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ON A SPACE OF BESICOVITCH FUNCTIONS

To the Bellaterrian cats in front of my window

Abstract

Let $C([0,1])$ be the set of all continuous functions mapping the unit interval $[0,1]$ into itself, equipped with the metric ρ of uniform convergence (and the induced topology τ). A function $f \in C([0,1])$ is called Besicovitch if it is nowhere one-sided differentiable (finite or infinite). For the Lebesgue measure λ we define the set $B(\lambda) \subset C([0,1])$ by

$$B(\lambda) = \{f \mid \forall \, \text{Borel } A \subset [0,1] \colon \; \lambda(A) = \lambda(f^{-1}(A)) \text{ and } f \text{ is Besicovitch}\}.$$

We construct a set $X \subset B(\lambda)$ such that the space $(X, \tau|X)$ is homeomorphic to the product topological space $(\prod_{i=0}^{\infty}[0,1), \mu)$.

1 Introduction.

The history of Besicovitch functions (a real-valued function of a real variable without finite or infinite one-sided derivatives) began many years ago with the classical work of Besicovitch [1]. From that time more authors have presented constructions of such functions and studied their properties (see e.g. [5], [6], [4], [3], [2]). The purpose of this note is not to give another new construction, but rather to provide an interesting insight into the set of all such functions. We restrict ourselves to the case of Besicovitch functions preserving Lebesgue measure. There are two main reasons why we prefer this restriction: First, we want to show that a coexistence of nowhere differentiability and a measure

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regularity is not so exceptional and "thin" as one could expect; second, as the reader will see, the measure regularity causes even simpler proofs of nowhere differentiability.

Let $C([0,1])$ be the set of all continuous functions mapping the unit interval $[0,1]$ into itself equipped with the metric ρ of uniform convergence. Denote τ the topology on $C([0,1])$ induced by ρ. A function $f \in C([0,1])$ is called Besicovitch if it has nowhere a finite or infinite one-sided derivative. For Lebesgue measure λ we put

$$B(\lambda) = \{f|\ \forall \text{ Borel } A \subset [0,1]\colon \lambda(A) = \lambda(f^{-1}(A)) \text{ and } f \text{ is Besicovitch}\}.$$

Our main result is the following.

Theorem 1.1. *There is a set* $X \subset B(\lambda)$ *such that the space* $(X, \tau|X)$ *is homeomorphic to the product topological space* $(\prod_{i=0}^{\infty}[0,1), \mu)$.

The paper is organized as follows.

In §2 we present a modified construction from [5]. We construct a function $f = f[\gamma] \in C([0,1])$ which depends on a parameter $\gamma \in \prod_{i=0}^{\infty}(4,\infty)$. In §3 we prove that f is Besicovitch and preserves Lebesgue measure. In §4 we give the proof of Theorem 1.1.

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2 Construction and Auxiliary Results.

2.1 Construction of a Canonical Step Triangle

For $k > 4$, let us construct in $[0,1/2]$ a discontinuum

$$D = [0,1/2] \setminus L, \text{ where } L = \bigcup_{m=1}^{\infty} \bigcup_{p=1}^{2^{m-1}} r_{m,p}, \tag{1}$$

and the open intervals $r_{m,p} = (a_{m,p}, b_{m,p})$ are constructed as follows:

(α) $d_{1,1} = [0,1/2]$, $r_{1,1} \subset d_{1,1}$, $\lambda(r_{1,1}) = \frac{1}{2k}$, $b_{1,1}$ is the center of $d_{1,1}$

(β) for $m > 1$, if $d_{m,1} \cdots d_{m,2^{m-1}}$ are (from left to right) the intervals of the set $[0,1/2] \setminus \bigcup_{q=1}^{m-1} \bigcup_{p=1}^{2^{q-1}} r_{q,p}$, then $r_{m,p} \subset d_{m,p}$, $b_{m,p}$ is the center of $d_{m,p}$ and $\lambda(r_{m,p}) = \frac{1}{2k^m}$.

Obviously, $\lambda(L) = \frac{1}{2(k-2)}$ and $\lambda(D) = \frac{k-3}{2(k-2)}$.

