Publ. Mat. **58** (2014), 193–212
DOI: 10.5565/PUBLMAT_58114_09

GENERALIZED QUASIDISKS AND CONFORMALITY

Chang-Yu Guo, Pekka Koskela, and Juhani Takkinen

Abstract: We introduce a weaker variant of the concept of linear local connectivity, sufficient to guarantee the extendability of a conformal map $f\colon \mathbb{D} \to \Omega$ to the entire plane as a homeomorphism of locally exponentially integrable distortion. Additionally, we show that a conformal map as above cannot necessarily be extended in this manner if we assume that Ω is the image of $\mathbb{D}$ under a self-homeomorphism of the plane that has locally exponentially integrable distortion.

2010 Mathematics Subject Classification: 30C62, 30C65.

Key words: Homeomorphism of finite distortion.

1. Introduction

The concept of a quasidisk is central in the theory of planar quasiconformal mappings; see, for example, [**2, 4, 7, 20**]. One calls a Jordan domain $\Omega \subset \mathbb{R}^2$ a quasidisk if it is the image of the unit disk $\mathbb{D}$ under a quasiconformal mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the entire plane. If f is K-quasiconformal, we say that Ω is a K-quasidisk. Another possibility is to require that f is additionally conformal in the unit disk $\mathbb{D}$. It is essentially due to Kühnau [**19**] that Ω is a K-quasidisk if and only if Ω is the image of $\mathbb{D}$ under a K^2-quasiconformal mapping $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ that is conformal in $\mathbb{D}$, see [**8**].

A substantial part of the theory of quasiconformal mappings has recently been shown to extend in a natural form to the setting of mappings of locally exponentially integrable distortion [**3, 4, 6, 9, 10, 12, 18, 22, 25**]. See Section 2 below for the definition of this class of mappings. However, very little is known about the analogues of the concept of a quasidisk. For the model domain

$$\Omega_s = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1,\, |x_2| < x_1^{1+s}\} \cup B(x_s, r_s), \tag{1.1}$$

where $x_s = (s+2, 0)$ and $r_s = \sqrt{(s+1)^2+1}$, $s > 0$, the situation is rather well understood: $\Omega_s = f(\mathbb{D})$ under a homeomorphism with

C.-Y. Guo and P. Koskela were partially supported by the Academy of Finland grant 131477.

locally λ-exponentially integrable distortion if $\lambda < 2/s$, but this cannot happen when $\lambda > 2/s$, see [**16**]. Moreover, if f is additionally required to be quasiconformal in $\mathbb{D}$, then the critical bound for λ is $1/s$, see [**15**]. Notice the difference to the setting of quasiconformal mappings: instead of the switch from K to K^2 under the additional conformality condition, one essentially switches from λ to $\lambda/2$. One might expect this to be the case in general, but this turns out not to hold.

Theorem 1.1. *Given $s > 0$, there is a homeomorphism $f_s\colon \mathbb{R}^2 \to \mathbb{R}^2$ of locally exponentially integrable distortion so that*

$$f_s(\mathbb{D}) = \Delta_s := B(x_s', r_s') \setminus \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\, |x_2| \leq x_1^{1+s}\},$$

where $x_s' = (-s, 0)$ and $r_s' = \sqrt{(s+1)^2 + 1}$. On the other hand, there is no homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ of locally exponentially integrable distortion such that f is quasiconformal in $\mathbb{D}$ and $f(\mathbb{D}) = \Delta_s$.

In fact, given $\lambda < 2/s$, we construct a homeomorphism $f_\lambda\colon \mathbb{R}^2 \to \mathbb{R}^2$ of locally λ-exponentially integrable distortion so that $f_\lambda(\mathbb{D}) = \Delta_s$. Suppose then that $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism of finite distortion $K_f(x)$ and that g is K-quasiconformal in $\mathbb{D}$ with $f(\mathbb{D}) = \Delta_s$. We prove below that $K_f \notin L^p_{\mathrm{loc}}(\mathbb{R}^2)$ if $p > K/s$.

Thus an inward pointing polynomial cusp rules out the extendability of a Riemann mapping function to a homeomorphism of locally exponentially integrable distortion, but such exterior cusps are not that dangerous.

It is then natural to ask for general sufficient conditions for extendability. Towards this end, let us describe the standard way of extending a conformal map $f\colon \mathbb{D} \to \Omega$, where Ω is a Jordan domain, to a mapping of the entire plane. First of all, f can be extended to a homeomorphism between $\overline{\mathbb{D}}$ and $\overline{\Omega}$. For simplicity, we denote this extended homeomorphism also by f. It follows from the Riemann Mapping Theorem there exists a conformal mapping $g\colon \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \mathbb{R}^2 \setminus \overline{\Omega}$ such that the complement of the closed unit disk gets mapped to the complement of $\overline{\Omega}$. In this correspondence the boundary curve $\Gamma = \partial\Omega$ is mapped homeomorphically onto the boundary circle $\partial\mathbb{D}$ and hence the composed mapping $G' = g^{-1} \circ f$ is a well-defined circle homeomorphism, called conformal welding. Suppose we are able to extend G' to the exterior of the unit disk, with the extension still denoted by G'. Then the mapping $G = g \circ G'$ will be well-defined outside the unit disk and it coincides with f on the boundary circle $\partial\mathbb{D}$. Finally, if we define

$$F(x) = \begin{cases} G(x) & \text{if } |x| \geq 1, \\ f(x) & \text{if } |x| \leq 1, \end{cases}$$

then we obtain an extension of f to the entire plane. In the case of a quasidisk, that is when Ω is linearly locally connected (LLC), the extension G' can be chosen to be quasiconformal and hence the obtained map F is also quasiconformal; see [**1**].

Before stating our extension result, let us stress that the extendability of a conformal mapping $f\colon \mathbb{D} \to \Omega$ to a homeomorphism $\hat{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ of locally integrable distortion is essentially equivalent to being able to extend the conformal welding G' above to this class. Indeed, if $\hat{f}$ extends f then $g^{-1} \circ \hat{f}$ extends G to the exterior of $\mathbb{D}$ and has the same distortion as $\hat{f}$. Reflecting (twice) with respect to the unit circle one then further obtains an extension to $\mathbb{D} \setminus \{0\}$. Hence, one obtains an extension $\hat{G}'$ of G' to $\mathbb{R}^2 \setminus \{0\}$ with distortion that has the same local integrability degree as the distortion of $\hat{f}$. If the latter distortion is sufficiently nice in a neighborhood of infinity (e.g. bounded, see Lemma 3.3 below for the locally exponentially integrable setting), then this holds in all of $\mathbb{R}^2$ as well. In the setting of Theorem 1.1, the welding G' is so strongly compressing at a single point that the known modulus of continuity estimate for (the inverse) of a mapping of locally exponentially integrable distortion gets violated. This argument also applies for the comment after Theorem 1.1.

