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BRIOT–BOUQUET'S THEOREM IN HIGH DIMENSION

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Abstract: Let X be a germ of holomorphic vector field at $0 \in \mathbb{C}^n$ and let E be a linear subspace of $\mathbb{C}^n$ which is invariant for the linear part of X at 0. We give a sufficient condition that imply the existence of a non-singular invariant manifold tangent to E at 0. It generalizes to higher dimensions the conditions in the classical Briot–Bouquet's Theorem: roughly speaking, we impose that the convex hull of the eigenvalues μ_i corresponding to E does not contain 0 and there are no resonances between the μ_i and the complementary eigenvalues. As an application, we propose an elementary proof of the analyticity of the local stable and unstable manifolds of a real analytic vector field at a singular point.

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1. Introduction

This work is framed in the local study of a complex holomorphic vector field X at a singular point $0 \in \mathbb{C}^n$. More precisely, in the problem of determining whether there exists a formal or analytic (germ of a) variety at 0 which is invariant for X.

In dimension $n = 2$, the problem is completely solved after Camacho and Sad's *Separatrix Theorem* [**4**] that asserts that at any singular point of a holomorphic planar vector field there is an analytic invariant curve (called a separatrix). The situation is much more complicated in higher dimension. For example, we can mention several works by Luengo et al. [**8, 12**], where some examples of singularities of holomorphic vector fields in dimension greater than two with no invariant analytic curve are exhibited.

By contrast, we have reasonable results concerning existence of invariant varieties if the singularity is assumed to be "non degenerated" (usually, that the linear part D_0X of the vector field at 0 is non-nilpotent) and "sufficiently generic" (that the eigenvalues of that linear part satisfy certain generic condition). For instance, returning to the planar case, if the eigenvalues λ, μ of the linear part D_0X satisfy the *non-resonance condition* $\lambda \neq k\mu$ for any natural number $k \geq 2$, it is easy to see, using indeterminate coefficients, that there exists a unique formal invariant curve $\widehat{\Gamma}_\mu$ at the origin which is non-singular and tangent to the eigendirection corresponding to μ. Moreover, we have a classical and well known theorem due to Briot and Bouquet [**3**] that asserts that if $\mu \neq 0$ then $\widehat{\Gamma}_\mu$ is convergent (and thus gives rise to a non-singular analytic separatrix). It is worth to mention that Briot–Bouquet's Theorem is an essential ingredient in the proof of Camacho–Sad's Separatrix Theorem: after a process of reduction of singularities of a planar vector field X by blowing-ups, due to Seidenberg [**15**], one finds a non-degenerate singular point with the hypothesis of Briot–Bouquet's Theorem in such a way that the non-singular separatrix assured by this theorem is not destroyed by the process and produces a separatrix at the initial singular point $0 \in \mathbb{C}^2$, which a priori is singular.

Different ways of generalizing Briot–Bouquet's Theorem to higher dimension are conceivable. This work is devoted to state and prove one of such generalizations, although we are not intended for the moment to use it for more general results about existence of invariant analytic varieties of holomorphic vector fields. The result is motivated by the fact that, in the planar case, the two hypothesis $\lambda \neq k\mu$ for $k \geq 2$ and $\mu \neq 0$, are equivalent to the condition

$$\inf_{k \in \mathbb{Z}_{\geq 2}} \left\{ \frac{|k\mu - \lambda|}{k} \right\} > 0.$$

Our main theorem is the following (see below for a more precise statement).

Main Theorem. *Let X be a holomorphic vector field at $0 \in \mathbb{C}^n$, E a linear r-dimensional subspace of $\mathbb{C}^n$ invariant for the linear part D_0X, and put $\operatorname{Spec}(D_0X|_E) = \{\mu_1, \ldots, \mu_r\}$ and $\operatorname{Spec}(D_0X) = \{\mu_1, \ldots, \mu_r, \lambda_1, \ldots, \lambda_{n-r}\}$ where the μ_i or the λ_j need not be all different. Assume that there is $\alpha > 0$ such that*

$$|q_1\mu_1 + \cdots + q_r\mu_r - \lambda_j| \geq \alpha(q_1 + \cdots + q_r) \tag{1}$$

for $j = 1, \ldots, n-r$ and non-negative integers q_i with $q_1 + \cdots + q_r \geq 2$. Then there exists a unique germ of analytic invariant r-manifold W_E for X at 0 which is tangent to E at the origin.

The hypothesis (1) implies in particular the non-resonance condition

$$\lambda_j \neq q_1\mu_1 + \cdots + q_r\mu_r \tag{2}$$

for any j and non-negative integers q_i with $q_1 + \cdots + q_r \geq 2$. This last condition is the only needed condition for having just a formal r-dimensional manifold $\widehat{W}_E$ tangent to the linear space E at 0 and invariant for X, but it is not sufficient for analyticity. We recall this well known result below (Theorem 3.1).

On the other hand, it is worth to mention that condition (1) is equivalent to the two conditions: (a) the non-resonance condition (2) and (b) the convex hull of $\{\mu_1, \ldots, \mu_r\}$ in $\mathbb{C}$ does not contain the origin. Expressed in these terms, our condition (1) is an analogous counterpart for the existence of analytic invariant manifolds to the condition assumed for the *Linearization Poincaré's Theorem* (see for instance [**1**], [**2**] or [**7**] for a proof): X is analytically equivalent to D_0X at the origin if the convex hull of the eigenvalues of D_0X does not contain the origin and they have no resonances (condition (2) holds when we take the μ's as the totality of the eigenvalues).

We cannot take for granted that our result is completely new and unknown. Surely many specialists in dynamical systems could be aware of it or of near formulations of it. However, we did not find any reference where the result is stated and proved in the way we do here.

