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PROPERTY OF RAPID DECAY FOR EXTENSIONS OF COMPACTLY GENERATED GROUPS

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Abstract: In the article we settle down the problem of permanence of property RD under group extensions. We show that if $1 \to N \to G \to Q \to 1$ is a short exact sequence of compactly generated groups such that Q has property RD, and N has property RD with respect to the restriction of a word-length on G, then G has property RD.

We also generalize the result of Ji and Schweitzer stating that locally compact groups with property RD are unimodular. Namely, we show that any automorphism of a locally compact group with property RD which distorts distances subexponentially, preserves the Haar measure.

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1. Introduction

The property of rapid decay (property RD for short) is a certain estimate for the norm of convolution operators in terms of a Sobolev-type norm corresponding to a length function on a group. Its study was begun by Haagerup, who established it for free groups in [**7**]. The basic theory was later developed by Jolissaint in [**9, 10**].

Property RD takes its name from an equivalent formulation, which states that the space of rapidly decreasing (with respect to some length) functions on a group naturally embeds in its reduced C^*-algebra. It turns out that this embedding induces isomorphisms in K-theory. This result found an application in the Connes–Moscovici proof of the Novikov conjecture for hyperbolic groups [**5**].

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In the paper [**10**] a partial result on permanence of property RD under group extensions is proved. Proposition 2.1.9 therein treats a special case of an extension $1 \to N \to G \to Q \to 1$ of finitely generated groups, satisfying technical conditions of polynomial amplitude and polynomial growth of certain associated functions. In Section 5 we show that these assumptions are equivalent to N being polynomially distorted in G. Other related results are [**3**, Proposition 1.14], dealing with split extensions of finitely generated groups, and [**4**, Proposition 5.5], dealing with polynomially distorted central extensions of compactly generated groups.

The main aim of this article is to provide a permanence result for general extensions of compactly generated groups. We show that for a compactly generated group G to have property RD, it is sufficient that it contains a normal subgroup N satisfying property RD with respect to the restriction of a word-length of G, such that the quotient G/N has property RD. The proof is an adaptation of the proof of [**10**, Proposition 2.1.9], based on a more careful choice of an auxiliary cross-section of the short exact sequence.

In the course of the proof, we first need to show that our extension is unimodular. We achieve this by proving a generalization of the result of Ji and Schweitzer [**8**, Theorem 2.2] stating that groups with property RD are unimodular. This is equivalent to saying that inner automorphisms are measure-preserving, and we extend this statement to all automorphisms which distort the length subexponentially in a certain sense.

The text is organized as follows. Section 2 introduces the basic notions associated to property RD. In Section 3 we prove an inequality about length distortion of automorphisms of groups with property RD, and use it to show that an extension satisfying the assumptions of our main theorem is unimodular. Section 4 contains the proof of the main result, while Section 5 is devoted to the discussion of some examples and the relation of our work to that of Jolissaint.

2. Property RD

Let G be a locally compact group. We will always endow groups with their right-invariant Haar measures. A *length* on G is a Borel function $\ell\colon G \to [0, \infty)$ such that

(1) $\ell(1) = 0$,

(2) $\ell(x^{-1}) = \ell(x)$,

(3) $\ell(xy) \le \ell(x) + \ell(y)$.

If G is generated by a subset S, then the word-length

$$\ell(x) = \min\{n : x \in (S^{\pm 1})^n\} \tag{1}$$

is an example of a length. When speaking about word-lengths we will always assume that the set S is relatively compact, and in particular, that the group is compactly generated.

Suppose ℓ_1 and ℓ_2 are two lengths on G. We say that ℓ_1 *dominates* ℓ_2 if there exist constants $r, C > 0$ such that

$$\ell_2(x) \leq C(1 + \ell_1(x))^r \tag{2}$$

for all $x \in G$. If ℓ_2 also dominates ℓ_1, they are said to be *equivalent*. By [**12**, Theorem 1.2.11], any length function is bounded on compact sets. Therefore, if G is compactly generated, all its word-lengths are equivalent and dominate all other lengths.

