

WEIGHTED ESTIMATES FOR DYADIC PARAPRODUCTS AND t -HAAR MULTIPLIERS WITH COMPLEXITY (m, n)

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Abstract: We extend the definitions of dyadic paraproduct and t -Haar multipliers to dyadic operators that depend on the complexity (m, n) , for m and n natural numbers. We use the ideas developed by Nazarov and Volberg to prove that the weighted $L^2(w)$ -norm of a paraproduct with complexity (m, n) , associated to a function $b \in BMO^d$, depends linearly on the A_2^d -characteristic of the weight w , linearly on the BMO^d -norm of b , and polynomially on the complexity. This argument provides a new proof of the linear bound for the dyadic paraproduct due to Beznosova. We also prove that the L^2 -norm of a t -Haar multiplier for any $t \in \mathbb{R}$ and weight w is a multiple of the square root of the C_{2t}^d -characteristic of w times the square root of the A_2^d -characteristic of w^{2t} , and is polynomial in the complexity.

2010 Mathematics Subject Classification: Primary: 42C99; Secondary: 47B38.

Key words: Operator-weighted inequalities, dyadic paraproduct, A_p -weights, Haar multipliers.

1. Introduction

In the past decade, many mathematicians have devoted their attention to finding out how the norm of an operator T on a weighted space $L^p(w)$ depends on the so called A_p -characteristic of the weight w . More precisely, is there some optimal growth function $\varphi: [0, \infty) \rightarrow \mathbb{R}$ such that for all functions $f \in L^p(w)$,

$$\|Tf\|_{L^p(w)} \leq C_{p,T}\varphi([w]_{A_p})\|f\|_{L^p(w)},$$

where $C_{p,T} > 0$ is a suitable constant?

The first result of this type was due to Buckley [Bu] in 1993; he showed that $\varphi(t) = t^{1/(p-1)}$ for the Hardy-Littlewood maximal function. Starting in 2000, one at a time, some dyadic model operators and some important singular integral operators (Beurling, Hilbert and Riesz transforms) were shown to obey a linear bound with respect to

the A_2 -characteristic of w in $L^2(w)$, meaning that for $p = 2$, the function $\varphi(t) = t$ is the optimal one, see [Wi1], [Wi2], [HTV], [PV], [Pt2], [Pt3], [Be2]. These linear estimates in $L^2(w)$ imply $L^p(w)$ -bounds for $1 < p < \infty$, by the sharp extrapolation theorem of Dragičević, Grafakos, Pereyra, and Petermichl, [DGPP]. All these papers used the Bellman function technique, see [Vo] for more insights and references.

The linear bound for H , the Hilbert transform, is based on a representation of H as an average of Haar shift operators of complexity $(0, 1)$, see [Pt1]. Haar shift operators with complexity (m, n) were introduced in [LPR]. Hytönen obtained a representation valid for *any* Calderón-Zygmund operator as an average of Haar shift operators of *arbitrary complexity*, paraproducts and their adjoints, and used this representation to prove the A_2 -conjecture, see [Hy]. Thus, he showed that for *all* Calderón-Zygmund operators T in \mathbb{R}^N , and all weights $w \in A_p$, there is a constant $C_{p,N,T} > 0$ such that,

$$\|Tf\|_{L^p(w)} \leq C_{p,N,T} [w]_{A_p}^{\max\{1, 1/p-1\}} \|f\|_{L^p(w)}.$$

See [La2] for a survey of the A_2 -conjecture including a rather complete history of most results that appeared up to November 2010, and that contributed to the final resolution of this mathematical puzzle. A crucial part of the proof was to obtain bounds for Haar shifts operators that depended linearly on the A_2 -characteristic and at most polynomially on the complexity (m, n) . In 2011, Nazarov and Volberg [NV1] provided a beautiful new proof that still uses Bellman functions, although minimally, and that can be transferred to geometric doubling metric spaces [NV2], [NRV]. Treil [Tr], independently Hytönen et al. [HLM+] obtained linear dependence on the complexity. Similar Bellman function techniques have been used to prove the Bump Conjecture in L^2 , see [NRTV].

