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ON THE MEASURABILITY OF FUNCTIONS SATISFYING SOME APPROXIMATE QUASICONTINUITY CONDITIONS

Abstract

In this article we investigate the smallest (in the sense of inclusion) σ-field of subsets of $\mathbb{R}$ in which all functions of some families of functions from $\mathbb{R}$ to $\mathbb{R}$ satisfying some approximate quasicontinuity conditions introduced in [2] are measurable.

Let $\mathbb{R}$ be the set of all reals and $\mathcal{H}$ a family of functions $f : \mathbb{R} \to \mathbb{R}$. Then there is the smallest (in the sense of inclusion) σ-field $\mathcal{A}(\mathcal{H})$ of subsets of $\mathbb{R}$ such that each function $f \in \mathcal{H}$ is $\mathcal{A}(\mathcal{H})$-measurable; i.e., for every Borel set $U \subset \mathbb{R}$ the preimage $f^{-1}(U) \in \mathcal{A}(\mathcal{H})$.

It is evident that for each family $\mathcal{H}$ of Borel measurable functions from $\mathbb{R}$ to $\mathbb{R}$ containing all continuous functions the σ-field $\mathcal{A}(\mathcal{H})$ is the σ-field $\mathcal{B}$ of all Borel subsets of $\mathbb{R}$.

Remark 1. *Suppose that $\mathcal{I}$ is an ideal of subsets of $\mathbb{R}$ and that $\mathcal{B}(\mathcal{I})$ is the σ-field generated by the union $\mathcal{B} \cup \mathcal{I}$. If $\mathcal{H}$ is the family of all functions $f : \mathbb{R} \to \mathbb{R}$ such that the set $D(f)$ of all discontinuity points of f belongs to $\mathcal{I}$, then $\mathcal{A}(\mathcal{H}) \subset \mathcal{B}(\mathcal{I})$.*

Proof. f $U \subset \mathbb{R}$ is an open set and $f \in \mathcal{H}$, then for each point $x \in f^{-1}(U) \cap C(f)$ ($C(f)$ denotes the set of all continuity points of f) we have $x \in \operatorname{int}(f^{-1}(U))$, where int denotes the interior operation. So $f^{-1}(U) \setminus \operatorname{int}(f^{-1}(U)) \in \mathcal{B}(\mathcal{I})$ and consequently $f^{-1}(W) \in \mathcal{B}(\mathcal{I})$ for every Borel set $W \in \mathcal{B}$. □

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Now we will consider some families of functions introduced in [2].

For this denote by μ the Lebesgue measure in $\mathbb{R}$ and by μ_e the outer Lebesgue measure in $\mathbb{R}$. For a set $A \subset \mathbb{R}$ and a point x we define the upper (lower) outer density $D_u(A,x)$ ($D_l(A,x)$) of the set A at the point x as

$$\limsup_{h\to 0^+} \frac{\mu_e(A\cap[x-h,x+h])}{2h}$$

$$(\liminf_{h\to 0^+} \frac{\mu_e(A\cap[x-h,x+h])}{2h} \text{ respectively}).$$

A point x is said to be an outer density point (a density point) of a set A if $D_l(A,x) = 1$ (if there is a Lebesgue measurable set $B \subset A$ such that $D_l(B,x) = 1$).

The family T_d of all sets A for which the implication

$$x \in A \Longrightarrow x \text{ is a density point of } A$$

is true, is a topology called the density topology ([1, 8]).

The sets $A \in T_d$ are Lebesgue measurable ([1, 8]).

Let T_e be the Euclidean topology in $\mathbb{R}$. The continuity of maps f from $(\mathbb{R},T_d)$ to $(\mathbb{R},T_e)$ is called approximate continuity ([1, 8]).

For an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ denote by $C_{ap}(f)$ the set of all approximate continuity points of f. Moreover let $D_{ap}(f) = \mathbb{R} \setminus C_{ap}(f)$.

In [2] the following properties were investigated.