On the other hand, we give an interesting application of the Main Theorem in the last section of the paper. We show that the stable and unstable manifolds (the so called *strong invariant manifolds*) of a real analytic vector field at a singular point are analytic. This result is quite well known for all specialists in dynamical systems or vector fields. It appears mentioned in many general references (see for instance [**1**] or [**14**]) and proved in several particular cases (see for instance Hadamard's work [**9**] or [**13**], which uses the same arguments). A complete recent proof can be deduced from a more general result established in [**10**].

In all these references, the proof consists in constructing the real strong manifold as the real trace of a complex manifold obtained as a graph of a uniform limit of a sequence of holomorphic maps. The proof that we present here, as a consequence of the Main Theorem, is more direct and (maybe) more elemental: we just prove that the (real) formal stable or unstable invariant manifold, which exists and it is unique by the general theory of existence of formal invariant manifolds, is convergent.

2. Some properties of formal power series

In this section we fix notations and establish very basic results about formal power series that will be used in the proof of the main result, Theorem 3.3 below.

If $\mathbf{x} = (x_1, \ldots, x_r)$ is an r-tuple of variables and $q = (q_1, \ldots, q_r) \in \mathbb{Z}^r_{\geq 0}$, we denote, as usual, $\mathbf{x}^q = x_1^{q_1} \cdots x_r^{q_r}$ and $|q| = q_1 + \cdots + q_r$. An element $a \in \mathbb{C}[[x_1, \ldots, x_r]]$ of the ring of complex formal power series in r variables, written as

$$a = \sum_{q \in \mathbb{Z}^r_{\geq 0}} a_q \mathbf{x}^q,$$

will be also written as the sum of its homogeneous components

$$a = \sum_{k=0}^{\infty} a^{(k)}, \quad \text{where } a^{(k)} = \sum_{|q|=k} a_q \mathbf{x}^q.$$

For each positive integer k, consider $\mathcal{P}_k$ the $\mathbb{C}$-vector space of homogeneous polynomials of total degree k in the variables $x_1, \ldots, x_r$ endowed with the norm $\| \, \|_k$ defined by

$$\|u\|_k = \sum_{|q|=k} |u_q| \quad \text{if } u = \sum_{|q|=k} u_q \mathbf{x}^q \in \mathcal{P}_k.$$

Notice that we have the inequality

$$\|u \cdot w\|_{k+l} \leq \|u\|_k \|w\|_l, \quad \forall\, u \in \mathcal{P}_k, \ \forall\, w \in \mathcal{P}_l. \tag{3}$$

To any $a \in \mathbb{C}[[x_1, \ldots, x_r]]$, we associate the real series in a single variable

$$\widehat{a} = \sum_{k=0}^{\infty} \|a^{(k)}\|_k t^k \in \mathbb{R}[[t]], \tag{4}$$

called the *majorant of* a. It is clear that a is a convergent series if and only if its majorant $\widehat{a}$ is a convergent series.

On the other hand, the operation of taking the majorant does not give a homomorphism between the rings of formal power series. It has, however, useful properties concerning domination of series. To be more precise, given series $f = \sum_{n=0}^{\infty} f_n t^n$, $g = \sum_{n=0}^{\infty} g_n t^n \in \mathbb{C}[[t]]$ in a single variable t, we say that g *dominates* f and we write $f \preceq g$ if $|f_n| \leq |g_n|$ for any n, where $|\cdot|$ denotes the usual norm on complex numbers. Then, using inequality (3), if $a, b \in \mathbb{C}[[x_1, \ldots, x_r]]$, we can assert

$$\widehat{a+b} \preceq \widehat{a} + \widehat{b} \quad \text{and} \quad \widehat{a \cdot b} \preceq \widehat{a} \cdot \widehat{b}.$$

In fact, we will need a more general result about domination of majorant series obtained by composition:

Proposition 2.1. *Let $g \in \mathbb{C}[[x_1, \ldots, x_r, y_1, \ldots, y_s]]$ and $h_1, \ldots, h_s \in \mathbb{C}[[x_1, \ldots, x_r]]$ be formal power series such that $h_1(0) = \cdots = h_s(0) = 0$. Denote by $|g| \in \mathbb{C}[[x_1, \ldots, x_r, y_1, \ldots, y_s]]$ the series obtained from g replacing any of its coefficients by its corresponding norm and consider the composition $G(x_1, \ldots, x_r) = g(x_1, \ldots, x_r, h_1, \ldots, h_s)$. Then we have*

$$\widehat{G} \preceq |g|(t, \ldots, t, \widehat{h}_1, \ldots, \widehat{h}_s).$$

Proof: Write $g = \sum_{q\in\mathbb{Z}^r_{\geq 0},\, J\in\mathbb{Z}^s_{\geq 0}} g_{q,J}\mathbf{x}^q\mathbf{y}^J$, where $\mathbf{x} = (x_1, \ldots, x_r)$, $\mathbf{y} = (y_1, \ldots, y_s)$. For each $J = (j_1, \ldots, j_s) \in \mathbb{Z}^s_{\geq 0}$ and $i = 1, \ldots, s$, write the j_i-th power of the series h_i as the sum of its homogeneous components $h_i^{j_i} = \sum_{k=j_i}^{\infty} (h_i^{j_i})^{(k_i)}$. Then we have

$$G = \sum_{q\in\mathbb{Z}^r_{\geq 0},\, J\in\mathbb{Z}^s_{\geq 0}} \; \sum_{k_1\geq j_1,\ldots,k_s\geq j_s} g_{q,J}\mathbf{x}^q (h_1^{j_1})^{(k_1)} \cdots (h_s^{j_s})^{(k_s)}.$$

Note that each term in this equation has degree $|q| + k_1 + \cdots + k_s \geq |q| + |J|$. The homogeneous component of G of degree m is given by

$$G^{(m)} = \sum_{\substack{q\in\mathbb{Z}^r_{\geq 0},\, J\in\mathbb{Z}^s_{\geq 0}\\ |q|+|J|\leq m}} \; \sum_{\substack{k_1\geq j_1,\ldots,k_s\geq j_s\\ k_1+\cdots+k_s=m-|q|}} g_{q,J}\mathbf{x}^q (h_1^{j_1})^{(k_1)} \cdots (h_s^{j_s})^{(k_s)}.$$