The algebra $C_c(G)$ of compactly supported continuous functions on G acts faithfully on the Hilbert space $L^2(G)$ by left convolution operators T_f, where

$$T_f g(x) = \int_G f(y^{-1}) g(yx)\, dy. \tag{3}$$

This induces the operator norm $\|\cdot\|_{\mathrm{op}}$ on $C_c(G)$. The group G is said to satisfy *property RD with respect to a length function* ℓ if there exist constants $s, C > 0$ such that for every $f \in C_c(G)$

$$\|f\|_{\mathrm{op}} \leq C(1 + \ell(f))^s \|f\|_2, \tag{4}$$

where $\ell(f) = \sup\{\ell(x) : f(x) \neq 0\}$, and $\|\cdot\|_2$ stands for the L^2-norm. We will also later employ the notation $\ell(U) = \sup\{\ell(x) : x \in U\}$ for $U \subseteq G$.

If ℓ_1 and ℓ_2 are two lengths on G such that ℓ_1 dominates ℓ_2 and G has property RD with respect to ℓ_2, then clearly it has property RD with respect to ℓ_1. Hence, if a compactly generated group has property RD with respect to one length, then it has property RD with respect to any of its word-lengths. In this case, it is said that G has *property RD*. Finally, since by [**8**] property RD implies unimodularity, using a left Haar measure leads to exactly the same notion.

3. Unimodularity of the extension

In this section we exhibit an inequality satisfied by automorphisms of groups with property RD, which, when applied to inner automorphisms, generalizes the result of Ji and Schweitzer, stating that groups with property RD are unimodular [**8**]. We use it to infer that in an extension $1 \to N \to G \to Q \to 1$, where Q has property RD, and N has property RD with respect to the restriction of the word-length of G, the group G is unimodular. This allows to apply a lemma of Jolissaint to show that G has property RD.

For a locally compact group G we will denote by $\mathrm{Aut}(G)$ the group of all topological automorphisms of G. If ρ is a right Haar measure on G, then the modular homomorphism $\Delta_G\colon \mathrm{Aut}(G) \to \mathbb{R}_+$ is defined by $\alpha_*\rho = \Delta_G(\alpha)\rho$.

Theorem 3.1. *Suppose that a locally compact group G satisfies property RD with respect to a length ℓ with exponent s. Let $\alpha \in \mathrm{Aut}(G)$. Then for any relatively compact open set $U \subseteq G$ there exists $D > 0$ such that*

$$\Delta_G(\alpha)^n \leq D(1 + \ell(\alpha^{-n}(U)))^{2s} \tag{5}$$

for all $n \in \mathbb{Z}$.

Proof: Observe that for $f, g \in C_c(G)$

$$(f \circ \alpha) * (g \circ \alpha) = \Delta_G(\alpha)(f * g) \circ \alpha, \tag{6}$$

and

$$\|f \circ \alpha\|_2 = \Delta_G(\alpha)^{1/2} \|f\|_2 . \tag{7}$$

It follows that

$$\begin{aligned} \Delta_G(\alpha)^{n/2}\|f*g\|_2 = \|(f * g) \circ \alpha^n\|_2 &= \Delta_G(\alpha)^{-n}\|(f \circ \alpha^n)*(g \circ \alpha^n)\|_2 \\ &\leq C(1 + \ell(f \circ \alpha^n))^s \Delta_G(\alpha)^{-n}\|f \circ \alpha^n\|_2\|g \circ \alpha^n\|_2 \\ &= C(1 + \ell(f \circ \alpha^n))^s \|f\|_2 \|g\|_2 . \end{aligned} \tag{8}$$

For arbitrarily chosen positive $f, g \in C_c(G)$ with f vanishing outside U, the convolution $f * g$ is nonzero, and

$$\Delta_G(\alpha)^n \leq (1 + \ell(f \circ \alpha^n))^{2s} \left(\frac{C \|f\|_2 \|g\|_2}{\|f * g\|_2} \right)^2 , \tag{9}$$

which, after noticing that $\ell(f \circ \alpha^n) \leq \ell(\alpha^{-n}(U))$, yields the desired inequality. □

If $\alpha(x) = axa^{-1}$ is an inner automorphism of G, then we have $\ell(\alpha^n(x)) \leq \ell(x) + 2|n|\ell(a)$. By Theorem 3.1, if G has property RD, then the function $(\rho(\alpha(U))/\rho(U))^n$ is bounded by a polynomial in $|n|$, and hence α is measure-preserving. Thus this theorem indeed generalizes the result on unimodularity of groups with property RD.