One then expects that the above procedure will produce a mapping of locally exponentially integrable distortion when the linear local connectivity property is relaxed to a suitable, slightly weaker condition. Our next result confirms this expectation.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a ψ-locally connected Jordan domain with $\psi(r) = \frac{Cr}{\log^s \log \frac{1}{r}}$ for some positive constant C and $s \in (0, \frac{1}{4})$. Then any conformal mapping $f\colon \mathbb{D} \to \Omega$ can be extended to the entire plane as a homeomorphism of locally exponential integrable distortion.*

Above, ψ-local connectivity requires that, for each x and all $r > 0$,

- each pair of points in $B(x, r) \cap \Omega$ can be joined by an arc in $B(x, \psi^{-1}(r)) \cap \Omega$, and
- each pair of points in $\Omega \backslash B(x, r)$ can be joined by an arc in $\Omega \setminus B(x, \psi(r))$.

If we were to choose $\psi(t) = Ct$, then this would reduce to the usual linear local connectivity condition. Some ψ-local connectivity condition is also necessary in the setting of Theorem 1.2. For example, the sharp modulus of continuity estimates [**10**], [**22**] for the extension and its inverse show that Ω must be ψ-locally connected for $\psi(t) = C\exp(-t^{4/\lambda})$, but there is no hope for an exact charaterization.

In the proof of Theorem 1.2, the extension of the circle homeomorphism is obtained via results by Zakeri [**25**]. In fact, we establish in Section 5 a slightly more general result than Theorem 1.2.

This paper is organized as follows. Section 2 contains the basic definitions and Section 3 some auxiliary results. We prove Theorem 1.1 in Section 4 and Theorem 1.2 in Section 5. In the final section, Section 6, we make some concluding remarks.

Acknowledgement. We wish to thank the anonymous referee for suggestions that both improved our presentation and simplified some of our reasoning.

2. Notation and Definitions

We sometimes associate the plane $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ for convenience and denote by $\hat{\mathbb{C}}$ the extended complex plane. The closure of a set $U \subset \mathbb{R}^2$ is denoted $\overline{U}$ and the boundary ∂U. The open disk of radius $r > 0$ centered at $x \in \mathbb{R}^2$ is denoted by $B(x,r)$ and simply write $\mathbb{D}$ for the unit disk. The boundary of $B(x,r)$ will be denoted by $S(x,r)$ and the boundary of the unit disk $\mathbb{D}$ is written as $\partial\mathbb{D}$. The symbol Ω always refers to a domain, i.e. a connected and open subset of $\mathbb{R}^2$. We call a homeomorphism $f\colon \Omega \to f(\Omega) \subset \mathbb{R}^2$ a homeomorphism of finite distortion if $f \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ and

$$\|Df(x)\|^2 \leq K(x)J_f(x) \text{ a.e. in } \Omega, \tag{2.1}$$

for some measurable function $K(x) \geq 1$ that is finite almost everywhere. Recall here that $J_f \in L^1_{\mathrm{loc}}(\Omega)$ for each homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ (cf. [**4**]). In the distortion inequality (2.1), $Df(x)$ is the formal differential of f at the point x and $J_f(x) := \det Df(x)$ is the Jacobian. The norm of $Df(x)$ is defined as

$$\|Df(x)\| := \max_{e\in\partial\mathbb{D}} |Df(x)e|.$$

For a homeomorphism of finite distortion it is convenient to write K_f for the optimal distortion function. This is obtained by setting $K_f(x) = \|Df(x)\|^2/J_f(x)$ when $Df(x)$ exists and $J_f(x) > 0$, and $K_f(x) = 1$ otherwise. The distortion of f is said to be locally λ-exponentially integrable if $\exp(\lambda K_f(x)) \in L^1_{\mathrm{loc}}(\Omega)$, for some $\lambda > 0$. Note that if we assume $K_f(x)$ to be bounded, $K_f \leq K$, we recover the class of K-quasiconformal mappings, see [**4**] for the theory of quasiconformal mappings.

Recall that a domain Ω is said to be linearly locally connected (LLC) if there is a constant $C \geq 1$ so that

- (LLC-1) each pair of points in $B(x,r) \cap \Omega$ can be joined by an arc in $B(x,Cr) \cap \Omega$, and
- (LLC-2) each pair of points in $\Omega \backslash B(x,r)$ can be joined by an arc in $\Omega \setminus B(x,C^{-1}r)$.

We need a weaker version of this condition, defined as follows. We say that Ω is (φ,ψ)-locally connected ((φ,ψ)-LC) if

- (φ-LC-1) each pair of points in $B(x,r) \cap \Omega$ can be joined by an arc in $B(x,\varphi(r)) \cap \Omega$, and
- (φ-LC-2) each pair of points in $\Omega \backslash B(x,r)$ can be joined by an arc in $\Omega \setminus B(x,\psi(r))$,

where $\varphi,\psi\colon [0,\infty) \to [0,\infty)$ are smooth increasing functions such that $\varphi(0) = \psi(0) = 0$, $\varphi(r) \geq r$ and $\psi(r) \leq r$ for all $r > 0$. For technical reasons, we assume that the function $t \mapsto \frac{t}{\varphi^{-1}(t)^2}$ is decreasing and that there exist constants C_1, C_2 so that $C_1\varphi(t) \leq \varphi(2t) \leq C_2\varphi(t)$ and $C_1\psi(t) \leq \psi(2t) \leq C_2\psi(t)$ for all $t > 0$. If $\varphi^{-1} = \psi$ above, as in the introduction, Ω will simply be called ψ-LC. One could relax joinability by an arc above to joinability by a continuum, but this leads to the same concept; see [**11**, Theorem 3-17]. Notice that if Ω is simply connected and bounded, then φ-LC-1 guarantees that Ω is a Jordan domain.

Finally we define the central tool for us – the modulus of a path family. A Borel function $\rho\colon \mathbb{R}^2 \to [0,\infty]$ is said to be admissible for a path family Γ if $\int_\gamma \rho\, ds \geq 1$ for each locally rectifiable $\gamma \in \Gamma$. The modulus of the path family Γ is then

$$\operatorname{mod}(\Gamma) := \inf\left\{\int_\Omega \rho^2(x)\, dx : \rho \text{ is admissible for } \Gamma\right\}.$$

For subsets E and F of $\overline{\Omega}$ we write $\Gamma(E,F,\Omega)$ for the path family consisting of all locally rectifiable paths joining E to F in Ω and abbreviate $\operatorname{mod}((\Gamma(E,F,\Omega))$ to $\operatorname{mod}(E,F,\Omega)$. In what follows, $\gamma(x,y)$ refers to a curve or an arc from x to y.

3. Auxiliary results

We begin this section by stating the following theorem from Zakeri [**25**], which plays a crucial role in the proof of Theorem 1.2 and Theorem 6.1 below.

Theorem 3.1. *Given a sense-preserving homeomorphism* $f\colon \partial\mathbb{D} \to \partial\mathbb{D}$ *and* $0 < t < \frac{\pi}{2}$, *set*

$$\delta_f(\theta,t) = \max\left\{\frac{|f(e^{i(\theta+t)}) - f(e^{i\theta})|}{|f(e^{i\theta}) - f(e^{i(\theta-t)})|}, \frac{|f(e^{i(\theta-t)}) - f(e^{i\theta})|}{|f(e^{i\theta}) - f(e^{i(\theta+t)})|}\right\} \tag{3.1}$$

and

$$\rho_f(t) = \sup_{\theta\in[0,2\pi]} \delta_f(\theta,t).$$

If

$$\rho_f(t) = O\left(\log\frac{1}{t}\right) \quad as \quad t \to 0, \tag{3.2}$$

then f *extends to a homeomorphism* $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ *of locally exponentially integrable distortion.*

Remark 3.2. If $\delta_f(\theta,t)$ or $\rho_f(t)$ is uniformly bounded by M, then by the results of Beurling and Ahlfors [**5**], f extends to a global quasiconformal mapping.