Taking norms and using (3), we have

$$\|G^{(m)}\|_m \leq \sum_{\substack{q\in\mathbb{Z}^r_{\geq 0},\, J\in\mathbb{Z}^s_{\geq 0}\\ |q|+|J|\leq m}} \; \sum_{\substack{k_1\geq j_1,\ldots,k_s\geq j_s\\ k_1+\cdots+k_s=m-|q|}} |g_{q,J}| \|(h_1^{j_1})^{(k_1)}\|_{k_1} \cdots \|(h_s^{j_s})^{(k_s)}\|_{k_s}. \tag{5}$$

On the other hand, if we write the j_i-th power of the majorant $\widehat{h}_i$ as $\widehat{h}_i^{j_i} = \sum_{k=j_i}^{\infty} (\widehat{h}_i^{j_i})_k t^k$, then, by definition,

$$\begin{aligned}
&|g|(t, \ldots, t, \widehat{h}_1, \ldots, \widehat{h}_s)\\
&\quad= \sum_{q\in\mathbb{Z}^r_{\geq 0},\, J\in\mathbb{Z}^s_{\geq 0}} |g_{q,J}| t^{|q|} \left(\sum_{k_1=j_1}^{\infty} (\widehat{h}_1^{j_1})_{k_1} t^{k_1}\right) \cdots \left(\sum_{k_s=j_s}^{\infty} (\widehat{h}_s^{j_s})_{k_s} t^{k_s}\right)\\
&\quad= \sum_{m=0}^{\infty} \left(\sum_{\substack{q\in\mathbb{Z}^r_{\geq 0},\, J\in\mathbb{Z}^s_{\geq 0}\\ |q|+|J|\leq m}} \; \sum_{\substack{k_1\geq j_1,\ldots,k_s\geq j_s\\ k_1+\cdots+k_s=m-|q|}} |g_{q,J}| (\widehat{h}_1^{j_1})_{k_1} \cdots (\widehat{h}_s^{j_s})_{k_s} \right) t^m.
\end{aligned} \tag{6}$$

Comparing (5) and (6), the result follows once we prove that, for any j_i and for any integer $k \geq j_i$, we have

$$\|(h_i^{j_i})^{(k)}\|_k \leq (\widehat{h_i^{j_i}})_k. \tag{7}$$

In order to prove (7), put for simplicity $h = h_i$, $j = j_i$ and write $h = \sum_{k\geq 1} h^{(k)}$, the sum of homogeneous components of h. The homogeneous component of degree k of h^j is then $(h^j)^{(k)} = \sum_{l_1+l_2+\cdots+l_j=k} h^{(l_1)}h^{(l_2)}\cdots h^{(l_j)}$.

Thus

$$\begin{aligned}\|(h^j)^{(k)}\|_k &\leq \sum_{l_1+l_2+\cdots+l_j=k} \|h^{(l_1)}h^{(l_2)}\cdots h^{(l_j)}\|_k \\ &\leq \sum_{l_1+l_2+\cdots+l_j=k} \|h^{(l_1)}\|_{l_1}\|h^{(l_2)}\|_{l_2}\cdots\|h^{(l_j)}\|_{l_j}.\end{aligned}$$

Equation (7) follows from this last equation together with

$$\widehat{h}^j = \sum_{k=j}^{\infty}\left(\sum_{l_1+l_2+\cdots+l_j=k} \|h^{(l_1)}\|_{l_1}\|h^{(l_2)}\|_{l_2}\cdots\|h^{(l_j)}\|_{l_j}\right)t^k. \qquad \square$$

Together with Proposition 2.1 we will use the following result about domination of the majorant of the partial derivatives of a series.

Proposition 2.2. *Let $a \in \mathbb{C}[[\mathbf{x}]]$ be a formal power series where $\mathbf{x} = (x_1,\ldots,x_r)$. For $i = 1,\ldots,r$, we have*

$$\widehat{\frac{\partial a}{\partial x_i}} \preceq \frac{d\widehat{a}}{dt}.$$

Proof: Write $a = \sum_{q\in\mathbb{Z}^r_{\geq 0}} a_q\mathbf{x}^q = \sum_{k\geq 0} a^{(k)}$ with the same notations introduced above. We have that, for any $k \geq 1$, $\frac{\partial a^{(k)}}{\partial x_i}$ is the homogeneous component of $\frac{\partial a}{\partial x_i}$ of order $k-1$. Therefore

$$\widehat{\frac{\partial a}{\partial x_i}} = \sum_{k\geq 1}\left\|\frac{\partial a^{(k)}}{\partial x_i}\right\|_{k-1} t^{k-1}. \tag{8}$$

On the other hand, we have $\frac{\partial a}{\partial x_i} = \sum_{\substack{q=(q_1,\ldots,q_r)\in\mathbb{Z}^r_{\geq 0}\\ |q|=k}} q_i a_q \mathbf{x}^{q-e_i}$, where e_i is the i-th vector of the standard basis of $\mathbb{C}^r$ and $|q_i a_q| \leq |q||a_q|$ for any $q \in \mathbb{Z}^r_{\geq 0}$. Thus

$$\left\|\frac{\partial a^{(k)}}{\partial x_i}\right\|_{k-1} \leq k\left\|\frac{\partial a}{\partial x_i}\right\|_k$$

and the proof follows from (8) and the definition of the majorant series (4). □

3. Invariant manifolds of holomorphic vector fields

During the rest of the paper we are interested in the following situation. Let X be a germ of a holomorphic vector field at $0 \in \mathbb{C}^n$ so that $X(0) = 0$ and denote by D_0X its linear part at the origin. Suppose that we have a linear subspace E of dimension r of $\mathbb{C}^n$, with $0 < r < n$, which is invariant for D_0X. The question is whether there exists a non-singular r-dimensional analytic manifold W_E through the origin, whose tangent space at 0 is equal to E and invariant for the vector field X.