Let $\phi : [0, 1/2] \to [0, 1]$ be a nondecreasing continuous function satisfying $\phi(0) = 0$, $\phi(1/2) = 1$, ϕ is constant on every interval $r_{m,p}$, $\phi(r_{m,p}) = \{(2p - 1)/2^m\}$. Define a function $p : [0, 1] \to [0, 1]$ by

$$p(x) = \begin{cases} \phi(x) & \text{if } x \in [0, 1/2] \\ \phi(1-x) & \text{if } x \in [1/2, 1]. \end{cases} \tag{2}$$

The function p and the interval $[0, 1]$ form the well-known (canonical) step triangle [5]. The base $[0, 1]$ is lower than the vertex $(1/2, 1)$ - in this case we say that a step triangle is positively oriented. The set $\{(x, p(x));\ x \in [0, 1/2]\}$, resp. $\{(x, p(x));\ x \in [1/2, 1]\}$ is the left, resp. right side of the triangle.

In order to explain what we mean by a step triangle with a general base, height and parameter we use linear transformations. For two compact intervals $[a, b], [c, d] \subset \mathbb{R}$ define the map

$$r_{a,b,c,d}(x) = \frac{d-c}{b-a}(x-a) + c,\ x \in [a, b].$$

Definition 2.1. Let $[a, b]$, $[c, d]$ be non-degenerate compact intervals. Consider the functions p, $s = r_{a,b,0,1}$, $t = r_{0,1,c,d}$. The function $t \circ p \circ s \colon [a, b] \to [c, d]$ and the segment $S = \{(x, c) : a \le x \le b\}$ will form a step triangle with base S, height $d - c$ and parameter k. An orientation and sides of such triangle are defined analogously as for p.

Further, for $f \in C([0, 1])$ put $u_y = \{(x, y); x \in [0, 1]\}$ and let $g(f)$ be a graph of the function f. Below we present the construction of a Besicovitch function preserving Lebesgue measure. This construction is composed of steps. In each of them we work with step triangles introduced above (triangles differ by their bases, heights and parameters). For our purpose it is not necessary to give a precise formula for the number of triangles managed in the nth step. However we will assume that there is an increasing sequence

$$0 = m(0) < 2 = m(1) < m(2) < \cdots \tag{3}$$

of positive integers such that for $n > 0$, $(m(n) - m(n-1))$ step triangles are constructed in the nth step. (For $n = 0$ it is exactly one triangle.) Let

$$\Gamma = \{\gamma = \{\gamma(i)\}_{i=0}^{\infty} \colon \gamma(i) > 4 \text{ for each } i\}. \tag{4}$$

Let $\gamma \in \Gamma$. We can carry out the following construction of a function $f = f[\gamma]$:

2.2 Construction

0th step: We construct a positively oriented (canonical) step triangle with base $[0,1]$, height 1 and parameter $\gamma(0)$; the sides of the step triangle define a function f_0. All D-contiguous intervals (see (1)) will be called 0th L-segments.

nth step: For $n > 0$ we construct $m(n)-m(n-1)$ step triangles (positively or negatively oriented, numbered by $i \in \{m(n-1)+1,\dots,m(n)\}$ from left to the right) whose bases are segments of the set $\bigcup_{p=1}^{2^{n-1}} u_{\frac{2p-1}{2^n}} \cap g(f_{n-1})$ and heights $\frac{1}{2^n}$. For $i \in \{m(n-1)+1,\dots,m(n)\}$, the corresponding triangle has a parameter $\gamma(i)$ and is placed inwards the bigger triangle on whose side has its base. The union of sides of all triangles constructed so far define a function f_n. All new contiguous intervals (subintervals of some previous L-segments) will be called nth L-segments.

Since for each $n \in \mathbb{N}$, f_n is continuous and

$$\rho(f_{n-1}, f_n) = \frac{1}{2^n}, \tag{5}$$

the continuous map $f = f[\gamma] = \lim_{n\to\infty} f_n$ is well defined.

Several auxiliary lemmas follow. In the first one, we pick up the rather trivial fact that two step triangles (canonical, with the base $[0,1]$) associated to distinct values of parameters differ.

