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NEW INVARIANTS AND ATTRACTING DOMAINS FOR HOLOMORPHIC MAPS IN $\mathbf{C}^2$ TANGENT TO THE IDENTITY

Feng Rong

Abstract: We study holomorphic maps of $\mathbf{C}^2$ tangent to the identity at a fixed point which have degenerate characteristic directions. With the help of some new invariants, we give sufficient conditions for the existence of attracting domains in these degenerate characteristic directions.

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1. Introduction

A holomorphic map f in $\mathbf{C}^n$ is said to be *tangent to the identity* at a fixed point p, if the Jacobian df_p is equal to the identity matrix. Choose local coordinates (z,w) with p as the origin O. Let $f(z,w) = (z+\sum_{i=2}^{\infty} p_i(z,w), w+\sum_{i=2}^{\infty} q_i(z,w))$ be the homogeneous expansion of f, where $p_i(z,w)$ and $q_i(z,w)$ are homogeneous of degree i. The *order* k of f at O is by definition the minimum of i such that either $p_i(z,w) \not\equiv 0$ or $q_i(z,w) \not\equiv 0$. A direction $[z:w]$ is called a *characteristic direction* of f, if there exists $\lambda \in \mathbf{C}$ such that $p_k(z,w) = \lambda z$ and $q_k(z,w) = \lambda w$. If $\lambda \neq 0$, then $[z:w]$ is said to be *non-degenerate*, otherwise *degenerate*.

The study of local dynamics of holomorphic maps tangent to the identity has been centered on generalizing the well-known Leau–Fatou Flower Theorem in the one-dimensional case (see e.g. [**4, 7**]). In [**11**], Hakim proved such a generalization for generic maps of $\mathbf{C}^n$, i.e. those with non-degenerate characteristic directions. In [**2**], Abate obtained a full generalization in dimension two. In particular, Abate introduced a "residual index" to each characteristic direction of a map tangent to the identity and showed that there exist "parabolic curves" in directions

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with residual index not in $\mathbf{Q}^+ \cup \{0\}$. In [**13**], Molino showed the existence of parabolic curves in directions with non-zero residual index, under a mild extra assumption. In [**12**], Hakim provided sufficient conditions for the existence of attracting domains in non-degenerate characteristic directions. In [**17**], the author introduced a new invariant, called the "non-dicritical order", associated to each non-degenerate characteristic direction and provided further sufficient conditions for the existence of attracting domains in such directions. More recently, there have been some studies on the local dynamics of so called "one-resonant" and "multi-resonant" biholomorphisms (see e.g. [**8, 9, 10, 14**]).

The general theory for local dynamics in degenerate characteristic directions with zero residual index is yet to be developed. Characteristic directions of a holomorphic map in $\mathbf{C}^2$ can be divided into three types: irregular, Fuchsian, and apparent (cf. [**5**]). In [**18, 19**], Vivas gave sufficient conditions for the existence of attracting domains in the irregular and Fuchsian directions. However, no general sufficient conditions are known for the existence of attracting domains in the apparent directions.

In this paper, we define some new invariants associated to a degenerate characteristic direction of holomorphic maps in $\mathbf{C}^2$ and give sufficient conditions for the existence of attracting domains in such a direction. The currently known invariants such as non-degeneracy and residual index are defined using only the leading nonlinear terms of the homogeneous expansion of a map. However, examples show that sometimes the higher order nonlinear terms also play an important role in the local dynamics of a map. Our new invariants capture exactly this feature.

Our main result is the following:

Theorem 1.1. *Let f be a holomorphic map in $\mathbf{C}^2$ tangent to the identity at a fixed point p. Assume that $[v]$ is a degenerate characteristic direction of f, which is essentially non-degenerate. If f is transversally attracting in the direction $[v]$, then there exists an attracting domain in the direction $[v]$ for f at p.*

In Section 2, we will define our new invariants and explain the terms *essentially non-degenerate* and *transversally attracting*. In Section 3, we prove Theorem 1.1.