It is well known that such an invariant manifold W_E does not exists in general, so that several conditions must be imposed. If we are only interested in formal invariant manifolds, there is a well known and satisfactory result which we want to recall here.

Put $s = n-r$ and $\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{C}^r$, $\lambda = (\lambda_1, \ldots, \lambda_s) \in \mathbb{C}^s$ where $\{\mu_1, \mu_2, \ldots, \mu_r, \lambda_1, \ldots, \lambda_s\}$ is the spectra of D_0X and $\{\mu_1, \ldots, \mu_r\}$ is the spectra of the restriction $D_0X|_E$ (where the μ_j or the λ_j are repeated according to their multiplicity and hence need not be all distinct). Denote finally by $\langle\, , \rangle$ the usual scalar product on $\mathbb{C}^r$.

Theorem 3.1. *Assume that the eigenvalues satisfy the* non-resonance conditions

$$(9) \qquad \lambda_j \neq \langle q, \mu\rangle, \quad \forall\, j \in \{1, \ldots, s\},\ \forall\, q \in \mathbb{Z}^r_{\geq 0} \text{ with } |q| \geq 2.$$

Then there exists a unique formal non-singular r-dimensional manifold $\widehat{W}_E$ tangent to E at 0 and invariant for X.

A proof of this result can be derived from the corresponding one in [**6**, Theorem 3.7, p. 417]. However, we outline here the main arguments for two reasons. On one hand, below we will explicitly use several notations appearing in the proof. On the other hand, our hypothesis are slightly weaker than the ones stated in the aforementioned reference (there, it is assumed that there exists another linear subspace F, invariant for D_0X and complementary to E, but one can see that this last condition is inessential).

Outline of the proof of Theorem 3.1: Choose analytic coordinates $(\mathbf{x}, \mathbf{y}) = (x_1, \ldots, x_r, y_1, \ldots, y_s)$ at $0 \in \mathbb{C}^n$ such that E is tangent to $\{\mathbf{y} = 0\}$. We look for $\widehat{W}_E$ given by the graph of a formal map

$$(10) \qquad \widehat{W}_E : \{\mathbf{y} = h(\mathbf{x})\},\ h(\mathbf{x}) = (h_1(\mathbf{x}), \ldots, h_s(\mathbf{x})), \quad h_j \in \mathbb{C}[[\mathbf{x}]],$$

where, for any j, $h_j(0) = 0$ and $h_j'(0) = 0$ (tangent to E), and such that, if the local expression of X is

$$X = a(\mathbf{x},\mathbf{y})\frac{\partial}{\partial \mathbf{x}} + b(\mathbf{x},\mathbf{y})\frac{\partial}{\partial \mathbf{y}},$$

where $a = (a_1,\dots,a_r)$ and $b = (b_1,\dots,b_s)$ are vectors of convergent power series, then $h = (h_1,\dots,h_s)$ is a formal solution of the the system of partial differential equations written in matricial notation as

$$\frac{\partial h}{\partial \mathbf{x}} a(\mathbf{x}, h(\mathbf{x})) = b(\mathbf{x}, h(\mathbf{x})). \tag{11}$$

Since E is invariant for D_0X we may write the linear part as a block-triangular matrix

$$D_0X = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where A is the matrix of the restriction $D_0X|_E$. Thus equation (11) becomes

$$\frac{\partial h}{\partial \mathbf{x}}\left(A\mathbf{x} + Ch(\mathbf{x}) + f(\mathbf{x}, h(\mathbf{x}))\right) = Bh(\mathbf{x}) + g(\mathbf{x}, h(\mathbf{x})), \tag{12}$$

where $f = (f_1,\dots,f_r)$ and $g = (g_1,\dots,g_s)$ are vectors of convergent power series in the variables $(\mathbf{x},\mathbf{y})$ of order greater or equal than two. Write each h_j as sum of its homogeneous components

$$h_j = \sum_{k=1}^{\infty} h_j^{(k)}, \quad h = \sum_{k=1}^{\infty} h^{(k)}, \quad h^{(k)} = (h_1^{(k)},\dots,h_s^{(k)}),$$

and let $F^{(k)}$ and $G^{(k)}$ be the homogeneous components of degree k of $f(\mathbf{x}, h(\mathbf{x}))$ and $g(\mathbf{x}, h(\mathbf{x}))$, respectively. Notice that $F^{(1)} = G^{(1)} = 0$. Recall also that we are looking for a solution h for which $h^{(1)} = 0$. Therefore, for $k \geq 2$, comparing the homogeneous components of degree k of both sides of equation (12), we obtain

$$\frac{\partial h^{(k)}}{\partial \mathbf{x}} A\mathbf{x} - Bh^{(k)} = G^{(k)} - \sum_{l=2}^{k-1} \frac{\partial h^{(k-l+1)}}{\partial \mathbf{x}} F^{(l)} - \sum_{l=1}^{k-2} \frac{\partial h^{(l+1)}}{\partial \mathbf{x}} Ch^{(k-l)}. \tag{13}$$

Denote by $\mathcal{P}_k$ the vector space over $\mathbb{C}$ of homogeneous polynomials of degree k. Notice that, due to the fact that f and g have order greater or equal than two as series in the variables $(\mathbf{x},\mathbf{y})$, in the right-hand side of (13) only the homogeneous components $h^{(2)},\dots,h^{(k-1)}$ are involved, but not $h^{(k)}$. On the other hand, we write the left-hand side of equation (13) as $\mathcal{L}_{k,A,B}(h^{(k)})$, where $\mathcal{L}_{k,A,B}\colon (\mathcal{P}_k)^s \to (\mathcal{P}_k)^s$ is the linear operator defined as

$$\mathcal{L}_{k,A,B}(v) = \frac{\partial v}{\partial \mathbf{x}} Ax - Bv.$$

The spectrum of $\mathcal{L}_{k,A,B}$ is precisely the set

$$\left\{ \langle q, \mu \rangle - \lambda_j \mid q \in \mathbb{Z}^r_{\geq 0},\, |q| = k,\, j = 1, \ldots, s \right\}$$