Lemma 2.2. *For $k < k'$ from $(4,\infty)$, let $p^k, p^{k'}$ be functions defined in (2). Then*

$$p^k(a^k_{1,1}) > p^{k'}(a^k_{1,1}).$$

More generally, we will need to be sure that distinct values γ, γ' give us distinct functions $f[\gamma], f[\gamma']$.

Lemma 2.3. *If $\gamma \neq \gamma'$, then $f[\gamma] \neq f[\gamma']$.*

PROOF. Let $k \in \mathbb{N} \cup \{0\}$ be the least value for which $\gamma(k) \neq \gamma'(k)$ and

$$k \in \{m(n-1)+1,\dots,m(n)\},$$

where the sequence $\{m(n)\}_{n=0}^{\infty}$ was defined in (3). Then by Lemma 2.2 the functions $f_n[\gamma]$ and $f_n[\gamma']$ differ in step triangles of common base associated to the values $\gamma(k), \gamma'(k)$. Assume that those triangles are built up on an L-segment S, they are positively oriented and $\gamma(k) < \gamma'(k)$. Using Definition 2.1 and Lemma 2.2, for the leftmost point a^k of maximal 0th L-segment (in S, for the parameter k) we have for some positive δ,

$$f_n[\gamma](a^k) - f_n[\gamma'](a^k) > \delta. \tag{6}$$

Since successive step triangles are placed inwards the precedent ones and $f_n[\gamma](a^k) = f[\gamma](a^k)$, we get from (6)

$$f[\gamma](a^k) - f[\gamma'](a^k) = f[\gamma](a^k) - f_n[\gamma'](a^k) > \delta.$$

In particular, $f[\gamma] \neq f[\gamma']$. The case when step triangles built up on an L-segment S are negatively oriented is analogous. $\square$

We end this section with the two obvious statements useful for our purpose. The first one will significantly simplify the proof that each $f = f[\gamma]$, $\gamma \in \Gamma$ is a Besicovitch function.

Lemma 2.4. *Let f preserve Lebesgue measure and assume that for some $x \in [0,1)$ ($\in (0,1]$), the function f has the right (left) derivative $a = f'_+(x)$ ($= f'_-(x)$) at x. Then $|a| \geq 1$.*

Proof. If $a = f'_+(x)$ exists and $a \in (-1,1)$, then there is an interval $[a, a+\varepsilon]$ such that $\lambda(f([a,a+\varepsilon])) < \lambda([a,a+\varepsilon])$. Since $[a,a+\varepsilon] \subset f^{-1}(f([a,a+\varepsilon]))$ and the map f preserves Lebesgue measure, we have

$$\lambda([a,a+\varepsilon]) \leq \lambda(f^{-1}(f([a,a+\varepsilon]))) = \lambda(f([a,a+\varepsilon])) < \lambda([a,a+\varepsilon]),$$

- a contradiction. The case when $a = f'_-(x) \in (-1,1)$ can be disproved analogously. $\square$

The second lemma gives two needed evaluations of lengths (slopes) of intervals (given by intervals) $d_{m,p}$ from Con. 2.1.

Lemma 2.5. *The following are true.*

1. *For $q = k/2$ and each $m > 1$,*

$$\min_p \lambda(d_{m,p}) = \frac{1}{(2q)^m}\ [q^m - \sum_{i=1}^{m-1} q^i], \ \ \max_p \lambda(d_{m,p}) = \frac{1}{2^m}.$$

2. *If $r_{m,p} = (a_{m,p}, b_{m,p}) \subset d_{m,p}$ and $x \in (b_{m,p}, \max d_{m,p}]$ is not a point of any 0th L-segment, then*

$$\frac{f(x) - f(b_{m,p})}{x - b_{m,p}} \leq \frac{2(k-2)}{k-4}. \tag{7}$$

Proof. (1) Since $\lambda(d_{m,1}) = \min_p \lambda(d_{m,p})$ and $\lambda(d_{m,2^{m-1}}) = \max_p \lambda(d_{m,p})$, both equalities can be easily verified. Let us show (2). By our definition,

$$f(\max d_{m,p}) - f(\min d_{m,p}) = \frac{1}{2^{m-1}}.$$

Let $A = \frac{1}{(2q)^m}\ [q^m - \sum_{i=1}^{m-1} q^i]$. Using $d_{m,1} = [0, a_{m-1,1}]$ and (1) for each m, p we get

$$\frac{f(\max d_{m,p}) - f(\min d_{m,p})}{\max d_{m,p} - \min d_{m,p}} \leq \frac{f(a_{m-1,1}) - f(0)}{a_{m-1,1}} = \frac{\frac{1}{2^{m-1}}}{A} = B(m).$$