A function $f : \mathbb{R} \to \mathbb{R}$ has the property (s_0) (resp. (s_5)) at a point x ($f \in s_0(x)$ or resp. $f \in s_5(x)$) if for each real $r > 0$ and for each set $U \in T_d$ containing x there is a point $t \in U \cap C(f)$ (resp. $t \in U \cap C_{ap}(f)$) with $|f(t) - f(x)| < r$.

A function $f : \mathbb{R} \to \mathbb{R}$ has the property (s_1) (resp. (s_3)) at a point x ($f \in s_1(x)$ or resp. $f \in s_3(x)$) if for each positive real r and for each set $U \in T_d$ containing x there is an open interval I such that $\emptyset \neq I \cap U \subset C(f)$ (resp. $\emptyset \neq I \cap U \subset C_{ap}(f)$) and $|f(t) - f(x)| < r$ for all points $t \in I \cap U$.

A function f has the property (s_i), where $i = 0, 1, 3, 5$, if $f \in s_i(x)$ for every point $x \in \mathbb{R}$.

A function $f : \mathbb{R} \to \mathbb{R}$ has the property (s_2) (resp. (s_4)) if for each nonempty open set $U \in T_d$ there is an open interval I such that $\emptyset \neq I \cap U \subset C(f)$ (resp. $\emptyset \neq I \cap U \subset C_{ap}(f)$).

Evidently each function f having the property (s_1) has also properties (s_2), (s_3), (s_4) and (s_0). Moreover the property (s_3) implies properties (s_0) ([2]), (s_4) and (s_5).

For each function f having the property (s_2) the set $D(f) = \mathbb{R} \setminus C(f)$ is nowhere dense and of Lebesgue measure 0. But the closure $\mathrm{cl}(D(f))$ of some functions f having the property (s_1) may be of positive measure. For example, if $A \subset (0,1)$ is a Cantor set of positive measure, (I_n) is a sequence of all components of the set $(0,1) \setminus A$ with $I_n \neq I_m$ for $n \neq m$ and (J_n) is a sequence of closed nondegenerate intervals $J_n \subset I_n$ with the same centers as I_n and such that

$$\frac{\mu(J_n)}{\mu(I_n)} < \frac{1}{n} \text{ for } n = 1, 2, \ldots,$$

then the function

$$f(x) = \frac{1}{n} \text{ for } x \in J_n, \ \ n = 1, 2, \ldots, \text{ and } f(x) = 0 \text{ otherwise on } \mathbb{R}$$

has the property (s_1) but $\mu($
$cl(D(f))) > 0$.

From [2] (p. 172) and [5] it follows that for each function f having property (s_0) the measure $\mu(D(f)) = 0$.

Let S_i, $i = 0, 1, 2, 3, 4, 5$ be the family of all functions $f : \mathbb{R} \to \mathbb{R}$ having property (s_i) and let P_0 denote the family of all functions f with $\mu(D(f)) = 0$.

Theorem 1. *For $i = 0, 1, 2, 3$ the equalities*

$$\mathcal{A}(S_i) = \mathcal{A}(P_0)$$

are true. Moreover, we have $\mathcal{A}(P_0) = \mathcal{B}(\mathcal{I}_0)$, where $\mathcal{I}_0$ is the σ-ideal of all subsets $A \subset \mathbb{R}$ such that there are F_σ-sets $E \subset \mathbb{R}$ of measure 0 with $A \subset E$.

PROOF. The inclusions

$$\mathcal{A}(S_i) \subset \mathcal{A}(P_0) \subset \mathcal{B}(\mathcal{I}_0) \text{ for } i = 0, 1, 2, 3,$$

follow from the inclusions $S_i \subset P_0$.

The identity $\text{id}(x) = x$ for $x \in \mathbb{R}$ is continuous and for every Borel set A we have $\text{id}^{-1}(A) = A$, so the inclusion $\mathcal{B} \subset \mathcal{A}(S_1)$ is true.

Let $E \subset \mathbb{R}$ be a set such that $\mu(\text{cl}(E)) = 0$ (cl denotes the closure operation). We will prove that there is a function $f \in S_1$ such that $f^{-1}(0) = E$.

In this construction we apply the following lemma from [4].