(see the proof in [**6**, Lemma 2.5, p. 409]). The hypothesis of non-resonance (9) implies then that $\mathcal{L}_{k,A,B}$ is non-singular and equation (13) can be solved recursively and uniquely for $h^{(k)}$ by means of the formula

$$h^{(k)} = \mathcal{L}^{-1}_{k,A,B} \left(G^{(k)} - \sum_{l=2}^{k-1} \frac{\partial h^{(k-l+1)}}{\partial \mathbf{x}} F^{(l)} - \sum_{l=1}^{k-2} \frac{\partial h^{(l+1)}}{\partial \mathbf{x}} C h^{(k-l)} \right). \tag{14}$$

This finishes the proof of Theorem 3.1. Notice that if A, B are real matrices then $\mathcal{L}_{k,A,B}$ preserves the set of real elements in $(\mathcal{P}_k)^s$. This remark will we used in Section 4. □

If all the components h_j of h in (10) are convergent power series, then their sum will give rise to a non-singular analytic manifold W_E, tangent to E at the origin and which is invariant for X in virtue of (11). Moreover, its germ at the origin is the unique germ of analytic r-manifold which is invariant for X and tangent to E, by uniqueness in Theorem 3.1. We will simply say that the *formal invariant manifold* $\widehat{W}_E$ *is convergent.*

It is well known that $\widehat{W}_E$ does not need to be convergent. A typical example is *Euler's equation*, associated to the planar vector field

$$X = x^2 \frac{\partial}{\partial x} + (y - x) \frac{\partial}{\partial y},$$

for which E is the line $\{y = x\}$. The eigenvalue, μ, associated to E is equal to 0; in particular we have the non-resonance conditions (9) and Theorem 3.1 applies. But the corresponding formal invariant curve $\widehat{W}_E$ is not convergent: one checks that $\widehat{W}_E$ is the graph of the series $h(x) = \sum_{n=0}^{\infty} n! x^{n+1}$.

A classical result due to Briot and Bouquet asserts that the annulation of the eigenvalue μ is the only obstruction to convergence in the planar situation.

Theorem 3.2 (Briot–Bouquet)**.** *Let X be a germ of holomorphic vector field at $0 \in \mathbb{C}^2$. Let $\mathrm{Spec}(D_0 X) = \{\mu, \lambda\}$ be the spectrum of the linear part and let E be an invariant line of $D_0 X$ associated to the eigenvalue μ. Assume that conditions* (9) *hold. If $\mu \neq 0$ then the formal curve $\widehat{W}_E$ is convergent.*

A modern proof of Briot and Bouquet's Theorem can be found in [**5**]. It is worth to mention that the situation considered in that reference is slightly more restrictive than the one stated here: the hypothesis about the eigenvalues is that $\mu \neq 0$ and λ/μ is not a positive rational number (X is said to have a *simple singularity* at the origin). However, their proof works as well with the weaker hypothesis (9); that is, $\lambda \neq k\mu$ for any integer $k \geq 2$. In fact, this last condition is the one stated in the original Briot and Bouquet's paper [**3**, Théorème XVIII, p. 168].

Notice that in the notations of Briot–Bouquet's Theorem, the non-resonance conditions (9), together with $\mu \neq 0$, are equivalent to the condition

$$\inf_{k \in \mathbb{Z}_{\geq 2}} \left\{ \frac{|k\mu - \lambda|}{k} \right\} > 0.$$

This property motivates our main result, a generalization of Theorem 3.2 in higher dimension.

Theorem 3.3 (Main). *Let X be a holomorphic vector field at $0 \in \mathbb{C}^n$ and E a linear r-dimensional subspace of $\mathbb{C}^n$ invariant for the linear part D_0X. Put $\mu = (\mu_1, \ldots, \mu_r)$ where* $\operatorname{Spec}(D_0X|_E) = \{\mu_1, \ldots, \mu_r\}$ *and denote* $\operatorname{Spec}(D_0X) = \{\mu_1, \ldots, \mu_r, \lambda_1, \ldots, \lambda_{n-r}\}$. *Assume that there is $\alpha > 0$ such that*

$$(15)\quad |\langle q, \mu\rangle - \lambda_j| \geq \alpha|q| \text{ for all } j = 1, \ldots, n-r \text{ and } q \in \mathbb{Z}^r_{\geq 0} \text{ with } |q| \geq 2.$$

(In particular the non-resonance conditions (9) *hold.) Then the formal invariant manifold $\widehat{W}_E$ given by Theorem 3.1 is convergent.*

The idea of the proof is to reduce the problem to the planar situation and apply Briot–Bouquet's Theorem (Theorem 3.2). We divide the proof in several steps.

Step 1. Choosing coordinates. Put $s = n - r$. Take coordinates $(\mathbf{x}, \mathbf{y})$ at 0 where $\mathbf{x} = (x_1, \ldots, x_r)$ and $\mathbf{y} = (y_1, \ldots, y_s)$ such that E is tangent to $\{\mathbf{y} = 0\}$. In these coordinates we may write

$$D_0X = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where A is a square r-dimensional matrix, the matrix of the restriction $D_0X|_E$ in the $\mathbf{x}$ coordinates.

Choose a constant $\sigma \in \mathbb{R}$ with $0 < \sigma < \frac{\alpha}{2r}$ and let $\tau = \frac{\alpha}{\sqrt{2}}$. Up to a linear transformation, we can assume the coordinates chosen so that

$$A = \begin{pmatrix} \mu_1 & 0 & \cdots & 0 & 0 \\ \sigma_2 & \mu_2 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & \mu_{r-1} & 0 \\ 0 & 0 & \cdots & \sigma_r & \mu_r \end{pmatrix}, \quad B = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ \tau_2 & \lambda_2 & \cdots & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & \cdots & \lambda_{s-1} & 0 \\ 0 & 0 & \cdots & \tau_s & \lambda_s \end{pmatrix},$$

with $\sigma_i \in \{0, \sigma\}$ and $\tau_j \in \{0, \tau\}$. To unify notation put also $\sigma_1 = \tau_1 = 0$.