It seems natural to study other dyadic operators with complexity (m, n) , and examine if we can recover the same dependence on the A_2 -characteristic that we have for the original operator (the one with complexity $(0, 0)$) times a factor that depends at most polynomially on the complexity of these operators. We will do this analysis for the dyadic paraproduct and for the t -Haar multipliers.

For $b \in BMO^d$, a function of dyadic bounded mean oscillation, $m, n \in \mathbb{N}$, the *dyadic paraproduct of complexity (m, n)* is defined by,

$$\pi_b^{m,n} f(x) = \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_n(L) \\ J \in \mathcal{D}_m(L)}} c_{I,J}^L m_I f \langle b, h_I \rangle h_J(x),$$

where $|c_{I,J}^L| \leq \sqrt{|I||J|}/|L|$, and $m_I f$ is the average of f on the interval I . Here \mathcal{D} denotes the dyadic intervals, $|I|$ the length of interval I , $\mathcal{D}_m(L)$ denotes the dyadic subintervals of L of length $2^{-m}|L|$, h_I are the Haar functions, and $\langle f, g \rangle$ denotes the L^2 -inner product on \mathbb{R} .

We prove that the dyadic paraproduct of complexity (m, n) obeys the same *linear bound* as obtained by Beznosova [Be2] for the dyadic paraproduct of complexity $(0, 0)$ (see [Ch] for the result in \mathbb{R}^N , $N > 1$), multiplied by a factor that depends polynomially on the complexity.

Theorem 1.1. *If $w \in A_2^d$, $b \in BMO^d$, then*

$$\|\pi_b^{m,n} f\|_{L^2(w)} \leq C(m + n + 2)^5 [w]_{A_2^d} \|b\|_{BMO^d} \|f\|_{L^2(w)}.$$

Our proof of Theorem 1.1 shows how to use the ideas in [NV1] for this setting, explicitly displaying the dependence on $\|b\|_{BMO^d}$ and bypassing the more complicated Sawyer two-weight testing conditions present in other arguments [HPTV], [La2], [HLM+]. From our point view, this makes the proof more transparent.

For $t \in \mathbb{R}$, $m, n \in \mathbb{N}$, and weight w , the t -Haar multiplier of complexity (m, n) is defined by

$$T_{t,w}^{m,n} f(x) = \sum_{L \in \mathcal{D}} \sum_{\substack{I \in \mathcal{D}_n(L) \\ J \in \mathcal{D}_m(L)}} c_{I,J}^L \frac{w^t(x)}{(m_L w)^t} \langle f, h_I \rangle h_J(x),$$

where $|c_{I,J}^L| \leq \sqrt{|I||J|}/|L|$. When $(m, n) = (0, 0)$ and $c_{I,J}^L = c^L = 1$ for all $L \in \mathcal{D}$, we denote the corresponding Haar multiplier by T_w^t . In addition, if $t = 1$, we denote the multiplier simply by T_w . A necessary condition for the boundedness of T_w^t on $L^2(\mathbb{R})$ is that $w \in C_{2t}^d$, that is,

$$[w]_{C_{2t}^d} := \sup_{I \in \mathcal{D}} \left(\frac{1}{|I|} \int_I w^{2t}(x) dx \right) \left(\frac{1}{|I|} \int_I w(x) dx \right)^{-2t} < \infty.$$

This condition is also sufficient for $t < 0$ and $t > 1/2$. For $0 \leq t \leq 1/2$ the condition C_{2t}^d is always fulfilled; in this case, boundedness of T_w^t is known when $w \in A_\infty^d$, see [KP]. The Haar multipliers T_w are closely related to the resolvent of the dyadic paraproduct [Pe1], and appeared in the study of Sobolev spaces on Lipschitz curves [Pe3]. It was proved in [Pe2] that the L^2 -norm for the Haar multiplier T_w depends linearly on the C_2^d -characteristic of the weight w . We show the following theorem that generalizes a result of Beznosova for T_w^t [Be1, Chapter 5].

