ there exists a perfect set P between x and y with $f(P) \subset C$.

It is easy to construct an additive function $f \in \mathrm{SZ} \cap \mathrm{PR}$. (See [BCN, Theorem 2].) Ciesielski and Jastrzębski constructed an additive function $f \in \mathrm{AC} \cap \mathrm{PR} \setminus \mathrm{CIVP}$ [CJ, Example 5.1]. Assuming the real line $\mathbb{R}$ is not a union of fewer than $\mathfrak{c}$-many of its meager subsets, Balcerzak, Ciesielski and Natkaniec show that there exists a function $f \in \mathrm{AC} \cap \mathrm{SZ} \cap \mathrm{PR}$ [BCN, Theorem 1]. Moreover, they show that some additional set-theoretic assumptions are necessary, because the existence of an SZ function which is almost continuous is independent of ZFC axioms [BCN, Section 5]. (See also [GN], [GN1], and [KP].)

The aim of this note is to find a single example having all these properties at once.

Theorem 1. *Assume that* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. *There exists an additive function* $f \in \mathrm{SZ} \cap \mathrm{AC} \cap \mathrm{PR} \setminus \mathrm{CIVP}$.

Proof. Let $C \subset (0,1)$ be a Cantor set which is linearly independent over $\mathbb{Q}$. (See, e.g., [MK, Theorem 2, p. 270].) Let p be a bilateral limit point of C and let $H = \{t_\alpha\colon \alpha < \mathfrak{c}\}$ be a Hamel basis such that $C \subset H$, $t_0 = p$, and $t_1 \in C$. Let $\{K_\alpha\colon \alpha < \mathfrak{c}\}$ be the collection of all perfect nowhere dense subsets of $\mathbb{R}$, $\mathcal{G} = \{g_\alpha\colon \alpha < \mathfrak{c}\}$ be the family of all continuous functions defined on G_δ subsets of the real line and let $\{I_n\colon n < \omega\}$ be a sequence of all open intervals with rational end-points. We will define a sequence f_α, $\alpha < \mathfrak{c}$, of linear functions defined on subspaces of $\mathbb{R}$ with the following properties.

(P1) $t_\alpha \in \mathrm{dom}\,(f_\alpha)$ and $|\mathrm{dom}\,(f_\alpha)| < \mathfrak{c}$.

(P2) $f_\beta \subset f_\alpha$ if $\beta < \alpha$.

(P3) If $\mathrm{dom}\,(g_\alpha)$ is residual in some interval I, then there is $x \in I \cap \mathrm{dom}\,(f_\alpha)$ with $f_\alpha(x) = \lim_{t\to x} g_\alpha(t)$.

(P4) $f_\alpha \cap g_\beta \subset f_\beta$ whenever $\beta < \alpha$.

(P5) $f_0(t_0) = 0$, $f_0(t_1) = 1$.

(P6) $f_\alpha|C$ is continuous at t_0.

(P7) There exists $x_\alpha \in K_\alpha \cap \mathrm{dom}\,(f_\alpha)$ with $f_\alpha(x_\alpha) \notin C$.

Then by properties (P1) and (P2), $f = \bigcup_{\alpha<\mathfrak{c}} f_\alpha$ is an additive function defined on all of $\mathbb{R}$. The property (P4) implies $f \cap g_\beta \subset f_\beta$ for each $\beta < \mathfrak{c}$, so $|f \cap g_\beta| < \mathfrak{c}$ and consequently, $f \in \mathrm{SZ}$. The condition (P6) implies that $f \in \mathrm{PR}$. In fact, fix $x \in \mathbb{R}$ and set $z = x - t_0$. Then $C + z$ is a perfect set containing x as a bilateral limit point, and $f|(C+z)$ is continuous at x because $f|(C+z) = (f|C) + \langle z, f(z)\rangle$. The statements (P5) and (P7) together with $C \subset (0,1)$ give $f \notin \mathrm{CIVP}$.