Hence after some standard computation we obtain

$$\frac{f(\max d_{m,p}) - f(\min d_{m,p})}{\max d_{m,p} - \min d_{m,p}} \leq B(m) = 2\ \frac{(k-2)(\frac{k}{2})^{m-1}}{(k-4)(\frac{k}{2})^{m-1} + 2} < \frac{2(k-2)}{k-4}.$$

Now the inequality (7) follows from Construction 2.1. □

Remark 2.6. *From Lemma 2.5(1) it follows that for $k > 4$, $\min_p \lambda(d_{m,p}) > 0$ for each $m \in \mathbb{N}$.*

3 Properties of $f[\gamma]$, $\gamma \in \Gamma$.

Let f be defined on a (one-sided) neighborhood of x. The derived numbers $D^+f(x)$, $D_+f(x)$ of f at x are

$$D^+f(x) = \limsup_{h \to 0_+} \frac{f(x+h) - f(x)}{h},\ \ D_+f(x) = \liminf_{h \to 0_+} \frac{f(x+h) - f(x)}{h}$$

and the analogous limits from the left define $D^-f(x)$, $D_-f(x)$. Obviously f has a one-sided derivative at a point x if and only if either $D^+f(x) = D_+f(x)$ or $D^-f(x) = D_-f(x)$.

Theorem 3.1. *For each $\gamma \in \Gamma$, $f = f[\gamma]$ is Besicovitch and preserves Lebesgue measure; i,e., $f \in B(\lambda)$.*

PROOF. The proof that f preserves Lebesgue measure is literally the same as the one of Th.6 in [2].

Let us show that f is Besicovitch. To this end we distinguish several cases.

I. First, we assume that $x \in [0,1]$ is not a point of any 0th L-segment. Because of symmetry we consider points from $[0, \frac{1}{2}]$ only.

I(+) Assume that $x \in [0, \frac{1}{2})$ is not the left endpoint of any 0th L-segment and show that $f'_+(x)$ does not exist. To this end we use values of f at endpoints and centers of 0th L-segments.

Fix $h > 0$, let $r_{m,p} = (\alpha, \beta)$ be the maximal 0th L-segment contained in $(x, x+h)$, Put $\gamma = \frac{\alpha+\beta}{2}$. Since ϕ is nondecreasing, $f(x) = \phi(x) < \phi(\alpha) = f(\alpha)$. Let us show that $f(x) \geq f(\gamma)$. Really, by our construction $f(\alpha) = \frac{2p-1}{2^m}$ and $f(\gamma) = \frac{2p-1}{2^m} - \frac{1}{2^m} = \frac{p-1}{2^{m-1}}$. It means that $f(\gamma)$ coincides with the image

$\phi(r_{m',p'})$ of some $r_{m',p'}$ with $m' < m$. Then $\lambda(r_{m',p'}) > \lambda(r_{m,p})$.Hence from our choice of $r_{m,p}$ we get $a_{m',p'} < b_{m'p'} \leq x$. Since ϕ is nondecreasing, $f(\gamma) = \phi(r_{m',p'}) \leq \phi(x) = f(x)$. The number h was chosen arbitrarily small, hence $D_+f(x) \leq 0 \leq D^+f(x)$. By Lemma 2.4, any derivative $f'_+(x)$ does not exist.

I(-) Assume that $x \in (0, \frac{1}{2}]$ is not a point or the right endpoint of any 0th L-segment.