We continue by proving the lemma that we referred to in connection with extendability of conformal weldings.

Lemma 3.3. *Suppose that* $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is a homeomorphism of locally exponentially integrable distortion. Then there is a homeomorphism* $\hat{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ *of locally exponentially integrable distortion so that the distortion is bounded outside a compact set and with* $\hat{f} = f$ *on* $\mathbb{D}$.

Proof: We choose $M \geq 1$ so large that $f(\mathbb{D}) \subset B \subset f(B(0,M))$ for some disk B. Let μ_f be the Beltrami coefficient of f and set $\mu = \chi_{B(0,2M)}\mu_f$. By [**4**, Theorem 20.4.9] there is a homeomorphism $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ of locally exponentially integrable distortion and with the Beltrami coefficient μ. Since μ vanishes outside $\overline{B}(0,2M)$, g is conformal there. On $B(0,2M)$, both f and $\hat{f}$ are solutions to the same Beltrami equation, and hence [**4**, Theorem 20.4.9] provides us with a conformal mapping $h\colon f(B(0,2M)) \to \Omega$ for some domain $\Omega \subset \mathbb{R}^2$ with $g = h \circ f$ on $B(0,2M)$.

It is easy to check that h is bi-Lipschitz on B. By [**24**], h extends to a bi-Lipschitz mapping $\hat{h}\colon \mathbb{R}^2 \to \mathbb{R}^2$. We define $\psi = \hat{h}^{-1}$. Then ψ is bi-Lipschitz. Finally, we set $\hat{f} = \psi \circ g$. Then $\hat{f}$ has all the desired properties. □

The following two modulus estimates are standard, see e.g. [**13**].

Lemma 3.4. *Let E, F be disjoint nondegenerate continua in $B(x, R)$. Then*

$$\operatorname{mod}(E, F, B(x, R)) \geq C_0 \log\left(1 + \frac{1}{t}\right), \tag{3.3}$$

where $t = \frac{\operatorname{dist}(E,F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}}$ and C_0 is an absolute constant.

Lemma 3.5. *Let $0 < r < R < \infty$. Then*

$$\operatorname{mod}(S(x, r), S(x, R), \overline{B}(x, R) \setminus B(x, r)) = \frac{2\pi}{\log \frac{R}{r}}. \tag{3.4}$$

Next, we recall the following result on the modulus of continuity of a quasiconformal mapping, whose proof can be found in [**14**]; also see [**21**].

Lemma 3.6. *Suppose $g\colon \Omega \to \mathbb{D}$ is a K-quasiconformal mapping from a simply connected domain Ω onto the unit disk. Then there exists a positive constant C, (depending on f), such that for any $\omega, \xi \in \Omega$,*

$$|g(\omega) - g(\xi)| \leq C d_I(\omega, \xi)^{\frac{1}{2K}}, \tag{3.5}$$

where $d_I(\omega, \xi)$ is defined as $\inf_{\gamma(\omega,\xi)\subset\Omega} \operatorname{diam}(\gamma(\omega, \xi))$. In particular, if Ω above is φ-LC-1, then

$$|g(\omega) - g(\xi)| \leq C\varphi(|\omega - \xi|)^{\frac{1}{2K}}. \tag{3.6}$$

Finally, we finish this section with a general upper modulus estimate for certain curve families.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^2$ be a φ-LC-1 Jordan domain. Let $\alpha \subset \partial\Omega$ be an arc and $\omega \in \partial\Omega$ be a point with $d = d(\omega, \alpha) > 0$. Moreover, let $\gamma' \subset \hat{\mathbb{C}}\setminus\Omega$ be the hyperbolic geodesic joining ω to ∞, i.e. $\gamma'(0) = \omega$ and $\gamma'(\infty) = \infty$. Then there exist a positive constant $\delta > 0$, depending only on Ω, and a positive constant C such that $d + \operatorname{diam}\alpha < \delta$ implies*

$$\operatorname{mod}(\gamma', \alpha, \mathbb{R}^2 \setminus \overline{\Omega}) \leq \frac{2\pi}{\log 2}\left(100C^2 \int_{\frac{d}{20}}^{d+\operatorname{diam}\alpha} \frac{t}{\varphi^{-1}(t)^2}\, dt + 2\right). \tag{3.7}$$

In particular, if Ω is LLC-1, then

$$\operatorname{mod}(\gamma', \alpha, \mathbb{R}^2 \setminus \overline{\Omega}) \leq \frac{2\pi}{\log 2}\left(100C^2 \log \frac{20(d + \operatorname{diam}(\alpha))}{d} + 2\right). \tag{3.8}$$

Proof: We claim first that there exist positive constants δ and C such that $d(\gamma'(t), \partial\Omega) < \delta$ implies

$$\operatorname{diam}(\gamma'([0, t])) \leq 2\varphi(Cd(\gamma'(t), \partial\Omega)) =: \Phi(d(\gamma'(t), \partial\Omega)). \tag{3.9}$$

Towards this end, let $g\colon \mathbb{R}^2 \setminus \overline{\mathbb{D}} \to \mathbb{R}^2 \setminus \overline{\Omega}$ be a conformal mapping. Then there exists a constant δ, depending on g, such that $g(B(0, 3/2) \setminus \overline{\mathbb{D}})$

contains Ω_δ, where Ω_δ consists of those points in the complement of $\overline{\Omega}$ whose distance to Ω is strictly less than δ. If $d(\gamma'(t), \partial\Omega) < \delta$, then standard harmonic measure estimates at $g^{-1}(\gamma'(t))$ provide us with a crosscut β of $\mathbb{R}^2 \setminus \overline{\Omega}$ in $g(B(0,3/2)\setminus\mathbb{D})$ that separates $\gamma'((0,t))$ from infinity in $\mathbb{R}^2 \setminus \overline{\Omega}$ so that $\gamma'(t) \in \beta$ and $\operatorname{diam}(\beta) \le Cd(\gamma'(t), \partial\Omega)$. Using the φ-LC-1 condition, on easily obtains

$$\operatorname{diam}(\gamma'([0,t])) \le Cd(\gamma'(t), \partial\Omega) + \varphi(Cd(\gamma'(t), \partial\Omega)) \le 2\varphi(Cd(\gamma'(t), \partial\Omega)),$$

as desired.

Next, notice that, by Lemma 3.5,

$$\operatorname{mod}(\gamma'([0,\infty)) \setminus B(\gamma'(0), 2d + 2\operatorname{diam}(\alpha)), \alpha, \mathbb{R}^2\setminus\overline{\Omega}) \le \frac{2\pi}{\log 2}. \tag{3.10}$$

To complete the proof of the lemma, we only need to estimate

$$\operatorname{mod}(\gamma'([0,\infty)) \cap B(\gamma'(0), 2d + 2\operatorname{diam}(\alpha)), \alpha, \mathbb{R}^2 \setminus \overline{\Omega}).$$

Observe that $d(\gamma'(t), \partial\Omega) < 2d + 2\operatorname{diam}(\alpha)$ if $\gamma'(t) \in B(\gamma'(0), 2d + 2\operatorname{diam}(\alpha))$. We thus choose the constant δ in our claim to be $\delta/2$ from the first paragraph of this proof.