Now, consider a short exact sequence $1 \to N \to G \xrightarrow{\pi} Q \to 1$ of compactly generated groups. By [**11**, Lemma 1.1], there exists a Borel cross-section $\sigma\colon Q \to G$. There are two Borel functions associated to the choice of σ, namely $\beta\colon Q \times Q \to N$ and $\theta\colon Q \to \mathrm{Aut}(N)$, defined by

$$\beta(p,q) = \sigma(p)\sigma(q)\sigma(pq)^{-1} \tag{10}$$

and

$$\theta(q)(n) = \sigma(q)n\sigma(q)^{-1}. \tag{11}$$

Using σ we obtain a Borel identification $N \times Q \to G$ given by $(n,q) \mapsto n\sigma(q)$, which we will use freely without any further mention. The multiplication on G can be now expressed by

$$(m,p)(n,q) = (m\theta(p)(n)\beta(p,q), pq). \tag{12}$$

Moreover, the right Haar measure of G is the product of right Haar measures of N and Q.

Lemma 3.2. *Suppose that $1 \to N \to G \to Q \to 1$ is a short exact sequence of compactly generated groups. If N and Q are unimodular, then G is unimodular if and only if the automorphisms $\theta(q)$ defined by* (11) *are measure-preserving.*

Proof: Let $f \in C_c(G)$. For $(m,p) \in G$ we have

$$\begin{aligned} &\int_N \int_Q f((m,p)(n,q))\, dq\, dn \\ &= \int_N \int_Q f(m\theta(p)(n)\beta(p,q), pq)\, dq\, dn \qquad (13)\\ &= \int_N \int_Q f(\theta(p)(n), q)\, dq\, dn = \Delta_N(\theta(p)) \int_N \int_Q f(n,q)\, dq\, dn, \end{aligned}$$

so the right Haar measure on G is left-invariant if and only if $\Delta_n(\theta(p)) = 1$ for all $p \in Q$. □

Corollary 3.3. *If in the short exact sequence $1 \to N \to G \to Q \to 1$ of compactly generated groups Q is unimodular, and N has property RD with respect to the restriction of a word-length of G, then G is unimodular.*

Proof: By Lemma 3.2, it is enough to observe that the automorphisms $\theta(q)$ associated with the section σ are measure-preserving. For $n \in N$ we have

$$\ell_G(\theta(q)^{-k}(n)) \leq \ell_G(n) + 2\,|k|\,\ell_G(\sigma(q)), \tag{14}$$

and thus by Theorem 3.1, the sequence $\Delta_N(\theta(q))^k$, with $k \in \mathbb{Z}$, is bounded by a polynomial in $|k|$. This is only possible if $\Delta_N(\theta(q)) = 1$. □

4. Permanence of property RD under extensions

Again, let $1 \to N \to G \xrightarrow{\pi} Q \to 1$ be a short exact sequence of compactly generated groups. Endow G with a word-length ℓ_G corresponding to a relatively compact generating set S. Denote by ℓ_Q the word-length on Q corresponding to $\pi(S)$. The main result of this article is the following theorem, the proof of which we postpone until the end of this section.

Theorem 4.1. *If in the short exact sequence* $1 \to N \to G \to Q \to 1$ *of compactly generated groups* Q *has property RD with respect to its word-length, and* N *has property RD with respect to the restriction of the word-length of* G*, then* G *has property RD.*

We have already remarked that there exists a Borel section of the quotient map π. The proof in [**11**] actually yields more. The next simple observation will be crucial – choosing the right cross-section will allow to drop the unnecessary assumptions of [**10**, Proposition 2.1.9].

Remark 4.2. There exists a Borel cross-section $\sigma\colon Q \to G$ of π such that for every $q \in Q$ we have $\ell_G(\sigma(q)) = \ell_Q(q)$.

Indeed, since we have $\ell_Q(\pi(g)) \leq \ell_G(g)$, and in consequence $\ell_Q(q) \leq \ell_G(\sigma(q))$, we need to construct a cross-section σ such that $\sigma(\pi(S)^k) \subseteq S^k$. But the proof of Mackey actually follows by first decomposing G into an increasing union of compact sets K_n such that every compact subset $K \subseteq G$ is contained in some K_n, and then constructing an increasing family of sections of the restrictions $\pi|_{K_n}$, using the Federer–Morse theorem. We may therefore put $K_n = S^n$, and we just need to notice that every compact subset $K \subseteq G$ is contained in S^n for some n. This is clear, as S^k has positive measure for some k, and the convolution $\chi_{S^k} * \chi_{S^k}$ is nonzero, continuous (as convolution of square-integrable functions), and vanishing outside S^{2k}. Hence S^{2k} has nonempty interior and generates G, so the claim follows.