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2. New invariants

Let f be a holomorphic map in $\mathbf{C}^2$, tangent to the identity at a fixed point p, with order $t+1$, $t \geq 1$. Assume that $[v]$ is a degenerate characteristic direction of f. In suitable local coordinates (z,w) around p, we can assume that p is the origin O and $[v]=[z:w]=[1:0]$. Write f as

$$\begin{cases} z_1 = z + wP_t(z,w) + O(t+2), \\ w_1 = w + wQ_t(z,w) + O(t+2), \end{cases} \tag{2.1}$$

where $P_t(z,w)$ and $Q_t(z,w)$ are homogeneous of degree t. We say that f is *generically degenerate* in the direction $[v]$ if $w \nmid Q_t(z,w)$.

Remark 2.1. A holomorphic map f of order $t+1$ at the origin, with a characteristic direction $[v]$ in suitable local coordinates (z,w), can be written as

$$\begin{cases} z_1 = z + p_{t+1}(z,w) + O(t+2), \\ w_1 = w + q_{t+1}(z,w) + O(t+2), \end{cases}$$

where $p_{t+1}(z,w) = \sum_{i=0}^{t+1} a_i z^{t+1-i} w^i$ and $q_{t+1}(z,w) = \sum_{j=1}^{t+1} b_j z^{t+1-j} w^j$. Then $[1:0]$ is a non-degenerate characteristic direction if and only if $a_0 \neq 0$, a generic condition. And $[1:0]$ is a generically degenerate characteristic direction if and only if $a_0 = 0$ and $b_1 \neq 0$, a generic condition among degenerate characteristic directions.

Remark 2.2. A generically degenerate characteristic direction is an apparent characteristic direction.

Let G be the group of local changes of coordinates which preserves the degenerate characteristic direction as $[1:0]$. Then each element $(Z,W) = \Phi(z,w)$ of G takes the form

$$\begin{cases} Z = lz + mw + O(2), \\ W = nw + O(2), \end{cases} \tag{2.2}$$

with $l, n \neq 0$. Under such a change of coordinates, we write f as

$$\begin{cases} Z_1 = Z + P_\Phi(Z) + WS_\Phi(Z,W), \\ W_1 = W + Q_\Phi(Z) + WR_\Phi(Z) + W^2 T_\Phi(Z,W). \end{cases} \tag{2.3}$$

It is easy to check that $\operatorname{ord} R_\Phi(Z) = t$ for every $\Phi \in G$, thus being generically degenerate is well defined.

We define the *virtual order* to be

$$s := \max_{\Phi \in G} \operatorname{ord} Q_\Phi(Z),$$

and set $\nu := s - t$, which we call the *weight*.

Remark 2.3. We allow $s = \infty$, which is the case if and only if there exists an f-invariant curve passing through p tangent to $[v]$.

Let H be the subgroup of G consisting of Φ's with $\operatorname{ord} Q_\Phi(Z) = s$. Then we define the *essential order* to be

$$\mu := \max_{\Phi \in H} \operatorname{ord} P_\Phi(Z).$$

Obviously both the virtual order and the essential order are invariants associated to f in the degenerate characteristic direction $[v]$. Finally, we say that $[v]$ is an *essentially non-degenerate characteristic direction* of f if

$$\mu < s.$$

Note that this implies $\nu \geq 3$.

Remark 2.4. For an essentially non-degenerate characteristic direction, we certainly have $\mu < \infty$. However, the case $\mu = \infty$ is also interesting. For instance, if $\mu = s = \infty$ then there exists a line of fixed points through p. For the dynamics in this case, see e.g. [**6**].

For later convenience, set $r := \mu - 1$. Rewrite (2.1) as

$$\begin{cases} z_1 = z + az^{r+1} + P(z) + wS(z,w), \\ w_1 = w + bz^t w + dz^s + Q(z) + wR(z) + w^2 T(z,w), \end{cases} \tag{2.4}$$

with $r \geq t+1$, $a, b \neq 0$, $d \neq 0$ if $s < \infty$, $P(z) = O(z^{r+2})$, $Q(z) = O(z^{s+1})$, $R(z) = O(z^{t+1})$, $S(z,w) = O(t)$, and $T(z,w) = O(t-1)$.