Write $r = \operatorname{diam}(\alpha)$, $x = \gamma'(0)$, and set $B(0) = B(x, d/2)$. Set $B(t) = B(\gamma'(t), d(\gamma'(t), \partial\Omega)/2)$ when $\gamma'(t) \in B(x, 2d + 2r) \setminus B(x, d/2)$. By the Vitali Covering Theorem we can find a subfamily of these balls, $B_0 = B(0)$, $B_1 = \overline{B}(\gamma'(t_1)), \dots, B_k = \overline{B}(\gamma'(t_k))$ covering $\gamma'([0,\infty)) \cap B(x, 2d + 2r)$ so that $\frac{1}{5}B_i \cap \frac{1}{5}B_j = \emptyset$ for $i \ne j$. Notice that each path that joins $\gamma'([0,\infty)) \cap B(\gamma'(0), 2d + 2\operatorname{diam}(\alpha))$ to α in $\mathbb{R}^2 \setminus \overline{\Omega}$ necessarily meets both B_i and exits $\mathbb{R}^2 \setminus 2B_i$ for some $0 \le i \le k$. Hence the modulus of our path family is no more than $\frac{2\pi}{\log 2}(k+1)$. Therefore, we only need to estimate the number $k+1$ of balls from above.

To this end, let $A_j = B(\gamma'(0), 2^{-j+1}(r+d)) \setminus B(\gamma'(0), 2^{-j}(r+d))$. Notice that $B_i \cap \frac{1}{5}B_0 = \emptyset$ for $i = 1, \dots, k$; actually even $B_i \cap B_0 = \emptyset$. From $2^{-j}(r+d) \ge d/10$, we deduce $j \le \log \frac{10(r+d)}{d} := \tau$. Let k_i be the number of the balls $\{B_m\}_{m=1}^k$ that intersect A_i, $i = 1, \dots, \tau$. Then (3.9) and a packing argument show that

$$k_i \Phi^{-1}(2^{-i}(r+d))^2 \le 25 \cdot 2^{-2(i+1)}(r+d)^2.$$

It follows immediately that

$$k \le \sum_{i=1}^{\tau} k_i \le 25 \sum_{i=1}^{\tau} \frac{2^{-2(i+1)}(r+d)^2}{\Phi^{-1}(2^{-i}(r+d))^2}.$$

Since $\frac{t}{\Phi^{-1}(t)^2}$ is non-decreasing, we have

$$k \leq 25 \int_{\frac{d}{10}}^{2(r+d)} \frac{t}{\Phi^{-1}(t)^2}\, dt = 100C^2 \int_{\frac{d}{20}}^{d+r} \frac{t}{\varphi^{-1}(t)^2}\, dt. \qquad \square$$

4. Proof of Theorem 1.1

We begin by recalling a modulus of continuity estimate from [**17**].

Lemma 4.1. *Let $f\colon G \to f(G)$ be a homeomorphism of finite distortion with $K_f \in L^p_{\mathrm{loc}}(G)$ for some $1 \leq p < \infty$. Then, for each compact set $F \subset f(G)$, there is a constant C_F so that*

$$|f^{-1}(x) - f^{-1}(y)| \leq C_F \log^{-p/2}(C_F/|x-y|)$$

in F.

We will need a modulus of continuity type estimate for our quasiconformal mapping f on the boundary of the unit disk. This is obtained via rather standard path family arguments. One could obtain the estimate below from the behavior of the corresponding conformal mapping and decomposition. Because of the lack of a good reference, we give a detailed proof.

Lemma 4.2. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism such that $f(\mathbb{D}) = \Delta_s$ and suppose that the restriction of f to $\mathbb{D}$ is K-quasiconformal. Let $E'_t = \{x \in \partial\Delta_s : |x| \leq t,\, x_1 \geq 0\}$ and $E_t = f^{-1}(E'_t)$. Then for all $\varepsilon > 0$, there exists $t_0 > 0$ and a constant $C > 0$, such that*

$$\operatorname{diam} E'_t \leq C(\operatorname{diam} E_t)^{\frac{2-\varepsilon}{K}} \tag{4.1}$$

for all $0 < t < t_0$.

Proof: Let $\varepsilon > 0$. As f is an homeomorphism, we may pick $0 < t_0 < 1/e^{2\pi}$ such that $\operatorname{diam} E_{t_0} < 1$. When $0 < t < t_0$ the set $E'_{t_0} \setminus E'_t \subset \partial\Delta_s$ consists of two separate continua that we denote F'_1 and F'_2 and we write F_1 and F_2 for their preimages with respect to f, respectively. As f is a homeomorphism, F_1 and F_2 are also two separate continua and $E_{t_0} \setminus E_t = F_1 \cup F_2 \subset \partial\mathbb{D}$. Denote $\Gamma := \Gamma(F_1, F_2, \mathbb{D})$ and $\Gamma' := \Gamma(F'_1, F'_2, \Delta_s)$. From the K-quasiconformality of f it follows that

$$\operatorname{mod}(\Gamma) \leq K \operatorname{mod}(\Gamma'). \tag{4.2}$$

Choose $a \in E_t$ such that $\operatorname{dist}(a, F_1) = \operatorname{dist}(a, F_2)$ and denote $d_1 := \operatorname{dist}(a, F_1)$ and $d_2 := d_1 + \frac{1}{2}\min_{i=1,2} \operatorname{diam} F_i$. If ρ is admissible for Γ, we may assume that it is also defined on $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$ and that $\rho(x) = 0$ for all $x \in \mathbb{R}^2 \setminus \overline{\mathbb{D}}$. Since $S(a, r)$ now intersects both F_1 and F_2 for all

$d_1 < r < d_2$, the previous assumptions on ρ and Hölder's inequality imply that for each such r

$$1 \leq \left(\int_{S(a,r)} \rho \, d\sigma \right)^2 \leq \pi r \int_{S(a,r)} \rho^2 \, d\sigma, \tag{4.3}$$

where $d\sigma$ is the length element of the circle $S(a, r)$. By applying Fubini's theorem together with (4.3) we obtain

$$\int_{\mathbb{D}} \rho^2 \, dx \geq \int_{d_1}^{d_2} \int_{S(a,r)} \rho^2 \, d\sigma \, dr \geq \int_{d_1}^{d_2} \frac{1}{\pi r} \, dr = \frac{\log(d_2/d_1)}{\pi}.$$

As $d_1 \leq 2 \operatorname{diam} E_t$ and ρ was an arbitrary admissible function for Γ, we readily obtain from the previous estimate that

$$\operatorname{mod}(\Gamma) \geq \frac{\log\left(1 + \frac{\min_{i=1,2} \operatorname{diam} F_i}{4 \operatorname{diam} E_t}\right)}{\pi}. \tag{4.4}$$

To obtain an upper bound, we define $\rho_t \colon \mathbb{R}^2 \to \mathbb{R}$ by setting

$$\rho_t(x) = \begin{cases} \frac{1}{(2\pi - \arctan t_0^s)|x|} & \text{if } t/e^{2\pi} < |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