Theorem 1.2. *If $w \in C_{2t}^d$ and $w^{2t} \in A_2^d$, then*

$$\|T_{t,w}^{m,n} f\|_2 \leq C(m+n+2)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_2^d}^{\frac{1}{2}} \|f\|_2.$$

The condition $w \in C_{2t}^d$ is necessary for the boundedness of $T_{t,w}^{m,n}$ when $c_{I,J}^L = \sqrt{|I||J|}/|L|$.

The result is optimal for $T_w^{\pm 1/2}$, see [Be1], [Pe2] and [BMP]. We expect that, for both the paraproducts and t -Haar multipliers with complexity (m, n) , the dependence on the complexity can be strengthened to be linear, in line with the best results for the Haar shift operators. However our methods yield polynomials of degree 5 and 3 respectively.

To simplify notation, and to shorten the exposition we analyze the one-dimensional case. Some of the building blocks in our arguments can be found in the literature in the case of \mathbb{R}^N , or even in the geometric doubling metric space case. As we go along we will note where such results can be found. For a complete presentation of these results in the geometric doubling metric spaces (in particular in \mathbb{R}^N) see [Mo2].

The paper is organized as follows. In Section 2 we provide the basic definitions and results that are used throughout this paper. In Section 3 we prove the lemmas that are essential for the main results. In Section 4 we prove the main estimate for the dyadic paraproduct with complexity (m, n) and present a new proof of the linear bound for the dyadic paraproduct. In Section 5 we prove the main estimate for the t -Haar multipliers with complexity (m, n) , also discussing necessary conditions for these operators to be bounded in $L^p(\mathbb{R})$, for $1 < p < \infty$.

Acknowledgements. The authors would like to thank Carlos Pérez, Rafael Espinola and Carmen Ortiz-Caraballo for organizing the Doc-course: *Harmonic analysis, metric spaces and applications to PDE*, held in Seville, at the Instituto de Matemáticas de la Universidad de Sevilla (IMUS) during the Summer of 2011. We are grateful to our thoughtful referees who pointed out multiple ways for improving this paper.

2. Preliminaries

2.1. Weights, maximal function and dyadic intervals. A weight w is a locally integrable function in \mathbb{R}^N taking values in $(0, \infty)$ almost everywhere. The w -measure of a measurable set E , denoted by $w(E)$, is $w(E) = \int_E w(x) dx$. For a measure σ , $\sigma(E) = \int_E d\sigma$, and $|E|$ stands for the Lebesgue measure of E . We define $m_E^\sigma f$ to be the integral average

of f on E , with respect to σ ,

$$m_E^\sigma f := \frac{1}{\sigma(E)} \int_E f(x) d\sigma.$$

When $d\sigma = dx$ we simply write $m_E f$; when $d\sigma = v dx$ we write $m_E^v f$.

Given a weight w , a measurable function $f: \mathbb{R}^N \rightarrow \mathbb{C}$ is in $L^p(w)$ if and only if $\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^N} |f(x)|^p w(x) dx\right)^{1/p} < \infty$.

For a weight v we define the *weighted maximal function of f* by

$$(M_v f)(x) := \sup_{Q \ni x} m_Q^v |f|,$$

where Q is a cube in \mathbb{R}^N with sides parallel to the axes. The operator M_v is bounded in $L^q(v)$ for all $q > 1$. Furthermore,

$$(2.1) \quad \|M_v f\|_{L^q(v)} \leq C_N q' \|f\|_{L^q(v)},$$

where q' is the dual exponent of q , that is $1/q + 1/q' = 1$. A proof of this fact can be found in [CMP2]. When $v = 1$, M_v is the usual Hardy-Littlewood maximal function, which we will denote by M . It is well-known that M is bounded on $L^p(w)$ if and only if $w \in A_p$ [Mu].