Define the *director* in the characteristic direction $[v]$ to be

$$\alpha := -b(-a)^{-t/r}.$$

We say that f is *transversally attracting* in $[v]$ if

$$\mathsf{Re}\, \alpha > 0. \tag{2.5}$$

Under a transformation Φ in H of the form (2.2), (2.4) becomes

$$\begin{cases} Z_1 = Z + al^{-r} Z^{r+1} + \tilde{P}(Z) + W\tilde{S}(Z,W), \\ W_1 = W + bl^{-t} Z^t W + dnl^{-s} Z^s + \tilde{Q}(Z) + W\tilde{R}(Z) + W^2 \tilde{T}(Z,W). \end{cases} \tag{2.6}$$

It is then easy to see that the director is well-defined and condition (2.5) does not depend on the choice of Φ.

Remark 2.5. The power function $x^{-t/r}$ is a multi-valued function with $r/\gcd(r,t)$ different values. Hence α has $r/\gcd(r,t)$ different values. Condition (2.5) should be interpreted as one of these values has positive real part. Such a condition is quite weak. In fact, one readily checks that the only case when this condition fails is when $t/r = 1/2$ and α is purely imaginary.

Remark 2.6. Although the definitions above were given under the assumption that $[v]$ is a degenerate characteristic direction for f, it is easy to see that the same definitions can be made replacing generically degenerate by non-degenerate. In the latter case, a non-degenerate characteristic direction is essentially non-degenerate and we have $r = t$. Then our director minus one is exactly the director of f in the non-degenerate characteristic direction defined in [**11**]. In this sense, our main theorem is a natural generalization of Hakim's results in [**11**] to the degenerate case.

3. Attracting domains

We now prove Theorem 1.1.

First assume that $s < \infty$. By scaling, we can rewrite (2.4) as

$$\begin{cases} z_1 = z - \dfrac{1}{r} z^{r+1} + P(z) + wS(z,w), \\ w_1 = w - cz^t w + \lambda z^s + Q(z) + wR(z) + w^2 T(z,w), \end{cases} \tag{3.1}$$

with $\beta := \mathsf{Re}\, c > 0$ and $0 < \lambda < \beta$.

For $0 < \delta \ll \epsilon$ small enough, set

$$V_{\epsilon,\delta} = \{\zeta \in \mathbf{C} : 0 < |\zeta| < \epsilon,\ |\mathsf{arg}\, \zeta| < \delta\}. \tag{3.2}$$

Denote by D the open set

$$\{(z,w) \in \mathbf{C}^2 : z \in V_{\epsilon,\delta},\ |w| < |z|^{\nu-\tau}\},$$

for $0 < \tau \ll 1$. Write $(z_n, w_n) = f^n(z,w)$. We want to show that $f(D) \subset D$ and $(z_n, w_n) \to (0,0)$ as $n \to \infty$.

Set $l := \min\{i + j\nu : z^i w^j \text{ in } S(z,w)\}$. By assumption, we have $\mu < s$, which easily implies that $\mu < l + \nu$. Thus for $(z,w) \in D$, from (3.1), we have

$$z_1 = z\left(1 - \frac{1}{r} z^r + o(z^r)\right) \tag{3.3}$$

and

$$w_1 = w(1 - cz^t + o(z^t)) + \lambda z^s + O(z^{s+1}). \tag{3.4}$$

Write $|w| = |z|^\gamma$ for some $\gamma = \gamma(z, w) > 1$. If $\gamma < \nu$, then

$$\frac{|w_1|}{|z_1|^{\nu-\tau}} \le \frac{|w|}{|z|^{\nu-\tau}}|1 - cz^t + \lambda z^{\nu-\gamma} z^t + o(z^t)| < \frac{|w|}{|z|^{\nu-\tau}} < 1.$$

If $\gamma \ge \nu$, then

$$\frac{|w_1|}{|z_1|^{\nu-\tau}} \le \frac{|w|}{|z|^{\nu}}|z|^\tau|1 + o(1)| + \lambda|z|^{t+\tau}|1 + o(1)| < 1.$$

Thus, we have $|w_1| < |z_1|^{\nu-\tau}$.