One easily observes that ρ_t is admissible for $\Gamma(F_1', F_2', \Delta_s)$ and thus

$$\begin{aligned} \operatorname{mod}(\Gamma') &\leq \int_{\mathbb{R}^2 \setminus \overline{\Omega}_s} \rho_t^2(x) \, dx \leq \int_{B(0,1) \setminus \overline{B}(0, t/e^{2\pi})} \rho_t^2(x) \, dx \\ &\leq \frac{2\pi}{(2\pi - \arctan t_0^s)^2} \int_{t/e^{2\pi}}^1 \frac{1}{r} \, dr \\ &\leq \frac{1}{2\pi - 2 \arctan t_0^s} \log\left(\frac{e^{2\pi}}{\operatorname{diam} E_t'} \right). \end{aligned} \tag{4.5}$$

Finally, by combining (4.2), (4.4) and (4.5) and taking t_0 sufficiently small in the beginning, the claim (4.1) readily follows. □

We need yet another modulus of continuity estimate. This can be proven via a modification to the proof of the preceding lemma. Recall the definition of E_t' from the previous lemma. The key is that $\operatorname{mod}(E_t', F, \mathbb{R}^2 \setminus \Delta_s) \leq C_F t^s$ for any fixed $F \subset \mathbb{R}^2 \setminus \Delta_s$ and all sufficiently small t. We omit the details.

Lemma 4.3. *Let* $g\colon \mathbb{R}^2\setminus\mathbb{D}\to\mathbb{R}^2\setminus\Delta_s$ *be conformal. Then* f *extends to a homeomorphism* $\hat{g}$ *between the closures of these domains and there is a constant* C *so that*

$$t^s \geq C\log^{-1}(C/\operatorname{diam}(\hat{g}^{-1}(E_t')))$$

for all sufficiently small $t>0$.

Theorem 1.1 is an immediate consequence of the following two results. First we give a stronger version of the non-existence part of Theorem 1.1 as Theorem 4.4 and after this the constructive proof for the existence of a suitable mapping onto the domain from Theorem 4.4 as Proposition 4.5.

Theorem 4.4. *Let* $f\colon\mathbb{R}^2\to\mathbb{R}^2$ *be a homeomorphism of finite distortion such that* $f(\mathbb{D})=\Delta_s$, $K_f\in L^p_{\mathrm{loc}}(\mathbb{R}^2)$ *for some* $1\leq p<\infty$, *and that the restriction of* f *to* $\mathbb{D}$ *is* K*-quasiconformal. Then necessarily* $p\leq K/s$.

Proof: The discussion after the Theorem 1.1 applies as well to the quasiconformal welding $G'=g^{-1}\circ f$. Hence, Lemma 4.1 guarantees that $h:=G'^{-1}$ has a modulus of continuity of the form

$$|h(x)-h(y)|\leq C_F\log^{-p/2}(C_F/|x-y|).$$

But $h=f^{-1}\circ g$ and hence the desired bound follows by combining Lemma 4.2 with Lemma 4.3. □

Proposition 4.5. *For the domain* Δ_s *and* $\lambda<2/s$, *there exists a homeomorphism* $f\colon\mathbb{R}^2\to\mathbb{R}^2$ *of finite distortion such that* $\exp(\lambda K_f)\in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ *and* $f(\mathbb{D})=\Delta_s$.

Proof: Notice that Δ_s is essentially the reflection of Ω_s with respect to a suitable circle. Thus the desired mapping can be obtained by reflection and suitable modifications to the mapping constructed in [**23**]. We leave the technical details to the interested reader. □

5. Proof of Theorem 1.2

Theorem 1.2 follows from the following more general result by choosing $\varphi=\psi^{-1}$ and $\psi(t)=Ct\log^{-s}\log\frac{1}{t}$, for $0<s<\frac{1}{4}$.

Theorem 5.1. *Let* $\Omega\subset\mathbb{R}^2$ *be a* (φ,ψ)*-locally connected Jordan domain with*

$$\lim_{r\to 0}\frac{r\cdot\varphi^{-1}\circ\psi(r)}{(\varphi^{-1}\circ\varphi^{-1}\circ\psi(r))^2\cdot\log\log\frac{1}{r}}=0, \tag{5.1}$$

where φ, ψ *satisfy the technical conditions from Section 2. Then any conformal mapping* $f\colon\mathbb{D}\to\Omega$ *can be extended to the entire plane as a homeomorphism of locally exponentially integrable distortion.*

Proof: Since Ω is a Jordan domain, f extends to a homeomorphism between $\mathbb{D}$ and $\overline{\Omega}$ and we denote also this extension by f. Let $e^{i(\theta-t)}$, $e^{i\theta}$ and $e^{i(\theta+t)}$ be three points on S. Since f is a sense-preserving homeomorphism, $f(e^{i(\theta-t)})$, $f(e^{i\theta})$ and $f(e^{i(\theta+t)})$ will be on the boundary of Ω in order. Let $g\colon \mathbb{R}^2\setminus\overline{\mathbb{D}} \to \mathbb{R}^2\setminus\overline{\Omega}$ be a conformal mapping from the Riemann Mapping Theorem. Then g extends to a homemorphism between $\mathbb{R}^2\setminus\mathbb{D}$ and $\mathbb{R}^2\setminus\Omega$. As before, we still denote this extension by g.

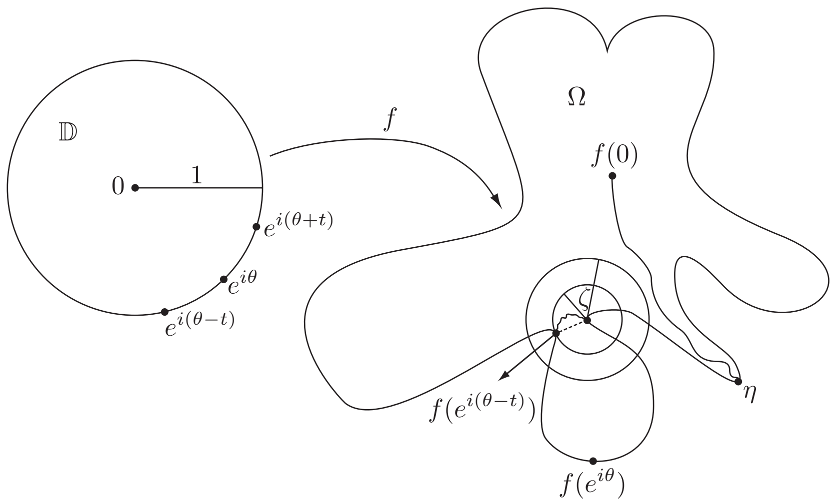

Figure 1. The first part of the proof.

For simplicity we denote by $\gamma_f(\theta-t,\theta)$ the arc from $f(e^{i\theta})$ to $f(e^{i(\theta-t)})$ on $\partial\Omega$ and analogously for the notation $\gamma_f(\theta,\theta+t)$. In view of Theorem 3.1, one aims to estimate $\delta_{g^{-1}\circ f}(\theta,t)$ in terms of t and θ. Theorem 3.1 guarantees that we may assume that $\mathrm{diam}(\gamma_f(\theta,\theta+t)) + \mathrm{diam}(\gamma_f(\theta-t,\theta)) \ll \mathrm{dist}(f(0),\partial\Omega)$.