Step 2. The equations of $\widehat{W}_E$. As we have done in the proof of Theorem 3.1, if the local expression of X in the chosen coordinates is

$$X = (A\mathbf{x} + C\mathbf{y} + f(\mathbf{x}, \mathbf{y}))\frac{\partial}{\partial \mathbf{x}} + (B\mathbf{y} + g(\mathbf{x}, \mathbf{y}))\frac{\partial}{\partial \mathbf{y}},$$

then the formal invariant manifold $\widehat{W}_E$ is given by the graph $\mathbf{y} = h(\mathbf{x})$ of the vector of formal power series $h = (h_1, \ldots, h_s) \in (\mathbb{C}[[\mathbf{x}]])^s$ which is a solution of the system (12) of PDEs. We have to show that any component h_i of h is a convergent series in the variables $\mathbf{x}$.

We write the system (12) in a different way. First put

$$\mathrm{U} = (u_1, \ldots, u_s) = \frac{\partial h}{\partial \mathbf{x}} A\mathbf{x} - Bh(\mathbf{x}) \in (\mathbb{C}[[\mathbf{x}]])^s.$$

Write $f = (f_1, \ldots, f_r)$, $g = (g_1, \ldots, g_s)$ the components of f and g, where $f_j, g_j \in \mathbb{C}\{\mathbf{x}, \mathbf{y}\}$ and write also $C = (c_{lm})_{\substack{1 \le l \le r \\ 1 \le m \le s}}$. Then (12) is equivalent to

$$u_i = g_i(\mathbf{x}, h(\mathbf{x})) - \sum_{j=1}^{r} \frac{\partial h_i}{\partial x_j}\left(f_j(\mathbf{x}, h(\mathbf{x})) + \sum_{m=1}^{s} c_{jm} h_m(\mathbf{x})\right) \quad \text{for } i = 1, \ldots, s. \tag{16}$$

Step 3. Formulas for the homogeneous components of u_i. By the especial form of the matrices A and B we have

$$u_i = \sum_{j=1}^{r} \frac{\partial h_i}{\partial x_j}(\sigma_j x_{j-1} + \mu_j x_j) - (\tau_i h_{i-1} + \lambda_i h_i).$$

Writing explicitly the homogeneous components $h_i = \sum_{k=2}^{\infty} h_i^{(k)}$, $h_i^{(k)} = \sum_{|q|=k} h_{i,q}^{(k)} \mathbf{x}^q$, we can write

$$(17)\quad u_i = \sum_{k=2}^{\infty} \left[\sum_{|q|=k} \sum_{j=1}^{r} \sigma_j q_j h_{i,q}^{(k)} \mathbf{x}^{q+e_{j-1}-e_j} + \sum_{|q|=k} (\langle q, \mu \rangle - \lambda_i) h_{i,q}^{(k)} \mathbf{x}^q - \tau_i h_{i-1}^{(k)} \right],$$

where e_i is the i-th vector of the standard basis of $\mathbb{C}^r$.

With the convention that $h_{i,q-e_{j-1}+e_j}^{(k)} = 0$ if $q_{j-1} = 0$, and changing the index q by $q+e_{j-1}-e_j$ in the first summand in (17), the homogeneous component of degree k of u_i is given by

$$(18)\quad u_i^{(k)} = \sum_{|q|=k} \left[\sum_{j=2}^{r} \sigma_j (q_j + 1) h_{i,q-e_{j-1}+e_j}^{(k)} + (\langle q, \mu \rangle - \lambda_i) h_{i,q}^{(k)} - \tau_i h_{i-1,q}^{(k)} \right] \mathbf{x}^q.$$

Step 4. Relation between the norms of $u_i^{(k)}$ and $h_j^{(k)}$. For each q in $\mathbb{Z}_{\geq 0}^r$ with $|q| = k$, we consider the triangle inequality

$$\left| (\langle q, \mu \rangle - \lambda_i) h_{i,q}^{(k)} + \sum_{j=2}^{r} \sigma_j (q_j + 1) h_{i,q-e_{j-1}+e_j}^{(k)} - \tau_i h_{i-1,q}^{(k)} \right| \geq |\langle q, \mu \rangle - \lambda_i| \left| h_{i,q}^{(k)} \right| - \sum_{j=2}^{r} \sigma (q_j + 1) \left| h_{i,q-e_{j-1}+e_j}^{(k)} \right| - \left| \tau_i h_{i-1,q}^{(k)} \right|.$$

Summing up over all q with $|q| = k$ and using the hypothesis (15), we obtain, from (18),

$$\|u_i^{(k)}\|_k \geq \alpha k \|h_i^{(k)}\|_k - \tau_i \|h_{i-1}^{(k)}\|_k - \sum_{|q|=k} \sum_{j=2}^{r} \sigma (q_j + 1) \left| h_{i,q-e_{j-1}+e_j}^{(k)} \right|.$$

Notice that if $q_j = k$ then $q_{j-1} = 0$ and so $h_{i,q-e_{j-1}+e_j}^{(k)} = 0$ by the convention we have established. Thus,

$$\sum_{|q|=k} \sum_{j=2}^{r} \sigma (q_j + 1) \left| h_{i,q-e_{j-1}+e_j}^{(k)} \right| < \sigma r k \|h_i^{(k)}\|_k.$$

Since we have chosen σ with $\sigma r < \frac{\alpha}{2}$ we have finally

$$\|u_i^{(k)}\|_k > \frac{\alpha}{2} k \|h_i^{(k)}\|_k - \tau_i \|h_{i-1}^{(k)}\|_k. \tag{19}$$

Step 5. Using majorants. In order to prove that the series $h_i \in \mathbb{C}[[\mathbf{x}]]$ is convergent, we will prove that its majorant $\widehat{h}_i \in \mathbb{C}[[t]]$, as defined in (4), is a convergent series in the (single) variable t. By the definition of the majorant, we have $t\dfrac{d\widehat{h}_i}{dt} = \sum_{k=1}^{\infty} k\|h_i^{(k)}\|_k t^k$. Therefore, we can write (19) as

$$\frac{\alpha}{2} t \frac{d\widehat{h}_i}{dt} \preceq \tau_i \widehat{h}_{i-1} + \widehat{u}_i. \tag{20}$$

It is enough to show that the series $H(t) = \widehat{h}_1(t) + \cdots + \widehat{h}_s(t)$ is convergent.