We work with the collection of all *dyadic intervals*, \mathcal{D} , given by: $\mathcal{D} = \cup_{n \in \mathbb{Z}} \mathcal{D}_n$, where $\mathcal{D}_n := \{I \subset \mathbb{R} : I = [k2^{-n}, (k+1)2^{-n}), k \in \mathbb{Z}\}$. For a dyadic interval L , let $\mathcal{D}(L)$ be the collection of its dyadic subintervals, $\mathcal{D}(L) := \{I \subset L : I \in \mathcal{D}\}$, and let $\mathcal{D}_n(L)$ be the n^{th} -generation of dyadic subintervals of L , $\mathcal{D}_n(L) := \{I \in \mathcal{D}(L) : |I| = 2^{-n}|L|\}$. Any two dyadic intervals $I, J \in \mathcal{D}$ are either disjoint or one is contained in the other. Any two distinct dyadic intervals $I, J \in \mathcal{D}_n$ are disjoint, furthermore \mathcal{D}_n is a partition of \mathbb{R} , and $\mathcal{D}_n(L)$ is a partition of L . For every dyadic interval $I \in \mathcal{D}_n$ there is exactly one $\hat{I} \in \mathcal{D}_{n-1}$, such that $I \subset \hat{I}$; \hat{I} is called the *parent of I* . Each dyadic interval I in \mathcal{D}_n is the union of two disjoint intervals in \mathcal{D}_{n+1} , the right and left halves, denoted I_+ and I_- respectively, and called the *children of I* .

A weight w is *dyadic doubling* if $w(\hat{I})/w(I) \leq C$ for all $I \in \mathcal{D}$. The smallest constant C is called the doubling constant of w and is denoted by $D(w)$. Note that $D(w) \geq 2$, and that in fact the ratio between the length of a child and the length of its parent is comparable to one; more precisely, $D(w)^{-1} \leq w(I)/w(\hat{I}) \leq 1 - D(w)^{-1}$.

2.2. Dyadic A_p^d , reverse Hölder RH_p^d and C_s^d classes. A weight w is said to belong to the *dyadic Muckenhoupt A_p^d -class* if and only if

$$[w]_{A_p^d} := \sup_{I \in \mathcal{D}} (m_I w^{\frac{-1}{p-1}})^{p-1} < \infty, \quad \text{for } 1 < p < \infty,$$

where $[w]_{A_p^d}$ is called the A_p^d -characteristic of the weight. If a weight is in A_p^d then it is dyadic doubling. These classes are nested: $A_p^d \subset A_q^d$ for all $p \leq q$. The class A_∞^d is defined by $A_\infty^d := \bigcup_{p>1} A_p^d$.

A weight w is said to belong to the *dyadic reverse Hölder RH_p^d -class* if and only if

$$[w]_{RH_p^d} := \sup_{I \in \mathcal{D}} (m_I w^p)^{\frac{1}{p}} (m_I w)^{-1} < \infty, \quad \text{for } 1 < p < \infty,$$

where $[w]_{RH_p^d}$ is called the RH_p^d -characteristic of the weight. If a weight is in RH_p^d then it is not necessarily dyadic doubling (in the non-dyadic setting reverse Hölder weights are always doubling). Also these classes are nested, $RH_p^d \subset RH_q^d$ for all $p \geq q$. The class RH_1^d is defined by $RH_1^d := \bigcup_{p>1} RH_p^d$. In the non-dyadic setting $A_\infty = RH_1$. In the dyadic setting the collection of dyadic doubling weights in RH_1^d is A_∞^d , hence A_∞^d is a proper subset of RH_1^d . See [BR] for some recent and very interesting results relating these classes.