Now we will verify that f is almost continuous. Fix a blocking set $K \subset \mathbb{R}^2$. Let α be the first ordinal for which there exist $q \in \mathbb{Q} \setminus \{0\}$, $n < \omega$ and $v \in f$, $v = \langle v_0, v_1\rangle$, such that $\mathrm{dom}\,[(qg_\alpha + v) \cap K]$ is residual in the interval I_n. Then $\mathrm{dom}\,(g_\alpha)$ is residual in the interval $J = q^{-1}(I_n - v_0)$. By (P3) there is $x \in J$ with $f_\alpha(x) = \lim_{t\to x} g_\alpha(t)$. Then $x' = qx + v_0 \in I_n$ and $\langle x', f(x')\rangle = \langle qx + v_0, qf(x) + v_1\rangle = q\langle x, f(x)\rangle + v = q\langle x, f_\alpha(x)\rangle + v \in \mathrm{cl}\,(qg_\alpha + v)$. Since $qg_\alpha + v$ is continuous and K is closed, this easily implies that $(I_n \times \mathbb{R}) \cap (qg_\alpha + v) = (I_n \times \mathbb{R}) \cap (qg_\alpha + v) \cap K$. Thus $\langle x', f(x')\rangle \in \mathrm{cl}\,((qg_\alpha + v) \cap K) \subset K$ and therefore $K \cap f \neq \emptyset$.

The functions f_α, $\alpha < \mathfrak{c}$, will be constructed by induction. Suppose α is fixed and all f_β, $\beta < \alpha$, are defined.

(i) Let $\bar{f}_\alpha = \mathrm{LIN}\,(\bigcup_{\beta<\alpha} f_\beta)$. We define a sequence $d_{\alpha,n}$, $n < \omega$, inductively in the following way. Let $D_{\alpha,n} = \{d_{\alpha,i} : i < n\} \setminus \{0\}$ and $f_{\alpha,n} = \mathrm{LIN}\,(\bar{f}_\alpha \cup (g_\alpha|D_{\alpha,n}))$. If

> **(∗)** $\mathrm{dom}\,(g_\alpha)$ is residual in I_n, and for all $\beta < \alpha$, $q \in \mathbb{Q}$ and $w \in f_{\alpha,n}$ the set $I_n \cap \mathrm{dom}\,[(qg_\beta + w) \cap g_\alpha]$ is nowhere dense,[1]

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[1]or equivalently, the set $I_n \cap \mathrm{dom}\,[(qg_\beta + w) \cap g_\alpha]$ is meager,

then $d_{\alpha,n} \in I_n \cap \mathrm{dom}\,(g_\alpha) \setminus \mathrm{LIN}\,(\mathrm{dom}\,(f_{\alpha,n}) \cup C)$ is such that

$$\mathrm{LIN}\ (\{\langle d_{\alpha,n}, g_\alpha(d_{\alpha,n})\rangle\} \cup f_{\alpha,n}) \cap \bigcup_{\beta<\alpha} g_\beta \subset f_{\alpha,n}. \tag{1}$$

Otherwise $d_{\alpha,n} = 0$.

(ii) Let $\tilde{f}_\alpha = \bigcup_{n<\omega} f_{\alpha,n}$. A real number t'_α has the following properties:

- **(a)** $t'_0 = 0$ and $t'_1 = 1$.
- **(b)** If $t_\alpha \in \mathrm{dom}\,(\tilde{f}_\alpha)$, then $t'_\alpha = \tilde{f}_\alpha(t_\alpha)$.
- **(c)** If $t_\alpha \in C \setminus \mathrm{dom}\,(\tilde{f}_\alpha)$, then $t'_\alpha \notin C$ and $|t'_\alpha - t'_0| < |t_\alpha - t_0|$.
- **(d)** For each $q \in \mathbb{Q}$, $\beta \leq \alpha$ and $x \in \mathrm{dom}\,(\tilde{f}_\alpha)$, if $qt_\alpha + x \notin \mathrm{dom}\,(\tilde{f}_\alpha)$, then the inequality $g_\beta(qt_\alpha + x) \neq qt'_\alpha + \tilde{f}_\alpha(x)$ holds.

(iii) Let $\hat{f}_\alpha = \mathrm{LIN}\,(\tilde{f}_\alpha \cup \{\langle t_\alpha, t'_\alpha\rangle\})$. Numbers $s_{\alpha,0}, \ldots, s_{\alpha,n}$, $s'_{\alpha,0}, \ldots, s'_{\alpha,n}$ have the following properties:

- **(a)** $s_{\alpha,0}, \ldots, s_{\alpha,n} \in H \setminus \mathrm{dom}\,(\hat{f}_\alpha)$ and there are $q_0, \ldots, q_n \in \mathbb{Q} \setminus \{0\}$ and $w \in \mathrm{dom}\,(\hat{f}_\alpha)$ such that $x_\alpha = \sum_{i=0}^n q_i s_{\alpha,i} + w \in K_\alpha \setminus \mathrm{dom}\,(\hat{f}_\alpha)$.
- **(b)** $\sum_{i=0}^n q_i s'_{\alpha,i} + \hat{f}_\alpha(w) \notin C$.
- **(c)** If $s_{\alpha,i} \in C$, then $|s'_{\alpha,i} - t'_0| < |s_{\alpha,i} - t_0|$.
- **(d)** $g_\beta(\sum_{i=0}^n p_i s_{\alpha,i} + x) \neq \sum_{i=0}^n p_i s'_{\alpha,i} + \hat{f}_\alpha(x)$ whenever $p_0, \ldots, p_n \in \mathbb{Q}$, $\sum_{i=0}^n p_i s_{\alpha,i} \neq 0$, $\beta \leq \alpha$, and $x \in \mathrm{dom}\,(\hat{f}_\alpha)$.

Put $f_\alpha = \mathrm{LIN}\,(\hat{f}_\alpha \cup \{\langle s_{\alpha,0}, s'_{\alpha,0}\rangle, \ldots, \langle s_{\alpha,n}, s'_{\alpha,n}\rangle\})$.

The existence of $s_{\alpha,0}, \ldots, s_{\alpha,n}$ follows from the fact that $\mathrm{dom}\,(\hat{f})$ is of size less than $\mathfrak{c}$, so $K_\alpha \not\subset \mathrm{dom}\,(\hat{f})$. The choice of t'_α is clear. Numbers $s'_{\alpha,i}$, $i \leq n$ are chosen by induction. We will show how to choose $d_{\alpha,n}$ in the case if $(*)$ holds. Observe that $\mathrm{dom}\,(f_{\alpha,n})$ is of size less than $\mathfrak{c}$, so the sets $A = I_n \cap \mathrm{dom}\left[\left(\mathbb{Q} \cdot \bigcup_{\beta<\alpha} g_\beta + f_{\alpha,n}\right) \cap g_\alpha\right]$ and $B = \mathrm{LIN}\,(\mathrm{dom}\,(f_{\alpha,n}) \cup C)$ are unions of fewer than $\mathfrak{c}$ many meager sets, and by $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$, the set $I_n \cap \mathrm{dom}\, g_\alpha \setminus (A \cup B)$ is non-empty. Choose $d_{\alpha,n}$ from this set. We have to verify that the condition (1) holds. Suppose there is $\beta < \alpha$ and $\langle x, y\rangle \in \mathrm{LIN}\,(\{\langle d_{\alpha,n}, g_\alpha(d_{\alpha,n})\rangle\} \cup f_{\alpha,n}) \cap g_\beta \setminus f_{\alpha,n}$. Then $\langle d_{\alpha,n}, g_\alpha(d_{\alpha,n})\rangle \in \mathbb{Q}g_\beta + f_{\alpha,n}$, so $d_{\alpha,n} \in \mathrm{dom}\,[(\mathbb{Q} \cdot \bigcup_{\beta<\alpha} g_\beta + f_{\alpha,n}) \cap g_\alpha]$, a contradiction.

It is easy to observe that f_α is a linear function having properties (P1), (P2) and (P5). (P4) is a consequence of (ii.d) and (iii.d). (P6) follows by (ii.c) and (iii.c), and (P7) by (iii.a) and (iii.b). To verify (P3) assume that $\mathrm{dom}\,(g_\alpha)$

is residual in I_n. If condition $(*)$ holds, then $d_{\alpha,n} \in \mathrm{dom}\,(f_\alpha \cap g_\alpha) \cap I_n$, and since g_α is continuous, $f_\alpha(d_{\alpha,n}) = \lim_{t\to d_{\alpha,n}} g_\alpha(t)$. Otherwise there are $\beta < \alpha$, $q \in \mathbb{Q}\setminus\{0\}$ and $w \in f_\alpha$, $w = \langle w_0, w_1\rangle$, such that $\mathrm{dom}\,[(qg_\beta + w)\cap g_\alpha]$ is residual in some interval $J \subset I_n$. (Note that for each $x \in J$, the limit of $qg_\beta + w$ at x exists iff the limit of g_α at x exists, and then those limits are equal.) Let $J' = q^{-1}(J - w_0)$. Then $\mathrm{dom}\,(g_\beta)$ is residual in J', so there is $x \in J' \cap \mathrm{dom}\,(f_\beta)$ with $f_\beta(x) = \lim_{t\to x} g_\beta(t)$. Therefore $x' = qx + w_0 \in J \cap \mathrm{dom}\,(f_\alpha)$ and $f_\alpha(x') = qf_\beta(x) + w_1 = \lim_{t\to x'} g_\alpha(t)$. □

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