We first show that there exists a constant $C_1>0$ such that

$$\varphi(\mathrm{dist}(f(e^{i(\theta-t)}),\gamma_f(\theta,\theta+t))) \geq C_1\psi(\mathrm{diam}(\gamma_f(\theta,\theta+t))/2). \tag{5.2}$$

Suppose that equation (5.2) fails to hold for $C_1=1/2$. Let $\zeta\in\gamma_f(\theta,\theta+t)$ be a point such that

$$|f(e^{i(\theta-t)})-\zeta| = \mathrm{dist}(f(e^{i(\theta-t)}),\gamma_f(\theta,\theta+t)) := d$$

and $\eta\in\gamma_f(\theta,\theta+t)$ such that $|\zeta-\eta|\geq\frac{\mathrm{diam}(\gamma_f(\theta,\theta+t))}{2}$. We consider the disk $B* = B(\zeta,\frac{\mathrm{diam}(\gamma_f(\theta,\theta+t))}{2})$. Clearly η, $f(0)\notin B*$, and thus the

ψ-LC-2 condition implies that we may find a curve $\gamma(f(0),\eta)$ joining $f(0)$ and η in Ω, with the property $\gamma(f(0),\eta)\cap B(\zeta,\psi(\frac{\operatorname{diam}(\gamma_f(\theta,\theta+t))}{2}))=\emptyset$. For simplicity, we denote by $B_\psi*$ the disk $B(\zeta,\psi(\frac{\operatorname{diam}(\gamma_f(\theta,\theta+t))}{2}))$. Since Ω is φ-LC-1, there is a curve $\gamma(f(e^{i(\theta-t)}),\zeta)\subset\Omega$ joining $f(e^{i(\theta-t)})$ and ζ, with diameter less than or equal to $\varphi(|f(e^{i(\theta-t)})-\zeta|)$.

Now consider the modulus $\operatorname{mod}(\gamma(f(e^{i(\theta-t)}),\zeta),\gamma(f(0),\eta),\Omega)$. Every curve joining $\gamma(f(e^{i(\theta-t)}),\zeta)$ and $\gamma(f(0),\eta),\Omega)$ has a subcurve joining $S(\zeta,\varphi(d))$ and $S_\psi*$, and hence Lemma 3.5 gives us the upper bound

$$\operatorname{mod}(\gamma(f(e^{i(\theta-t)}),\zeta),\gamma(f(0),\eta),\Omega)\le\frac{2\pi}{\log\frac{\psi(\operatorname{diam}(\gamma_f(\theta,\theta+t))/2)}{\varphi(d)}}. \tag{5.3}$$

On the other hand, let

$$\gamma(1,2)=f^{-1}(\gamma(f(e^{i(\theta-t)}),\zeta)),\quad \gamma(0,4)=f^{-1}(\gamma(f(0),\eta)).$$

Then $\gamma(1,2)$ is an arc from $e^{i(\theta-t)}$ to $f^{-1}(\zeta)$ and $\gamma(0,4)$ an arc from 0 to $f^{-1}(\eta)$. Invoking Lemma 3.4, one concludes that

$$\operatorname{mod}(\gamma(1,2),\gamma(0,4),\mathbb{D})\ge C_2, \tag{5.4}$$

for some constant $C_2>0$.

By conformal invariance, estimate (5.3) together with (5.4) implies that

$$\psi\left(\frac{\operatorname{diam}(\gamma_f(\theta,\theta+t))}{2}\right)\le C_3\varphi(d), \tag{5.5}$$

as desired.

Analogously, one concludes that also

$$\varphi(\operatorname{dist}(f(e^{i(\theta+t)}),\gamma_f(\theta-t,\theta)))\ge C_1\psi(\operatorname{diam}(\gamma_f(\theta-t,\theta))/2).$$

We now estimate $\delta_{g^{-1}\circ f}(\theta,t))$, applying (5.2) if $\operatorname{diam}(g^{-1}(\gamma_f(\theta,\theta+t)))\ge\operatorname{diam}(g^{-1}(\gamma_f(\theta-t,\theta)))$ and its analogue otherwise. Let us assume that we are in the former case.

Let $\gamma'\colon[0,\infty)\to\mathbb{R}^2\backslash\Omega$ be the hyperbolic geodesic joining $f(e^{i(\theta-t)})$ to ∞, i.e. $\gamma'(0)=f(e^{i(\theta-t)})$ and $\gamma'(\infty)=\infty$. Lemma 3.7 gives the modulus bound

$$\operatorname{mod}(\gamma',\gamma_f(\theta,\theta+t),\mathbb{R}^2\setminus\overline{\Omega})\le C_1\int_{\frac{d}{20}}^{d+\operatorname{diam}(\gamma_f(\theta,\theta+t))}\frac{t}{\varphi^{-1}(t)^2}\,dt. \tag{5.6}$$

Set $r=\operatorname{diam}(\gamma_f(\theta,\theta+t))$. Then it follows from (5.5) that

$$\varphi^{-1}(C_3^{-1}\psi(r/2))\le d\le r.$$

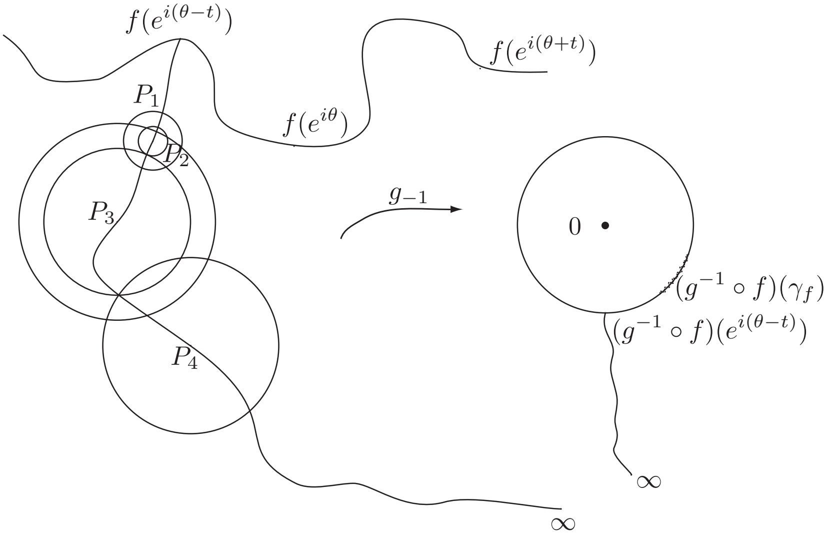

FIGURE 2. The second part of the proof.