The next ingredient in the proof of Theorem 4.1 is a generalization of [**10**, Lemma 2.1.2], which was formulated in the setting of discrete groups. The proof, adapted from [**10**], works for arbitrary lengths on G and Q, not only for word-lengths.

Lemma 4.3. *Let* $1 \to N \to G \to Q \xrightarrow{\pi} 1$ *be a short exact sequence of compactly generated groups, endowed with lengths* ℓ_N, ℓ_G, *and* ℓ_Q, *where* ℓ_N *is arbitrary,* ℓ_G *is a word-length corresponding to a relatively compact generating set* S, *and* ℓ_Q *is the word-length corresponding to* $\pi(S)$. *Suppose that*

(1) *N has property RD with respect to ℓ_N,*
(2) *Q has property RD with respect to ℓ_Q,*
(3) *G is unimodular,*
(4) *there exist constants $D, r > 0$ such that for any $(n, q) \in G$ we have*

$$\ell_N(n) + \ell_Q(q) \leq D\ell_G(n, q)^r. \tag{15}$$

Then G has property RD with respect to ℓ_G.

Proof: Let $f, g \in C_c(G)$. We get, using unimodularity of N and G, that

(16)

$$\begin{aligned} f * g(n, q) &= \int_Q \int_N f((m, p)^{-1}) g((m, p)(n, q))\, dm\, dp \\ &= \int_Q \int_N f(\theta(p)^{-1}(m^{-1}\beta(p, p^{-1})^{-1}), p^{-1}) g(m\theta(p)(n)\beta(p, q), pq)\, dm\, dp \\ &= \Delta_N(\theta(p)^{-1}) \int_Q \int_N f_p(m^{-1}) g_{p,q}(mn)\, dm\, dp = \int_Q f_p * g_{p,q}(n)\, dp, \end{aligned}$$

where

$$f_p = f(m, p^{-1}), \tag{17}$$

and

$$g_{p,q}(m) = g(\beta(p, p^{-1})^{-1}\theta(p)(m)\beta(p, q), pq). \tag{18}$$

Now, let N and Q satisfy property RD with constants C and s. Using the triangle inequality, we may estimate the norm of $f * g$ by

$$\begin{aligned} \|f * g\|_2^2 &\leq \int_Q \left[\int_Q \|f_p * g_{p,q}\|_2\, dp\right]^2 dq \\ &\leq \int_Q \left[C \int_Q (1 + \ell_N(f_p))^s \|f_p\|_2 \|g_{p,q}\|_2\, dp\right]^2 dq \qquad (19) \\ &\leq C^2(1 + D\ell_G(f)^r)^{2s} \int_Q \left[\int_Q \|f_p\|_2 \|g_{p,q}\|_2\, dp\right]^2 dq. \end{aligned}$$

If we put

$$\phi(p) = \|f_{p^{-1}}\|_2 \tag{20}$$

and

$$\psi(p) = \left[\int_N |g(m, p)|^2\, dm\right]^{1/2}, \tag{21}$$

we have, using unimodularity of N, that $\|g_{p,q}\|_2 = \psi(pq)$, and therefore

$$\begin{aligned} \|f * g\|_2 &\leq C(1 + D\ell_G(f)^r)^s \|\phi * \psi\|_2 \\ &\leq C^2(1 + D\ell_G(f)^r)^s (1 + \ell_Q(\phi))^s \|\phi\|_2 \|\psi\|_2 \\ &\leq C^2(1 + D\ell_G(f)^r)^{2s} \|f\|_2 \|g\|_2 \\ &\leq C'(1 + \ell_G(f))^{2rs} \|f\|_2 \|g\|_2 \end{aligned} \tag{22}$$

for some constant $C' > 0$. □

Now, it turns out that if we use the cross-section from Remark 4.2, the inequality in Lemma 4.3 is satisfied by the restriction of the word-length of G.