We look for a differential inequality for H. In one hand, adding the inequalities (20) for any $i = 1, \ldots, s$, we have

$$\frac{\alpha}{2} t \frac{dH}{dt} \preceq \tau H + (\widehat{u}_1 + \cdots + \widehat{u}_s). \tag{21}$$

On the other hand, take $c > 0$ so that $|c_{lm}| \leq c$ for any l, m and apply Propositions 2.1 and 2.2 to equation (16) to obtain

$$\begin{aligned} \widehat{u}_i &\preceq |g_i|(t, \ldots, t, \widehat{h}_1, \ldots, \widehat{h}_s) \\ &\quad + \frac{d\widehat{h}_i}{dt} \sum_{j=1}^{r} \left(|f_j|(t, \ldots, t, \widehat{h}_1, \ldots, \widehat{h}_s) + c \sum_{m=1}^{s} \widehat{h}_m \right) \\ &\preceq |g_i|(t, \ldots, t, H, \ldots, H) \\ &\quad + \frac{d\widehat{h}_i}{dt} \left(\sum_{j=1}^{r} |f_j|(t, \ldots, t, H, \ldots, H) + cH \right). \end{aligned} \tag{22}$$

Define

$$\begin{aligned} F(t, z) &= \sum_{j=1}^{r} |f_j|(t, \ldots, t, z, \ldots, z) \quad \text{and} \\ G(t, z) &= \sum_{i=1}^{s} |g_i|(t, \ldots, t, z, \ldots, z). \end{aligned} \tag{23}$$

They are convergent power series in the two variables (t, z) with order greater or equal than two and with non-negative real coefficients. Adding inequalities (22) for $i = 1, \dots, r$ and using (21) we conclude finally

$$\frac{\alpha}{2} t \frac{dH}{dt} \preceq \tau H + G(t, H) + \frac{dH}{dt} \left(F(t, H) + cH \right). \tag{24}$$

Step 6. Reduction to Briot–Bouquet's Theorem. Consider the analytic vector field

$$Y = \left(\frac{\alpha}{2} t - cz - F(t, z) \right) \frac{\partial}{\partial t} + (\tau z + G(t, z)) \frac{\partial}{\partial z}$$

at the origin of $\mathbb{C}^2$. The linear part of Y has eigenvalues $\alpha/2, \tau$. Since $\alpha \neq 0$ and $\dfrac{\tau}{\alpha/2} = \sqrt{2} \notin \mathbb{Q}_{>0}$, Y is in the hypothesis of Briot–Bouquet's Theorem 3.2 with $E = \{z = 0\}$. Therefore, the (unique) formal series $z(t) = \sum_{n=2}^{\infty} z_n t^n \in \mathbb{C}[[t]]$ which is a solution of the differential equation

$$\frac{\alpha}{2} t \frac{dz}{dt} = \tau z + G(t, z) + \frac{dz}{dt} \left(F(t, z) + cz \right) \tag{25}$$

is a convergent series.

To finish the proof, we are going to show that any coefficient z_n of the series $z(t)$ is non-negative and that, if we write $H = \sum_{n=2}^{\infty} H_n t^n$, then $H_n \leq z_n$ for any $n \geq 2$.

We proceed by induction on n. Looking more carefully how the series $F(t, z)$ and $G(t, z)$ are constructed (23), we can write

$$F(t, z(t)) = \sum_{n=2}^{\infty} \overline{F}_n(z_2, \dots, z_{n-1}) t^n, \quad G(t, z(t)) = \sum_{n=2}^{\infty} \overline{G}_n(z_2, \dots, z_{n-1}) t^n,$$

where, for each $n \geq 2$, $\overline{F}_n$ and $\overline{G}_n$ are polynomials in $n-2$ variables with non-negative coefficients ($\overline{F}_2$ and $\overline{G}_2$ are non-negative constants). Consequently, we have also

$$F(t, H) = \sum_{n=2}^{\infty} \overline{F}_n(H_2, \dots, H_{n-1}) t^n, \quad G(t, H) = \sum_{n=2}^{\infty} \overline{G}_n(H_2, \dots, H_{n-1}) t^n.$$

Inequality (24) and equation (25) give for $n = 2$,

$$(\alpha - \tau) H_2 \leq \overline{G}_2, \quad (\alpha - \tau) z_2 = \overline{G}_2.$$

Then $H_2 \leq z_2$. Suppose we have shown that $H_k \leq z_k$ for $k \leq n-1$ where $n \geq 3$. Since the coefficients of $\overline{F}_k$ and $\overline{G}_k$ are non-negative, using again (24) and (25), we obtain finally

$$\begin{aligned}
\left(\frac{\alpha}{2}n - \tau\right) H_n &\leq \overline{G}_n(H_2, \dots, H_{n-1}) \\
&\quad + \sum_{k=1}^{n-2} (k+1) H_{k+1} \left(\overline{F}_{n-k}(H_2, \dots, H_{n-k-1}) + cH_{n-k}\right) \\
&\leq \overline{G}_n(z_2, \dots, z_{n-1}) \\
&\quad + \sum_{k=1}^{n-2} (k+1) z_{k+1} \left(\overline{F}_{n-k}(z_2, \dots, z_{n-k-1}) + cz_{n-k}\right) \\
&= \left(\frac{\alpha}{2}n - \tau\right) z_n
\end{aligned}$$

as was to be proved. □

4. Application to real analytic vector fields

In this final section we give an application of our main result, Theorem 3.3, to (germs of) real analytic vector fields X at $0 \in \mathbb{R}^n$.