Thus, (5.6) yields that

$$\operatorname{mod}(\gamma', \gamma_f(\theta, \theta+t), \mathbb{R}^2 \setminus \overline{\Omega}) \leq C_1 \int_{\frac{1}{20}\varphi^{-1}(C_3^{-1}\psi(r/2))}^{2r} \frac{t}{\varphi^{-1}(t)^2}\, dt.$$

Monotonicity of $\frac{t}{\varphi^{-1}(t)^2}$ implies the further upper bound

$$40 \cdot \frac{r \cdot \varphi^{-1}(C_3^{-1}\psi(r/2))}{\varphi^{-1}(\varphi^{-1}(C_3^{-1}\psi(r/2)))^2} := \Pi(r).$$

Our assumption (5.1) together with the doubling and inverse doubling conditions for φ and ψ show that for all sufficiently small r

$$\Pi(r) \leq \frac{C_0}{2} \log\log\frac{1}{r}, \tag{5.7}$$

where C_0 is the constant from Lemma 3.4.

Recalling that $\operatorname{diam}(g^{-1}(\gamma_f(\theta, \theta+t))) \geq \operatorname{diam}(g^{-1}(\gamma_f(\theta-t, \theta)))$, Lemma 3.4 gives

$$\log \delta_{g^{-1}\circ f}(\theta, t) \leq C_0^{-1} \operatorname{mod}(g^{-1}(\gamma'), g^{-1}(\gamma_f(\theta, \theta+t)), \mathbb{R}^2 \setminus \overline{\mathbb{D}}). \tag{5.8}$$

Since modulus is conformally invariant, combining (5.6), (5.7) and (5.8), we arrive at

$$\delta_{g^{-1}\circ f}(\theta,t) \leq \exp(C_0^{-1}\Pi(r)) \leq \exp\left(\frac{1}{2}\log\log(1/r)\right). \tag{5.9}$$

On the other hand, by applying Lemma 3.6 and noticing that our technical assumptions on φ implies that $\varphi^{-1}(t) \geq Ct^\alpha$ for some $\alpha > 0$, we obtain that

$$r \geq C\varphi^{-1}(t^2) \geq Ct^{2\alpha}.$$

In conclusion, for sufficiently small t,

$$\rho_{g^{-1}\circ f}(t) = \sup_{\theta\in[0,2\pi]} \delta_{g^{-1}\circ f}(\theta,t) = O\left(\log\frac{1}{t}\right). \tag{5.10}$$

Therefore, Theorem 3.1 allows us to complete the proof. □

6. Concluding remarks

Let us begin by pointing out that our extension results also hold for quasiconformal mappings in the sense that any quasiconformal mapping $f\colon \mathbb{D} \to \Omega$ extends to the entire plane as a homeomorphism of locally exponentially integrable distortion in the setting of Theorem 5.1 and consequently also in the setting of Theorem 1.2. This is easily seen by analyzing the arguments that we have used in the proof of Theorem 5.1: the only essential change is that the constant 2π in (5.3) gets changed to $2K\pi$.

Secondly, as mentioned in the introduction, polynomial interior cusps rule out the possibility of a locally exponentially integrable distortion extension, but polynomial exterior cusps do not. Thus one expects for a similar phenomenon in our general extension result. The following result shows that this is indeed the case, but one needs to examine the proof of Theorem 5.1.

Theorem 6.1. *There exists $\delta > 0$ so that the following holds. Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain that satisfies one of the following conditions:*

(a) *Ω is LLC-1 and ψ-LC-2 with $\psi(r) = \frac{Cr}{\log^\alpha \frac{1}{r}}$ for some positive constant C and some $0 < \alpha < \delta$.*

(b) *Ω is LLC-2 and φ-LC-1 with $\varphi^{-1}(r) = \frac{Cr}{\log^s \log \frac{1}{r}}$ for some positive constant C and some $0 < s < \frac{1}{3}$.*

Then any conformal mapping $f\colon \mathbb{D} \to \Omega$ can be extended to the entire plane as a homeomorphism of locally exponentially integrable distortion.

Proof of Theorem 6.1: First of all, the claim under assumption (b) directly follows from Theorem 5.1. For the proof of our claim under assumption (a), we need to examine the proof of Theorem 5.1. We only point out the crucial differences in what follows.

In this case, the estimate (5.5) becomes

$$\psi\left(\frac{\operatorname{diam}(\gamma_f(\theta,\theta+t))}{2}\right)\leq Cd, \tag{6.1}$$

and the modulus estimate (5.6) becomes

$$\operatorname{mod}(\gamma',\gamma_f(\theta,\theta+t),\mathbb{R}^2\setminus\overline{\Omega})\leq C_1'\log\frac{\operatorname{diam}(\gamma_f(\theta,\theta+t))}{d}. \tag{6.2}$$

Using conformal invariance as in the proof of Theorem 5.1 and the above estimate, we arrive at

$$\delta_{g^{-1}\circ f}(\theta,t)\leq C_1'\left(\frac{r}{d}\right)^{C_2'}\leq C_1'\left(\frac{r}{\psi(r)}\right)^{C_2'}\left(\frac{\psi(r)}{d}\right)^{C_2'}, \tag{6.3}$$

where $r=\operatorname{diam}(\gamma_f(\theta,\theta+t))$, Estimate (6.1) together with the fact that $\frac{r}{\psi(r)}$ is non-increasing further implies that

$$\delta_{g^{-1}\circ f}(\theta,t)\leq C_1'\left(\frac{|f(e^{i(\theta+t)})-f(e^{i\theta})|}{\psi(|f(e^{i(\theta+t)})-f(e^{i\theta})|)}\right)^{C_2'}. \tag{6.4}$$

Using our assumption that $\frac{r}{\psi(r)}$ is non-increasing one more time together with Lemma 3.6, we obtain

$$\delta_{g^{-1}\circ f}(\theta,t)\leq C_1'\left(\frac{|f(e^{i(\theta+t)})-f(e^{i\theta})|}{\psi(|f(e^{i(\theta+t)})-f(e^{i\theta})|)}\right)^{C_2'}\leq C_1'\left(\frac{t^2}{\psi(t^2)}\right)^{C_2'}. \tag{6.5}$$

It follows easily from our assumption on ψ that

$$\rho_{g^{-1}\circ f}(t)=\sup_{\theta\in[0,2\pi]}\delta_{g^{-1}\circ f}(\theta,t)=O\left(\log\frac{1}{t}\right) \tag{6.6}$$

as $t\to 0$. Once again, Theorem 3.1 allows us to complete the proof. □

Thirdly, recall from Section 4 that extendability is not possible for the model domain that contains a single inward polynomial cusp. The following observation shows that the critical case here is logarithmic. We consider the domain

$$\Delta=B((-1,0),\sqrt{5})\setminus\{(x_1,x_2)\in\mathbb{R}^2:x_1>0,\ |x_2|<x_1\log^{-1}e/x_1\}. \tag{6.7}$$

This domain is (φ, ψ)-locally connected with $\varphi(t) = Ct \log \frac{3}{t}$ and $\psi(t) = Ct$, and hence not covered by Theorem 5.1 that allows for possibly infinitely many cusps, but of lower order. Therefore we construct our mapping by hand.

Theorem 6.2. *There is a homeomorphism $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ of locally exponentially integrable distortion such that $f \colon \mathbb{D} \to \Delta$ is quasiconformal. Conversely, the degree of the local exponential integrability of such an extension cannot exceed a bound that depends on the quasiconformality constant.*

Proof: The construction of the desired mapping f is quite similar to that used in [**23**], but we need to ensure that f is quasiconformal in $\mathbb{D}$. For this we rely on ideas from [**15, 23**] and only point out the difference with construction from the proof of Proposition 4.5.