Given $s \in \mathbb{R}$, a weight w is said to satisfy the C_s^d -condition if

$$[w]_{C_s^d} := \sup_{I \in \mathcal{D}} (m_I w^s) (m_I w)^{-s} < \infty.$$

The quantity defined above is called the C_s^d -characteristic of w . The class C_s^d was defined in [KP]. Let us analyze this definition. For $0 \leq s \leq 1$, we have that any weight satisfies the condition with C_s^d -characteristic 1, being just a consequence of Hölder's inequality (cases $s = 0, 1$ are trivial). When $s > 1$, the condition is equivalent to the dyadic reverse Hölder condition and $[w]_{C_s^d}^{1/s} = [w]_{RH_s^d}$. For $s < 0$, we have that $w \in C_s^d$ if and only if $w \in A_{1-1/s}^d$. Moreover $[w]_{C_s^d} = [w]_{A_{1-1/s}^d}^{-s}$.

2.3. Weighted Haar functions. For a given weight v and an interval I define the corresponding *weighted Haar function* by

$$(2.2) \quad h_I^v(x) = \frac{1}{v(I)} \left(\sqrt{\frac{v(I_-)}{v(I_+)}} \chi_{I_+}(x) - \sqrt{\frac{v(I_+)}{v(I_-)}} \chi_{I_-}(x) \right),$$

where χ_I is the characteristic function of the interval I .

If v is the Lebesgue measure on \mathbb{R} , we will denote the *Haar function* simply by h_I . It is an important fact that $\{h_I^v\}_{I \in \mathcal{D}}$ is an orthonormal system in $L^2(v)$, with the inner product $\langle f, g \rangle_v = \int_{\mathbb{R}} f(x) \overline{g(x)} v(x) dx$.

It is a simple exercise to verify that the weighted and unweighted Haar functions are related linearly as follows:

Proposition 2.1. *For any weight v , there are numbers α_I^v, β_I^v such that*

$$h_I(x) = \alpha_I^v h_I^v(x) + \beta_I^v \chi_I(x) / \sqrt{|I|}$$

where (i) $|\alpha_I^v| \leq \sqrt{m_I v}$, (ii) $|\beta_I^v| \leq |\Delta_I v| / m_I v$, $\Delta_I v := m_{I_+} v - m_{I_-} v$.

For a weight v and a dyadic interval I , $|\Delta_I v| / m_I v = 2|1 - m_{I_-} v / m_I v| \leq 2$. If the weight v is dyadic doubling then we get an improvement on the above upper bound, $|\Delta_I v| / m_I v \leq 2(1 - 2/D(v))$.

2.4. Dyadic BMO and Carleson sequences. A locally integrable function b is a function of *dyadic bounded mean oscillation*, $b \in BMO^d$, if and only if

$$(2.3) \quad \|b\|_{BMO^d} := \left(\sup_{J \in \mathcal{D}} \frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} |\langle b, h_I \rangle|^2 \right)^{\frac{1}{2}} < \infty.$$

Note that if $b_I := \langle b, h_I \rangle$ then $|b_I| |I|^{-\frac{1}{2}} \leq \|b\|_{BMO^d}$, for all $I \in \mathcal{D}$.

If v is a weight, a positive sequence $\{\alpha_I\}_{I \in \mathcal{D}}$ is called a *v-Carleson sequence with intensity B* if for all $J \in \mathcal{D}$,

$$(2.4) \quad (1/|J|) \sum_{I \in \mathcal{D}(J)} \lambda_I \leq B m_{Jv}.$$

When $v = 1$ we call a sequence satisfying (2.4) for all $J \in \mathcal{D}$ a *Carleson sequence with intensity B*. If $b \in BMO^d$ then $\{|b_I|^2\}_{I \in \mathcal{D}}$ is a Carleson sequence with intensity $\|b\|_{BMO^d}^2$.