Fix $h > 0$ and denote $r_{m,p} = (\alpha, \beta)$ the maximal 0th L-segment contained in $(x-h, x)$, let $\gamma = \frac{\alpha+\beta}{2}$. Since $r_{m,p}$ is maximal, $x \in (\beta, \max d_{m,p}]$ and from Con. 2.1 and Lemma 2.5 we obtain

$$0 < x - \alpha = \beta - \alpha + x - \beta \leq \frac{1}{2k^m} + \frac{1}{2}\lambda(d_{m,p}) \leq \frac{1}{2k^m} + \frac{1}{2^{m+1}}. \tag{8}$$

Now, computing the difference $\Theta(\alpha, \gamma) = \frac{f(x)-f(\gamma)}{x-\gamma} - \frac{f(x)-f(\alpha)}{x-\alpha}$, again from Con. 2.1 and (8) we get

$$\Theta(\alpha,\gamma) > \frac{f(\alpha)-f(\gamma)}{x-\alpha} = \frac{1}{2^m(x-\alpha)} \geq \frac{1}{2^m(\frac{1}{2k^m} + \frac{1}{2^{m+1}})} = \frac{2}{1+(\frac{2}{k})^m}; \tag{9}$$

at the same time from Lemma 2.5(2) we have

$$\frac{f(x)-f(\beta)}{x-\beta} \leq \frac{2(k-2)}{k-4}. \tag{10}$$

Since h can be chosen arbitrarily small, the inequalities (9), (10) used for an increasing sequence of m's imply

$$D^-f(x) - D_-f(x) \geq 2 \quad \& \quad 0 \leq D_-f(x) \leq \frac{2(k-2)}{k-4};$$

i.e., $f'_-(x)$ does not exist.

II Second, we assume that for some $n \in \mathbb{N}$, $x \in [0,1]$ is a point of some $(n-1)$st L-segment (a,b) and does not belong to any nth L-segment. Then the point $(x, f(x))$ lies on the side of a step triangle Δ with the base $\{(z, f_{n-1}(z)) \colon z \in [a,b]\}$, height $\frac{1}{2^n}$ and some parameter $k > 4$. Suppose that the step triangle Δ is positively oriented. Using the fact - see Definition 2.1 - that Δ was created with the help of linear transformations $r_{a,b,0,1}$, $r_{0,1,f(a),f(a)+1/2^n}$, for one-sided derivatives of f at x we get: if $x \in [a, \frac{a+b}{2})$ is not the left endpoint of any nth L-segment, then $D_+f(x) \leq 0 \leq D^+f(x)$ and we can use Lemma 2.4 again; if $x \in (a, \frac{a+b}{2}]$ is not the right endpoint of any nth L-segment, then

$$D^-f(x) - D_-f(x) \geq 2\,\frac{1}{2^n(b-a)} \quad \& \quad 0 \leq D_-f(x) \leq \frac{2(k-2)}{k-4}\,\frac{1}{2^n(b-a)};$$

i.e., no one-sided derivative at x exists. As above we can use the symmetry to conclude the same for points from $[\frac{a+b}{2}, b]$.

The case when the step triangle is negatively oriented is analogous: if $x \in [a, \frac{a+b}{2})$ is not the left endpoint of any nth L-segment, then $D_+f(x) \le 0 \le D^+f(x)$ and we can use Lemma 2.4 again; if $x \in (a, \frac{a+b}{2}]$ is not the right endpoint of any nth L-segment, then

$$D^-f(x) - D_-f(x) \ge 2\ \frac{1}{2^n(b-a)} \quad \& \quad 0 \ge D^-f(x) \ge \frac{-2(k-2)}{k-4}\ \frac{1}{2^n(b-a)};$$

i.e., as above no one-sided derivative at x exists.

III Let $x \in [0,1]$ belong to L-segments of all orders. Then $(x, f(x))$ has to lie on sides of infinitely many step triangles whose bases (their lengths) and heights converge to zero. Moreover, by Con. 2.2 infinitely many of them are positively/negatively oriented. The reader can easily verify alone that in such a case we get $D_+f(x) \le 0 \le D^+f(x)$ and $D_-f(x) \le 0 \le D^-f(x)$; i.e., by Lemma 2.4 no one-sided derivative at x exists. □

Remark 3.2. The proof of (2) is in fact simpler than the original one in [5]. This is caused by the fact that our map preserves the Lebesgue measure.