Differently from [**23**], we set

$$g(r) = r, \quad H(r) = \log^{-1}(e/r)$$

and define linear functions $L_r^i \colon \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}$ and $L_r^o \colon [\frac{\pi}{2}, \frac{3\pi}{2}] \to \mathbb{R}$ by setting

$$L_r^i(\theta) = \frac{2\theta}{\pi} \arctan H(r) \qquad \text{and}$$

$$L_r^o(\theta) = 2\theta - \pi + \left(2 - \frac{2\theta}{\pi}\right) \arctan H(r).$$

Continuing all the constructions as in [**23**], one concludes that

$$K_{f_2}(x) \leq C \log(e/|x|) + E(x),$$

for all $x \in B \cap H_R$, where $E \colon \mathbb{R}^2 \to \mathbb{R}$ is some bounded function and that $K_{f_2}(x) \leq C$ for $x \in B \backslash H_R$. The first claim then follows.

The second claim is obtained via a modification to the proof of Theorem 1.2. We only indicate the necessary changes. First of all, the upper bound for the analog of Lemma 4.3 is now $C \log^{-2}(\frac{3}{t})$. Secondly, instead of relying on Lemma 4.1, one uses the estimate the modulus of continuity from [**10**]. Finally, we combine these estimates with Lemma 4.2. □

References

[1] L. V. Ahlfors, Quasiconformal reflections, *Acta Math.* **109** (1963), 291–301. `DOI: 10.1007/BF02391816`.

[2] L. V. Ahlfors, *"Lectures on quasiconformal mappings"*, Second edition, With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard, University Lecture Series **38**, American Mathematical Society, Providence, RI, 2006.

[3] K. Astala, J. T. Gill, S. Rohde, and E. Saksman, Optimal regularity for planar mappings of finite distortion, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27(1)** (2010), 1–19. DOI: 10.1016/j.anihpc.2009.01.012.

[4] K. Astala, T. Iwaniec, and G. Martin, *"Elliptic partial differential equations and quasiconformal mappings in the plane"*, Princeton Mathematical Series **48**, Princeton University Press, Princeton, NJ, 2009.

[5] A. Beurling and L. Ahlfors, The boundary correspondence under quasiconformal mappings, *Acta Math.* **96** (1956), 125–142. DOI: 10.1007/BF02392360.

[6] G. David, Solutions de l'équation de Beltrami avec $\|\mu\|_\infty = 1$, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **13(1)** (1988), 25–70. DOI: 10.5186/aasfm.1988.1303.

[7] F. W. Gehring, *"Characteristic properties of quasidisks"*, Séminaire de Mathématiques Supérieures **84**, Presses de l'Université de Montréal, Montreal, Que., 1982.

[8] F. W. Gehring and K. Hag, Reflections on reflections in quasidisks, in: *"Papers on analysis"*, Rep. Univ. Jyväskylä Dep. Math. Stat. **83**, Univ. Jyväskylä, Jyväskylä, 2001, pp. 81–90.

[9] S. Hencl and P. Koskela, Regularity of the inverse of a planar Sobolev homeomorphism, *Arch. Ration. Mech. Anal.* **180(1)** (2006), 75–95. DOI: 10.1007/s00205-005-0394-1.

[10] D. A. Herron and P. Koskela, Mappings of finite distortion: gauge dimension of generalized quasicircles, *Illinois J. Math.* **47(4)** (2003), 1243–1259.

[11] J. G. Hocking and G. S. Young, *"Topology"*, Second edition, Dover Publications, Inc., New York, 1988.

[12] T. Iwaniec, P. Koskela, and J. Onninen, Mappings of finite distortion: compactness, *Ann. Acad. Sci. Fenn. Math.* **27(2)** (2002), 391–417.

[13] P. Koskela, *"Lectures on quasiconformal and quasisymmetric mappings"*, Jyväskylä Lecures in Mathematics **1**, University of Jyväskylä, Jyväskylä, 2010.

[14] P. KOSKELA, J. ONNINEN, AND J. T. TYSON, Quasihyperbolic boundary conditions and capacity: Hölder continuity of quasiconformal mappings, *Comment. Math. Helv.* **76(3)** (2001), 416–435. DOI: 10.1007/PL00013214.

[15] P. KOSKELA AND J. TAKKINEN, Mappings of finite distortion: formation of cusps. *Publ. Mat.* **51(1)** (2007), 223–242. DOI: 10.5565/PUBLMAT_51107_10.

[16] P. KOSKELA AND J. TAKKINEN, A note to "Mappings of finite distortion: formation of cusps II" [MR2354095], *Conform. Geom. Dyn.* **14** (2010), 184–189. DOI: 10.1090/S1088-4173-2010-00211-0.

[17] P. KOSKELA AND J. TAKKINEN, Mappings of finite distortion: formation of cusps. III, *Acta Math. Sin.* **26(5)** (2010), 817–824. DOI: 10.1007/s10114-010-7565-1.

[18] P. KOSKELA, A. ZAPADINSKAYA, AND T. ZÜRCHER, Mappings of finite distortion: generalized Hausdorff dimension distortion, *J. Geom. Anal.* **20(3)** (2010), 690–704. DOI: 10.1007/s12220-010-9121-8.

[19] R. KÜHNAU, Möglichst konforme Spiegelung an einer Jordankurve, *Jahresber. Deutsch. Math.-Verein.* **90(2)** (1988), 90–109.

[20] O. LEHTO AND K. I. VIRTANEN, *"Quasiconformal mappings in the plane"*, Second edition, Translated from the German by K. W. Lucas, Die Grundlehren der mathematischen Wissenschaften **126**, Springer-Verlag, New York-Heidelberg, 1973.

[21] R. NÄKKI AND B. PALKA, Lipschitz conditions and quasiconformal mappings, *Indiana Univ. Math. J.* **31(3)** (1982), 377–401. DOI: 10.1512/iumj.1982.31.31033.

[22] J. ONNINEN AND X. ZHONG, A note on mappings of finite distortion: the sharp modulus of continuity, *Michigan Math. J.* **53(2)** (2005), 329–335. DOI: 10.1307/mmj/1123090772.

[23] J. TAKKINEN, Mappings of finite distortion: formation of cusps. II, *Conform. Geom. Dyn.* **11** (2007), 207–218. DOI: 10.1090/S1088-4173-07-00170-1.

[24] P. TUKIA, The planar Schönflies theorem for Lipschitz maps, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **5(1)** (1980), 49–72.

[25] S. ZAKERI, On boundary homeomorphisms of trans-quasiconformal maps of the disk, *Ann. Acad. Sci. Fenn. Math.* **33(1)** (2008), 241–260.

Chang-Yu Guo and Pekka Koskela:
Department of Mathematics and Statistics
University of Jyväskylä
P.O. Box 35
FI-40014 University of Jyväskylä
Finland
E-mail address: changyu.c.guo@jyu.fi
E-mail address: pkoskela@maths.jyu.fi

Juhani Takkinen:
Mäntymäentie 5
41310 Leppävesi
Finland
E-mail address: juhani.takkinen@kolumbus.fi

Primera versió rebuda el 29 d'octubre de 2012,
darrera versió rebuda el 8 de març de 2013.