Proposition 2.2. *Let v be a weight, $\{\lambda_I\}_{I \in \mathcal{D}}$ and $\{\gamma_I\}_{I \in \mathcal{D}}$ be two v-Carleson sequences with intensities A and B respectively then for any $c, d > 0$ we have that*

- (i) $\{c\lambda_I + d\gamma_I\}_{I \in \mathcal{D}}$ is a v-Carleson sequence with intensity $cA + dB$.
- (ii) $\{\sqrt{\lambda_I} \sqrt{\gamma_I}\}_{I \in \mathcal{D}}$ is a v-Carleson sequence with intensity \sqrt{AB} .
- (iii) $\{(c\sqrt{\lambda_I} + d\sqrt{\gamma_I})^2\}_{I \in \mathcal{D}}$ is a v-Carleson sequence with intensity $2c^2A + 2d^2B$.

The proof of these statements is quite simple. To prove the first one we just need properties of the supremum, for the second one we apply the Cauchy-Schwarz inequality, and the third one is a consequence of the first two statements combined with the fact that $2cd\sqrt{A}\sqrt{B} \leq c^2A + d^2B$.

3. Main tools

In this section, we state and prove the lemmas and theorems necessary to obtain the estimates for the paraproduct and the t -Haar multipliers of complexity (m, n) . The Weighted Carleson Lemma 3.1, α -Lemma 3.4 and Lift Lemma 3.7 are fundamental for all our estimates.

3.1. Carleson lemmas. We present some weighted Carleson lemmas that we will use. Lemma 3.3 was introduced and used in [NV1], it was called a folklore lemma in reference to the likelihood of having been known before. Here we obtain Lemma 3.3 as an immediate corollary of the Weighted Carleson Lemma 3.1 and what we call the Little Lemma 3.2, introduced by Beznosova in her proof of the linear bound for the dyadic paraproduct.

3.1.1. Weighted Carleson Lemma. The Weighted Carleson Lemma we present here is a variation in the spirit of other weighted Carleson embedding theorems that appeared before in the literature [NV1], [NTV2]. All the lemmas in this section hold in \mathbb{R}^N or even geometric doubling metric spaces, see [Ch], [NRV].

Lemma 3.1 (Weighted Carleson Lemma). *Let v be a dyadic doubling weight, then $\{\alpha_L\}_{L \in \mathcal{D}}$ is a v -Carleson sequence with intensity B if and only if for all non-negative v -measurable functions F on the line,*

$$(3.1) \quad \sum_{L \in \mathcal{D}} \alpha_L \inf_{x \in L} F(x) \leq B \int_{\mathbb{R}} F(x) v(x) dx.$$

Proof: (\Rightarrow) Assume that $F \in L^1(v)$ otherwise the first statement is automatically true. Setting $\gamma_L = \inf_{x \in L} F(x)$, we can write

$$\sum_{L \in \mathcal{D}} \gamma_L \alpha_L = \sum_{L \in \mathcal{D}} \int_0^\infty \chi(L, t) dt \alpha_L = \int_0^\infty \left(\sum_{L \in \mathcal{D}} \chi(L, t) \alpha_L \right) dt,$$

where $\chi(L, t) = 1$ for $t < \gamma_L$ and zero otherwise, and the last equality follows by the monotone convergence theorem. Define $E_t = \{x \in \mathbb{R} : F(x) > t\}$. Since F is assumed to be a v -measurable function, E_t is a v -measurable set for every t . Moreover, since $F \in L^1(v)$ we have, by Chebychev's inequality, that the v -measure of E_t is finite for all real t . If $\chi(L, t) = 1$ then $L \subset E_t$. Moreover, there is a collection of maximal disjoint dyadic intervals \mathcal{P}_t that are contained in E_t . Then we can write

$$(3.2) \quad \sum_{L \in \mathcal{D}} \chi(L, t) \alpha_L \leq \sum_{L \subset E_t} \alpha_L = \sum_{L \in \mathcal{P}_t} \sum_{I \in \mathcal{D}(L)} \alpha_I \leq B \sum_{L \in \mathcal{P}_t} v(L) \leq Bv(E_t),$$

where, in the second inequality, we used the fact that $\{\alpha_I\}_{I \in \mathcal{D}}$ is a v -Carleson sequence with intensity B . Thus we can estimate

$$\sum_{L \in \mathcal{D}} \gamma_L \alpha_L \leq B \int_0^\infty v(E_t) dt = B \int_{\mathbb{R}} F(x) v(x) dx.$$

The last equality follows from the layer cake representation.