For such a vector field X, denote by D_0X its linear part at the origin and consider the decomposition

$$\mathbb{R}^n = E^s \oplus E^c \oplus E^u$$

into the linear subspaces associated to the eigenvalues of D_0X with positive, zero and negative real parts, respectively. Put also $E^{cs} = E^s \oplus E^c$, $E^{cu} = E^u \oplus E^c$. A well known result in the theory of invariant manifolds (see for instance [**11**] or [**7**]) asserts that, for any $k \in \mathbb{Z}_{\geq 1}$, there are smooth embedded manifolds W_k^s, W_k^{cs}, W_k^c, W_k^{cu}, W_k^u of class C^k in a neighborhood of the origin which are invariant by X and such that $T_0W_k^\epsilon = E^\epsilon$ with $\epsilon = s, cs, c, cu, u$. They are called *local stable, center-stable, center, center-unstable* and *unstable manifold of class C^k of X at* 0, respectively. Moreover, the stable and unstable manifolds are uniquely determined by dynamical properties, so that they do not depend on k (thus they are of class C^∞). Both are called *strong invariant manifolds*. In contrast, the *weak invariant manifolds* W_k^{cs}, W_k^c, W_k^{cu}, are not unique, may depend on k and may not be even of class C^∞ at the origin.

The existence (and uniqueness) of such invariant manifolds in the formal setting is easier, however much less treated in the literature. Also, the strong manifolds are (real) analytic at the origin. We summarize these results in the following statement, which can be proved as a corollary of Theorem 3.1 and our main result, Theorem 3.3.

Theorem 4.1. *There are unique formal non-singular manifolds*

$$\widehat{W}^s, \quad \widehat{W}^{cs}, \quad \widehat{W}^c, \quad \widehat{W}^{cu}, \quad \widehat{W}^u$$

at the origin of $\mathbb{R}^n$ *which are invariant for* X *and tangent to* E^s*,* E^{cs}*,* E^c*,* E^{cu} *and* E^u*, respectively. Moreover, the formal stable and unstable manifolds,* $\widehat{W}^s$ *and* $\widehat{W}^u$ *are convergent.*

Proof: Fix $\epsilon \in \{s, cs, c, cu, u\}$ and let $r = \dim(E^\epsilon)$. Let $E^{\epsilon'}$ with $\epsilon' \in \{s, cs, c, cu, u\}$ be a complementary of E^ϵ. In order to obtain $\widehat{W}^\epsilon$, we take real analytic coordinates $(\mathbf{x}, \mathbf{y})$ such that E^ϵ is tangent to $\{\mathbf{y} = 0\}$ and write in this coordinates

$$X = a(\mathbf{x}, \mathbf{y})\frac{\partial}{\partial \mathbf{x}} + b(\mathbf{x}, \mathbf{y})\frac{\partial}{\partial \mathbf{y}},$$

where $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_s)$ are vectors of real convergent power series. Considering a, b also as convergent complex power series, we can think of the real analytic vector field X as a holomorphic one in a neighborhood of $0 \in \mathbb{C}^n$ (with the same notation for complex coordinates $(\mathbf{x}, \mathbf{y})$). By the definition of the linear subspaces E^ϵ, if we denote by μ_i the eigenvalues of D_0X corresponding to E^ϵ and λ_j the ones corresponding to $E^{\epsilon'}$ then the non-resonance conditions (9) in Theorem 3.1 hold. We obtain a (complex) formal invariant manifold $\widehat{W}_{E^\epsilon}$, tangent to E^ϵ, given as a graph

$$\mathbf{y} = h(\mathbf{x}) = (h_1(\mathbf{x}), \ldots, h_{n-r}(\mathbf{x})) \in (\mathbb{C}[[\mathbf{x}]])^{n-r}$$

formally satisfying the system of partial differential equations (11). It only remains to show that any component of h is a real power series, that is, that $h_i(\mathbf{x}) \in \mathbb{R}[[\mathbf{x}]]$. Notice that, if we write in homogeneous components

$$h_i = \sum_{k=2}^{\infty} h_i^{(k)}, \quad h_i^{(k)} = \sum_{|q|=k} h_{i,q}^{(k)} \mathbf{x}^q, \quad h_{i,q}^{(k)} \in \mathbb{C},$$

then formula (14) gives recursively for $k \geq 2$ that the values $\{h_{i,q}^{(k)}\}_{i,q}$ form a solution of a system of linear equations where the coefficients are real polynomials in the values $\{h_{i,q}^{(l)}\}_{i,q,l<k}$, since a and b are real power series. We conclude that there exists a solution of (11) which is a vector of real power series and, by the uniqueness of the formal manifold $\widehat{W}_{E^\epsilon}$ stated in Theorem 3.1, each $h_i \in \mathbb{R}[[\mathbf{x}]]$.

The second part of the theorem, i.e. that $\widehat{W}^s$ and $\widehat{W}^u$ are convergent, follows directly from Theorem 3.3 and the above observation: for example in the case of the stable manifold, we have that $\widehat{W}^s$ is the formal invariant manifold $\widehat{W}_{E^s}$ given by Theorem 3.1 and, being $\mathrm{Re}(\mu_i) < 0$ for any eigenvalue μ_i corresponding to E^s and $\mathrm{Re}(\lambda_j) \geq 0$ for any eigenvalue corresponding to the complementary space E^{cu}, the condition (15) in Theorem 3.3 holds and thus $\widehat{W}^s$ is convergent. $\square$

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