Lemma 1. *If $A \subset \mathbb{R}$ is a nonempty compact set of Lebesgue measure zero, $U \supset A$ is an open set, then there is a family $\{K_{i,j}; i, j = 1, 2, \ldots\}$ of pairwise disjoint nondegenerate closed intervals $K_{i,j} \subset U \setminus A$ such that for each positive integer i and for each point $x \in A$ the upper density*

$$D_u(\bigcup_{j=1}^{\infty} K_{i,j}, x) = 1$$

and for each positive real r the set of all pairs (i, j) for which there are points $t \in K_{i,j}$ and $x \in A$ with $|t - x| \geq r$ is empty or finite.

Continuation of the proof of Theorem 1.

In the beginning we suppose that the set E is bounded. By Lemma 1 there is a family of pairwise disjoint closed intervals

$$K_{i,j} \subset \mathbb{R} \setminus \mathrm{cl}(E),$$

$i, j = 1, 2, \ldots$ such that for each $i = 1, 2, \ldots$ and for each $x \in \mathrm{cl}(E)$ the upper density $D_u(\bigcup_{j=1}^{\infty} K_{i,j}, x) = 1$ and for each positive real r the set of pairs (i, j) such that there are points $x \in \mathrm{cl}(E)$ and $y \in K_{i,j}$ with $|x - y| \geq r$ is empty or finite.

In the interiors $\mathrm{int}(K_{i,j})$ we find closed intervals $I_{i,j} \subset \mathrm{int}(K_{i,j})$ such that for each point $x \in \mathrm{cl}(E)$ and for each integer $i = 1, 2, \ldots$ the upper density

$$D_u(\bigcup_{j=1}^{\infty} I_{i,j}, x) = 1.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} \frac{1}{i} & \text{for } x \in I_{i,j}, \quad i, j = 1, 2, \ldots \\ 1 & \text{for } x \in \mathbb{R} \setminus (\mathrm{cl}(E) \cup \bigcup_{i,j=1}^{\infty} \mathrm{int}(K_{i,j})) \\ 1 & \text{for } x \in \mathrm{cl}(E) \setminus E \\ 0 & \text{for } x \in E, \end{cases}$$

and let g be linear on all components of the sets $K_{i,j} \setminus \mathrm{int}(I_{i,j})$, $i, j = 1, 2, \ldots$.

We will prove that the function g has the property (s_1). For this, fix a positive real r, a point $x \in \mathbb{R}$ and a set $U \in T_d$ containing x. If $x \in \mathbb{R} \setminus \mathrm{cl}(E)$, then g is continuous at x and consequently $g \in s_1(x)$.

If $x \in \mathrm{cl}(E) \setminus E$, then

$$g(x) = 1 \text{ and } D_u(\bigcup_{j=1}^{\infty} I_{1,j}, x) = 1.$$

So there is an index j_0 such that $U \cap \mathrm{int}(I_{1,j_0}) \neq \emptyset$. Since $g(t) = 1$ for $t \in I_{1,j_0}$, we have $g \in s_1(x)$.

If $x \in E$, then $g(x) = 0$ and there is a positive integer i_1 with $\frac{1}{i_1} < r$. Since

$$g(t) = \frac{1}{i_1} \text{ for } t \in \bigcup_{j=1}^{\infty} I_{i_1,j},$$

and

$$D_u(\bigcup_{j=1}^{\infty} I_{i_1,j}, x) = 1,$$

there is an index j_1 with

$$U \cap \text{int}(I_{i_1,j_1}) \neq \emptyset$$

and

$$|g(t) - g(x)| = g(t) = \frac{1}{i_1} < r \text{ for } t \in \text{int}(I_{i_1,j_1}).$$

So $g \in S_1$.