4 Properties of $(X, \tau|X)$.

For $\tilde\Gamma \subset \Gamma$ (see (4)) defined by

$$\tilde\Gamma = \{\gamma = \{\gamma(i)\}_{i=0}^{\infty} \colon \gamma(i) \ge 5 \text{ for each } i\}$$

let us put $X = X(\tilde\Gamma) = \{f \in C([0,1]) \colon f = f[\gamma] \text{ for some } \gamma \in \tilde\Gamma\}$. We know from Theorem 3.1 that $X \subset B(\lambda)$. Consider the topological space $(X, \tau|X)$, where τ is the topology on $C([0,1])$ induced by the metric ρ of uniform convergence. In this section we show that the product space $(\prod_{i=0}^{\infty}[0,1), \mu)$ and $(X, \nu = \tau|X)$ are homeomorphic. Without loss of generality and to simplify our notation we identify $(\prod_{i=0}^{\infty}[0,1), \mu)$ with $(\tilde\Gamma, \mu)$.

From Lemma 2.3 we obtain the following.

Lemma 4.1. *The map $F\colon\ \tilde\Gamma \to X$ defined by $F(\gamma) = f[\gamma]$ is a bijection.*

We need to show that F is even a homeomorphism. Since both the topological spaces $(\tilde\Gamma, \mu)$, (X, ν) are second countable, checking the continuity of F, F^{-1} we can restrict ourselves to the case of convergent sequences.

We let the reader prove the next lemma. We suppose that its validity can be easily seen from the construction of f_n but a complete formal proof would be rather long and vain.

Lemma 4.2. *For every $n \in \mathbb{N} \cup \{0\}$ and sequence $\{\gamma_m\}_m \subset \tilde{\Gamma}$,*

$$\gamma_m \to_\mu \gamma \iff f_n[\gamma_m] \to_\nu f_n[\gamma]. \tag{11}$$

The final property we need to show is the following.

Theorem 4.3. *For every sequence $\{\gamma_m\}_m \subset \tilde{\Gamma}$,*

$$\gamma_m \to_\mu \gamma \iff f[\gamma_m] \to_\nu f[\gamma].$$

Proof. Suppose that $\gamma_m \to_\mu \gamma$ and fixed $\varepsilon > 0$. Using (11), we can find an n and m_0 such that $\frac{1}{2^n} < \frac{\varepsilon}{3}$ and for each $m > m_0$ also $\rho(f_n[\gamma_m], f_n[\gamma]) < \frac{\varepsilon}{3}$. Then from (5) we obtain

$$\begin{aligned}\rho(f[\gamma_m], f[\gamma]) &\le \rho(f[\gamma_m], f_n[\gamma_m]) + \rho(f_n[\gamma_m], f[\gamma]) \\ &\le \rho(f[\gamma_m], f_n[\gamma_m]) + \rho(f_n[\gamma_m], f_n[\gamma]) + \rho(f_n[\gamma], f[\gamma]) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Hence $f[\gamma_m] \to_\nu f[\gamma]$.

On the other hand, when $\gamma_m \nrightarrow_\mu \gamma$, there is the least value $k \in \mathbb{N} \cup \{0\}$ (let us take it as in Lemma 2.3) for which $\gamma_m(k) \nrightarrow \gamma(k)$ and $k \in \{m(n-1)+1, \dots, m(n)\}$ for the sequence (3). Without loss of generality we can assume that for γ' from Lemma 2.3, $\gamma(i) = \gamma'(i)$ for each $i \in \{0, \dots, k-1\}$ and $\liminf_m \gamma_m(k) \ge \gamma'(k)(> \gamma(k))$. Then for the same L-segment S, its leftmost point a^k and δ from Lemma 2.3 we get

$$f[\gamma](a^k) - \delta > \limsup_m f[\gamma_m](a^k);$$

i.e., $f[\gamma_m] \nrightarrow_\nu f[\gamma]$. This proves the theorem. □

Now the proof of Theorem 1.1 directly follows from Theorem 3.1 and Theorem 4.3.

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