Write $z = \epsilon(z)e^{i\delta(z)}$ with $0 < \epsilon(z) < \epsilon$ and $|\delta(z)| < \delta$. Denote $x = 1 - z^r(1/r + o(1))$. Then it is easy to see that $|x| < 1$ and $\arg x$ is of different sign as $\delta(z)$ with $|\arg x| < |\delta(z)|$. From (3.3) we have

$$|z_1| = \epsilon(z)|x| < \epsilon(z) < \epsilon, \quad |\arg z_1| = |\delta(z) + \arg x| < |\delta(z)| < \delta. \tag{3.5}$$

Therefore, we have shown that $f(D) \subset D$.

From (3.3) we have

$$\frac{1}{z_1^r} = \frac{1}{z^r} + 1 + o(1),$$

from which we get the estimate

$$z_n \sim \frac{1}{n^{1/r}}. \tag{3.6}$$

Set $b_k = 1 - cz_k^t + o(z_k^t)$. From (3.4) we have

$$w_n = w\prod_{k=0}^{n-1} b_k + \lambda\sum_{l=0}^{n-1} z_l^s \prod_{m=l+1}^{n-1} b_m + \text{h.o.t.} \tag{3.7}$$

For z small enough, we have

$$\prod_{m=l+1}^{n-1} b_m = e^{\sum_{m=l+1}^{n-1} \log b_m} \sim e^{-c\sum_{m=l+1}^{n-1} z_m^t}. \tag{3.8}$$

From (3.6) and (3.8) we get

$$\prod_{m=l+1}^{n-1} |b_m| \sim e^{-\beta\sum_{m=l+1}^{n-1} m^{-\frac{t}{r}}} \sim e^{-\beta(n^{1-\frac{t}{r}} - l^{1-\frac{t}{r}})}. \tag{3.9}$$

Therefore

$$\sum_{l=0}^{n-1} |z_l|^s \prod_{m=l+1}^{n-1} |b_m| \sim \sum_{l=1}^{n-1} l^{-\frac{s}{r}} e^{-\beta(n^{1-\frac{t}{r}} - l^{1-\frac{t}{r}})} = e^{-\beta n^{1-\frac{t}{r}}} \sum_{l=1}^{n-1} l^{-\frac{s}{r}} e^{\beta l^{1-\frac{t}{r}}}. \tag{3.10}$$

For n large, we have

$$\sum_{l=1}^{n-1} l^{-\frac{s}{r}} e^{\beta l^{1-\frac{t}{r}}} \sim \int_1^n x^{-\frac{s}{r}} e^{\beta x^{1-\frac{t}{r}}}\, dx \sim n^{-\frac{s}{r}+\frac{t}{r}} e^{\beta n^{1-\frac{t}{r}}}. \tag{3.11}$$

From (3.7), (3.9), (3.10), and (3.11), we finally have the estimate

$$|w_n| \sim \frac{1}{n^{\nu/r}}, \quad (s < \infty). \tag{3.12}$$

For $s = \infty$, denote by D the open set

$$\{(z, w) \in \mathbf{C}^2 : z \in V_{\epsilon,\delta},\ |w| < |z|^{\kappa}\},$$

for $\kappa \gg 1$. Then a similar (and simpler) argument as above shows that $f(D) \subset D$ and

$$z_n \sim \frac{1}{n^{1/r}}; \quad |w_n| \sim e^{-\beta n^{(r-t)/r}}, \quad (s = \infty). \tag{3.13}$$

This completes the proof of Theorem 1.1. □

Remark 3.1. For $r = 2$ and $t = 1$, a special case of maps as in (2.4) was studied in [**15**]. (As noted there, we do not get a "flower" of attracting domains, but only some "petals".) Such maps also show up in the study of the local dynamics of holomorphic maps of $\mathbf{C}^2$ with a Jordan fixed point (cf. [**16**], see also [**1, 3**]).

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Department of Mathematics
Shanghai Jiao Tong University
800 Dong Chuan Road
Shanghai, 200240
P. R. China
E-mail address: frong@sjtu.edu.cn

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