Lemma 4.4. *If in the short exact sequence $1 \to N \to G \to Q \to 1$ of compactly generated groups, where G is endowed with a word-length ℓ_G corresponding to a relatively compact generating set S, and Q is endowed with the word-length ℓ_Q corresponding to $\pi(S)$, the group G is identified with $N \times Q$ using a cross-section $\sigma\colon Q \to G$ satisfying $\ell_G(\sigma(q)) = \ell_Q(q)$ for all $q \in Q$, then for all $(n, q) \in G$ we have*

$$\ell_G(n) + \ell_Q(q) \leq 3\ell_G(n, q). \tag{23}$$

Proof: Let $(n, q) \in G$. We have

$$\begin{aligned} \ell_G(n) + \ell_Q(q) &= \ell_G((n, q)\sigma(q)^{-1}) + \ell_Q(q) \\ &\leq \ell_G(n, q) + 2\ell_Q(q) \leq 3\ell_G(n, q). \end{aligned} \tag{24}$$

□

All these considerations sum up to the proof of our main theorem.

Proof of Theorem 4.1: Suppose that N has property RD with respect to the restriction of ℓ_G, and Q has property RD with respect to ℓ_Q. By Remark 4.2 there exists a cross-section $\sigma\colon Q \to G$ such that $\ell_G(\sigma(q)) = \ell_Q(\sigma)$ for all $q \in Q$. Then, by Lemma 4.4, the inequality $\ell_N(n) + \ell_Q(q) \leq 3\ell_G(n, q)$ is satisfied. Moreover, by Corollary 3.3, the group G is unimodular. Hence, Lemma 4.3 applies, and G has property RD. □

5. Final remarks

In Theorem 4.1 we required N to have property RD with respect to the restriction of the word-length on G. It might be tempting to ask whether this can be weakened to having property RD with respect to a word-length on N. Such a strengthening is easily seen to be false. It suffices to consider any metabelian group with exponential growth, e.g. the

Baumslag–Solitar group $\langle a, b \mid bab^{-1} = a^2\rangle$. Such a group is amenable, and by [**10**, Corollary 3.1.8], no amenable group with superpolynomial growth can satisfy property RD.

Even in the case of finitely generated groups, Theorem 4.1 is strictly stronger than Jolissaint's result, which assumes that N has property RD with respect to its own word-length ℓ_N associated to a generating set S_N, and the associated functions θ and β satisfy the inequalities

$$\ell_N(\beta(p,q)) \leq A(1+\ell_Q(p))^\alpha(1+\ell_Q(q))^\alpha \tag{25}$$

and

$$\ell_N(\theta(q)(s)) \leq B(1+\ell_Q(q))^\beta \tag{26}$$

for some $A, B, \alpha, \beta > 0$. As the next proposition shows, existence of a cross-section σ for which β and θ satisfy these conditions is equivalent to N being *polynomially distorted* in G, i.e. to the estimate

$$\ell_N(n) \leq C(1+\ell_G(n))^r \tag{27}$$

for some $C, r > 0$.

Proposition 5.1. *For a short exact sequence* $1 \to N \to G \to Q \to 1$ *of finitely generated groups the following conditions are equivalent:*

(1) *there exists a section* $\sigma\colon Q \to G$ *of* π *such that the corresponding functions* β *and* θ *satisfy conditions* (25) *and* (26)*;*
(2) N *has polynomial distortion in* G*.*

Proof: To show (1) $\Rightarrow$ (2), assume that inequalities (25) and (26) hold for functions β and θ associated with a section σ. Observe that these conditions are independent of the choice of particular finite generating sets and corresponding word-lengths. Hence, without loss of generality, we may choose generating sets satisfying $S_G = S_N \cup \sigma(S_Q)$. We will prove (27) by induction on $\ell_G(n)$. The constants C and r will be fixed later.

Take $n \in N$. If $n = 1$, inequality (27) is satisfied for any C and r, so assume that $n \neq 1$, and let $n = s_1 s_2 \cdots s_k$ be a minimal representation in S_G. First, suppose that $s_1, \ldots, s_k \in \sigma(S_Q)$, and write $s_i = \sigma(q_i)$ with $q_i \in S_Q$. Observe that $q_1 \cdots q_k = \pi(n) = 1$, and therefore

$$\begin{aligned} s_1 \cdots s_k &= \left(\prod_{i=1}^{k-1} \beta(q_1 \cdots q_i, q_{i+1})\right) \sigma(q_1 \cdots q_k) \\ &= \left(\prod_{i=1}^{k-1} \beta(q_1 \cdots q_i, q_{i+1})\right) \sigma(1), \end{aligned} \tag{28}$$

so

$$\ell_N(n) \le \sum_{i=1}^{k-1} A(1+i)^\alpha 2^\alpha + \ell_N(\sigma(1)) \le D_1(1+k)^{\gamma_1}. \tag{29}$$