Up to now we have supposed that the set E is bounded. Now we consider the general case. We have

$$\mathbb{R} = \bigcup_{k=-\infty}^{\infty} [x_k, x_{k+1}],$$

where $x_k \in \mathbb{R} \setminus \text{cl}(E)$ and

$$-\infty \leftarrow x_{-k} < x_{-k+1} < \cdots < x_0 < \cdots x_k < x_{k+1} \rightarrow \infty.$$

For every integer $k = 0, 1, -1, 2, -2, \dots$ there is a function $g_k : [x_k, x_{k+1}] \rightarrow [0, 1]$ having property (s_1) such that

$$g_k^{-1}(0) = E \cap [x_k, x_{k+1}] \text{ and } D(g_k) = \text{cl}(E) \cap (x_k, x_{k+1}).$$

Putting

$$f(x) = g_k(x) \text{ for } x \in [x_k, x_{k+1}), \quad k = 0, 1, -1, 2, -2, \dots$$

we obtain a function g having property (s_1) such that $g^{-1}(0) = E$.

Since each set E with $\mu(\text{cl}(E)) = 0$ belongs to $\mathcal{A}(S_1)$ and since $\mathcal{B} \subset \mathcal{A}(S_1)$, we obtain

$$\mathcal{B} \cup \mathcal{I}_0 \subset \mathcal{A}(S_1),$$

and consequently

$$\mathcal{B}(\mathcal{I}_0) \subset \mathcal{A}(S_1).$$

But $\mathcal{A}(S_1) \subset \mathcal{A}(S_i)$ for $i = 0, 1, 2, 3$, so the proof is completed. □

Now we will describe the field $\mathcal{A}(S_4)$. For this we put

$$\mathcal{I}_1 = \{A \subset \mathbb{R} : \text{if} T_d \ni B \subset \text{cl}(A), \text{ then } A \cap B \text{ is nowhere dense in } B\}.$$

Evidently $\mathcal{I}_1$ is an ideal of subsets of $\mathbb{R}$ (see [6]), but it is not an σ-ideal. Let $\mathcal{I}_2$ be the smallest σ-ideal generated by $\mathcal{I}_1$. Since each closed set of measure zero belongs to $\mathcal{I}_1$, we have $\mathcal{I}_0 \subset \mathcal{I}_2$.

Example. If $C \subset (0,1)$ is a Cantor set of positive measure and A is the set of the centers of all components of the set $(0,1) \setminus C$, then $\mathrm{cl}(A) \supset C$ and $\mu(\mathrm{cl}(A)) > 0$. Consequently, A is not in $\mathcal{I}_0$, but evidently $A \in \mathcal{I}_1 \subset \mathcal{I}_2$.

Since the sets belonging to $\mathcal{I}_1$ are nowhere dense and of measure zero, we have $\mathcal{I}_2 \subset \mathcal{M} \cap \mathcal{L}$, where $\mathcal{M}$ (and resp. $\mathcal{L}$) denotes the σ-field of all subsets with the Baire property (resp. the σ-field of all subsets which are measurable in the Lebesgue sense).

Theorem 2. *There are sets $H \in (\mathcal{M} \cap \mathcal{L}) \setminus \mathcal{B}(\mathcal{I}_1)$.*

PROOF. Let $C \subset (0,1)$ be a nowhere dense closed set of positive measure, let

$$A = \{x \in C : D_l(C,x) = 1\}$$

and let $B \subset \mathrm{cl}(A)$ be a G_δ-set of measure zero dense in A. Assume the Continuum Hypothesis (CH) and enumerate all uncountable Borel subsets of B in a transfinite sequence

$$A_0, A_1, \ldots, A_\alpha, \ldots, \quad \alpha < \omega_1,$$

(ω_1 denotes the first uncountable ordinal), such that

$$A_\alpha \neq A_\beta \text{ for } \alpha < \beta < \omega_1.$$

Now, by transfinite induction, we construct two disjoint sets $H, G \subset B$ such that for each $\alpha < \omega_1$ we have

$$A_\alpha \cap H \neq \emptyset \text{ and } A_\alpha \cap G \neq \emptyset.$$

Then the set H is nowhere dense set of measure zero, so it belongs to $\mathcal{M} \cap \mathcal{L}$. Observe that B is a residual subset of $\mathrm{cl}(A)$ and H, G are of the second category in $\mathrm{cl}(A)$.