(\Leftarrow) Assume (3.1) is true; in particular it holds for $F(x) = \chi_J(x)/|J|$. Since $\inf_{x \in I} F(x) = 0$ if $I \cap J = \emptyset$, and $\inf_{x \in I} F(x) = 1/|J|$ otherwise,

$$\frac{1}{|J|} \sum_{I \in \mathcal{D}(J)} \alpha_I \leq \sum_{I \in \mathcal{D}} \alpha_I \inf_{x \in I} F(x) \leq \int_{\mathbb{R}} F(x) v(x) dx = m_J v. \quad \square$$

3.1.2. Little Lemma. The following lemma was proved by Beznosova in [Be2] using the Bellman function $B(u, v, l) = u - 1/v(1 + l)$.

Lemma 3.2 (Little Lemma [Be2]). *Let v be a weight, such that v^{-1} is a weight as well, and let $\{\lambda_I\}_{I \in \mathcal{D}}$ be a Carleson sequence with intensity B . Then $\{\lambda_I/m_I v^{-1}\}_{I \in \mathcal{D}}$ is a v -Carleson sequence with intensity $4B$, that is for all $J \in \mathcal{D}$,*

$$(1/|J|) \sum_{I \in \mathcal{D}(J)} \lambda_I/m_I v^{-1} \leq 4B m_J v.$$

For a proof of this result we refer [Be1, Proposition 3.4], or [Be2, Proposition 2.1]. For an \mathbb{R}^N version of this result see [Ch, Proposition 4.6].

Lemma 3.2 together with Lemma 3.1 immediately yield the following:

Lemma 3.3 ([NV1]). *Let v be a weight such that v^{-1} is also a weight. Let $\{\lambda_J\}_{J \in \mathcal{D}}$ be a Carleson sequence with intensity B , and let F be a non-negative measurable function on the line. Then*

$$\sum_{J \in \mathcal{D}} (\lambda_J/m_J v^{-1}) \inf_{x \in J} F(x) \leq C B \int_{\mathbb{R}} F(x) v(x) dx.$$

Note that Lemma 3.2 can be deduced from Lemma 3.3 with $F(x) = \chi_J(x)$.

3.2. α -Lemma. The following lemma was proved by Beznosova for $\alpha = 1/4$ in [Be1], and by Nazarov and Volberg for $0 < \alpha < 1/2$ in [NV1], using the Bellman function $B(u, v) = (uv)^\alpha$.

Lemma 3.4 (α -Lemma). *Let $w \in A_2^d$ and then for any $\alpha \in (0, 1/2)$, the sequence $\{\mu_I^\alpha\}_{I \in \mathcal{D}}$, where*

$$\mu_I^\alpha := (m_I w)^\alpha (m_I w^{-1})^\alpha |I| \left(\frac{|\Delta_I w|^2}{(m_I w)^2} + \frac{|\Delta_I w^{-1}|^2}{(m_I w^{-1})^2} \right),$$

is a Carleson sequence with intensity $C_\alpha [w]_{A_2^d}^\alpha$, with $C_\alpha = 72/(\alpha - 2\alpha^2)$.

A proof of this lemma that works in \mathbb{R}^N (for $\alpha = 1/4$) can be found in [Ch, Proposition 4.8], and one that works in geometric doubling metric spaces can be found in [NV2], [Vo].

The following lemmas simplify the exposition of the main theorems (this was pointed to us by one of our referees). We deduce these lemmas from the α -Lemma. According to our kind anonymous referee, one can also deduce Lemma 3.5 from a pure Bellman-function argument without reference to the α -Lemma.