Now, assume that $s_1, \dots, s_m \in \sigma(S_Q)$ and $s_{m+1} \in S_N$ for some $m<k$. We then have

$$n = (s_1 \cdots s_m) s_{m+1} (s_1 \cdots s_m)^{-1} n', \tag{30}$$

with $n' = s_1 \cdots \hat{s}_{m+1} \cdots s_k$ satisfying $\ell_G(n') \le k-1$, and obtain

$$\begin{aligned} \ell_N(n) &\le \ell_N(n') + \ell_N((s_1 \cdots s_m) s_{m+1} (s_1 \cdots s_m)^{-1}) \\ &\le Ck^r + \ell_N((s_1 \cdots s_m) s_{m+1} (s_1 \cdots s_m)^{-1}). \end{aligned} \tag{31}$$

To estimate the second summand, we proceed similarly as in the previous case. Write $s_i = \sigma(q_i)$ for $i = 1, \dots, m$ and observe that, using the first part of (28), we get

$$\begin{aligned} &\ell_N((s_1 \cdots s_m) s_{m+1} (s_1 \cdots s_m)^{-1}) \\ &\quad \le 2 \sum_{i=1}^{m-1} \ell_N(\beta(q_1 \cdots q_i, q_{i+1})) + \ell_N(\theta(q_1 \cdots q_m)(s_{m+1})) \\ &\quad \le 2 \sum_{i=1}^{m-1} A(1+i)^\alpha 2^\alpha + B(1+m)^\beta \le D_2 k^{\gamma_2}. \end{aligned} \tag{32}$$

If we take $C = \max\{D_1, D_2\}$ and $r = 1 + \max\{\gamma_1, \gamma_2\}$, we finally get

$$\ell_N(n) \le Ck^r + D_2 k^{r-1} \le C(1+k)^r, \tag{33}$$

which, together with (29), ends the inductive step and the proof of (1) $\Rightarrow$ (2).

In order to prove the implication (2) $\Rightarrow$ (1), suppose that N is polynomially distorted in G, and take a section $\sigma\colon Q \to G$ and generating sets S_N, S_G, and S_Q, such that $S_G = S_N \cup \sigma(S_Q)$ and $\ell_G(\sigma(q)) = \ell_Q(q)$ for all $q \in Q$. We have

$$\begin{aligned} \ell_N(\beta(p,q)) &\le C(1+\ell_G(\sigma(p)\sigma(q)\sigma(pq)^{-1}))^r \\ &\le C(1+2\ell_Q(p)+2\ell_Q(q))^r \le 2^r C(1+\ell_Q(p))^r (1+\ell_Q(q))^r \end{aligned} \tag{34}$$

and

$$\ell_N(\theta(q)s) \le C(1+\ell_G(\sigma(q)s\sigma(q)^{-1}))^r \le 2^r C(1+\ell_Q(q))^r, \tag{35}$$

which completes the proof. □

In particular, polynomial distortion of N implies that its word-length is dominated by the restriction of the word-length of G, so the assumptions of Jolissaint can be formulated equivalently as requiring N to be polynomially distorted in G, and satisfy property RD with respect to the restriction of the word-length of G.

In [**1**] and [**2**] the authors construct hyperbolic semidirect products $N \rtimes \mathbb{Z}$ with N free, such that the distortion of N is superpolynomial. Such extensions do not fall into the scope of Jolissaint's theorem, while they still satisfy assumptions of Theorem 4.1. Indeed, since $G = N \rtimes \mathbb{Z}$ is hyperbolic, it satisfies property RD by [**6**], and therefore N has RD with respect to the restriction of the word-length of G. This example can be seen as somewhat unsatisfactory, as we already know that G has RD, and we use this to show that N has RD with respect to the restricted length.

Apart from being a subgroup of a group with property RD, the only other criterion for having RD with respect to a length not equivalent to a word-length that we are aware of states that an amenable group has property RD with respect to a length ℓ if and only if it has polynomial growth with respect to ℓ. Therefore, one way to construct a potentially nontrivial example leads through solving the following problem.

Problem. Construct a finitely generated group G with an amenable normal subgroup N, which is superpolynomially distorted, but its relative growth in G is polynomial, and such that the quotient G/N has property RD.

Of course, it would be best to construct such a group which does not satisfy the assumptions of other known criteria for property RD, such as hyperbolicity or admitting a proper cocompact action on a CAT(0) cube complex.

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