We will prove that H is not in $\mathcal{B}(\mathcal{I}_1)$. Assume to the contrary that $H \in \mathcal{B}(\mathcal{I}_1)$. Then

$$H = (H_1 \cup \ldots \cup H_n \cup \ldots) \cup E,$$

where E is a Borel set and $H_n \in \mathcal{I}_1$ for $n \geq 1$. Since $G \subset B \setminus H$ cuts each uncountable Borel subset of B, the set E must be countable. But H is of the second category in $\mathrm{cl}(A)$, so there is a positive integer k such that the set H_k also is of the second category in $\mathrm{cl}(A)$. Thus there is an open interval J such that

$$J \cap H_k \neq \emptyset \text{ and } J \cap \mathrm{cl}(A) \subset \mathrm{cl}(J \cap H_k).$$

Then $T_d \ni A \cap J \subset \mathrm{cl}(H_k \cap J)$ and consequently H_k is not in $\mathcal{I}_1$. This contradiction finishes the proof. □

In the proof of the next theorem we will use the following lemma.

Lemma 2. *If $A \subset \mathbb{R}$ is a Borel set such that each point $x \in A$ is a density point of A and if $f : A \to \mathbb{R}$ is a function approximately continuous at each point $x \in A$, then f is a Borel function on A.*

PROOF. Without loss of generality we can assume that f is bounded, since in the opposite case we can consider the function $\arctan(f)$.

Let

$$g(x) = f(x) \text{ for } x \in A \text{ and let } g(x) = 0 \text{ for } x \in \mathbb{R} \setminus A$$

and let

$$F(x) = \int_0^x g(t)dt.$$

Then the functions

$$F_n(x) = \frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{n}}, \quad n \geq 1,$$

are continuous and for $x \in A$ we have

$$f(x) = F'(x) = \lim_{n \to \infty} F_n(x),$$

so f is a Borel function on A. □

Theorem 3. *The equality*

$$\mathcal{A}(S_4) = \mathcal{B}(\mathcal{I}_1)$$

is true.

PROOF. For the proof of the inclusion

$$\mathcal{B}(\mathcal{I}_1) \subset \mathcal{A}(S_4)$$

observe that

$$\mathcal{B} \subset \mathcal{A}(S_1) \subset \mathcal{A}(S_4),$$

and that for each set $A \in \mathcal{I}_1$ the function

$$f(x) = 1 \text{ for } x \in A, \text{ and } f(x) = 0 \text{ for } x \in \mathbb{R} \setminus A$$

belongs to S_4. Of course, fix a nonempty set $B \in T_d$. If $B \setminus \mathrm{cl}(A) \neq \emptyset$, then there is an open interval J such that $J \setminus \mathrm{cl}(A) = \emptyset$ and $B \cap J \neq \emptyset$. Evidently $f|(J \cap B) = 0$.

If $B \subset \mathrm{cl}(A)$, then the set $B \cap A$ is nowhere dense in B and there is an open interval J such that $J \cap B \neq \emptyset$ and $J \cap A = \emptyset$. Consequently, $f|(J \cap B) = 0$. This proves that $f \in S_4$. Consequently $A = f^{-1}(1) \in \mathcal{A}(S_4)$.

For the proof of the inverse inclusion assume that $f \in S_4$ is a function and that a is a real. We apply transfinite induction.

Let

$$A_0 = \mathrm{int}(C_{ap}(f)), \text{ and } B_1 = \mathbb{R} \setminus A_0.$$

If $\mu(B_1) > 0$, then the set

$$E_1 = \{x \in B_1 : D_l(B_1, x) = 1\} \in T_d.$$

Since $E_1 \neq \emptyset$ and $f \in S_4$, there is an open interval K_1 with rational endpoints such that

$$\emptyset \neq K_1 \cap E_1 \subset C_{ap}(f).$$

If $B_2 = B_1 \setminus K_1$ and $\mu(B_2) > 0$, then we put

$$E_2 = \{x \in B_2 : D_l(B_2, x) = 1\}$$

and observe that $\emptyset \neq E_2 \in T_d$. Since $f \in S_4$, there is an open interval K_2 with rational endpoints such that

$$\emptyset \neq K_2 \cap E_2 \subset C_{ap}(f).$$

Suppose that $\alpha < \omega_1$ (ω_1 denotes the first uncountable ordinal) and for each ordinal β with $\beta < \alpha$ there is an open interval K_β with rational endpoints such that the set

$$B_\beta = B_1 \setminus \bigcup_{\gamma < \beta} K_\gamma$$

is of positive measure and for the set

$$E_\beta = \{x \in B_\beta : D_l(B_\beta, x) = 1\}$$

we have

$$\emptyset \neq K_\beta \cap E_\beta \subset C_{ap}(f).$$

If the set

$$B_\alpha = B_1 \setminus \bigcup_{\beta < \alpha} K_\beta$$

is of positive measure, then we put

$$E_\alpha = \{x \in B_\alpha : D_l(B_\alpha, x) = 1\}.$$

Since $f \in S_4$ and $\emptyset \neq E_\alpha \in T_d$, there is an open interval K_α with rational endpoints such that

$$\emptyset \neq K_\alpha \cap E_\alpha \subset C_{ap}(f).$$

But the family of all open intervals with rational endpoints is countable, so the smallest ordinal α_0 such that $\mu(B_{\alpha_0}) = 0$ is countable.

Since the set of all density points of a Borel set is a Borel set (see [7]), for all ordinals $\alpha < \alpha_0$ the set E_α is a Borel set. By Lemma 2 the restricted functions $f|(K_\alpha \cap E_\alpha)$, $\alpha < \alpha_0$, are Borel measurable. Consequently, for $\alpha < \alpha_0$ the sets

$$G_\alpha = \{x \in K_\alpha \cap E_\alpha : f(x) < a\}$$

are Borel sets and the set

$$A_0 \cup \bigcup_{\alpha < \alpha_0} G_\alpha$$

is the same.

Let

$$H = (\mathbb{R} \setminus A_0) \setminus \bigcup_{\alpha < \alpha_0} (K_\alpha \cap E_\alpha).$$

If $\mu(\mathrm{cl}(H)) = 0$, then evidently $H \in \mathcal{I}_1$. So we assume that $\mu(\mathrm{cl}(A)) > 0$. We will prove that $H \in \mathcal{I}_1$. For this let $U \subset \mathrm{cl}(H)$ be a nonempty set belonging to T_d. Evidently $U \cap A_0 = \emptyset$. Let J be an open interval with $U_1 = U \cap J \neq \emptyset$. Let $\alpha_1 < \alpha_0$ be the first ordinal with

$$K_{\alpha_1} \cap U_1 \neq \emptyset.$$

If there is a point

$$x \in (K_{\alpha_1} \cap U_1) \setminus E_{\alpha_1}$$

then

$$D_u(\bigcup_{\alpha < \alpha_1} K_\alpha, x) > 0, \text{ and } D_l(U_1, x) = 1,$$

and consequently, there is an ordinal $\alpha_2 < \alpha_1$ with $H \cap K_{\alpha_2} \neq \emptyset$. This contradicts to the choice of α_1. So,

$$U_1 \cap K_{\alpha_1} \subset U_1 \cap K_{\alpha_1} \cap E_{\alpha_1} \subset U_1 \setminus H,$$

and

$$U_1 \cap H \cap K_{\alpha_1} = U \cap J \cap H \cap K_{\alpha_1} = \emptyset.$$

Consequently $H \cap U$ is a nowhere dense subset of U and $H \in \mathcal{I}_1$.

Since

$$\{x \in \mathbb{R} : f(x) < a\} = \{x \in A_0 : f(x) < a\} \cup \bigcup_{\alpha < \alpha_0} G_\alpha \cup \{x \in H : f(x) < a\} \in \mathcal{B}(\mathcal{I}_1),$$

the proof is finished. □

In article [3] it is proved that the σ-field $\mathcal{A}(S_5)$ coincides with the σ-field $\mathcal{L}$ of all Lebesgue measurable subsets of $\mathbb{R}$.

There are sets belonging to $I_2 \setminus I_0$ (see example in [6], pp. 310–311).

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