Lemma 3.5. *Let $w \in A_2^d$ and let $\nu_I = |I|(m_I w)^2(\Delta_I w^{-1})^2$. The sequence $\{\nu_I\}_{I \in \mathcal{D}}$ is a Carleson sequence with intensity at most $C[w]_{A_2^d}^2$ for some numerical constant C ($C = 288$ works).*

Proof: Multiply and divide ν_I by $(m_I w^{-1})^2$ to get for any $0 < \alpha < 1/2$,

$$\nu_I = |I|(m_I w)^2 (m_I w^{-1})^2 (|\Delta_I w^{-1}|/m_I w^{-1})^2 \leq [w]_{A_2^d}^{2-\alpha} \mu_I^\alpha.$$

But $\{\mu_I^\alpha\}_{I \in \mathcal{D}}$ is a Carleson sequence with intensity $C_\alpha [w]_{A_2^d}^\alpha$ by Lemma 3.4, therefore by Proposition 2.2(i) $\{\nu_I\}_{I \in \mathcal{D}}$ is a Carleson sequence with intensity at most $C_\alpha [w]_{A_2^d}^2$ as claimed. \square

It is well known that if $w \in A_2^d$ then $\{|I||\Delta_I w|^2/(m_I w)^2\}_{I \in \mathcal{D}}$ is a Carleson sequence with intensity $\log[w]_{A_2^d}$, see [Wi1]. This estimate together with Proposition 2.2(i), give intensities $[w]_{A_2^d}^\alpha \log[w]_{A_2^d}$ and $[w]_{A_2^d}^2 \log[w]_{A_2^d}$ respectively for the sequences $\{\mu_I^\alpha\}_{I \in \mathcal{D}}$ and $\{\nu_I\}_{I \in \mathcal{D}}$. The lemmas show we can improve the intensities by dropping the logarithmic factor. Even more generally, we can show the following lemma, which extends the α -Lemma 3.4 to the range $\alpha \geq 1/2$. It also refines it for the range $\alpha \in (1/4, 1/2)$ and shows that the blow up of the constant C_α for $\alpha = 1/2$ is an artifact of the proof.

Lemma 3.6. *Let $w \in A_2^d$, $s > 0$, and*

$$\tau_I^s := |I|(m_I w)^s (m_I w^{-1})^s \left(\frac{|\Delta_I w|^2}{(m_I w)^2} + \frac{|\Delta_I w^{-1}|^2}{(m_I w^{-1})^2} \right).$$

Then for $0 < \alpha < \min\{1/2, s\}$, the sequence $\{\tau_I^s\}_{I \in \mathcal{D}}$ is a Carleson sequence with intensity at most $C_\alpha[w]_{A_2^s}^s$ where C_α is the constant in Lemma 3.4 (when $s > 1/4$ can choose $\alpha = 1/4$ and $C_\alpha = 576$).

3.3. Lift Lemma. Given a dyadic interval L , and weights u, v , we introduce a family of stopping time intervals \mathcal{ST}_L^m such that the averages of the weights over any stopping time interval $K \in \mathcal{ST}_L^m$ are comparable to the averages on L , and $|K| \geq 2^{-m}|L|$. This construction appeared in [NV1] for the case $u = w, v = w^{-1}$. We also present a lemma that lifts w -Carleson sequences on intervals to w -Carleson sequences on “ m -stopping intervals”. We present the proofs for the convenience of the reader.

Lemma 3.7 (Lift Lemma [NV1]). *Let u and v be weights, L be a dyadic interval and m, n be fixed natural numbers. Let \mathcal{ST}_L^m be the collection of maximal stopping time intervals $K \in \mathcal{D}(L)$, where the stopping criteria are either (i) $|\Delta_K u|/m_K u + |\Delta_K v|/m_K v \geq 1/(m+n+2)$, or (ii) $|K| = 2^{-m}|L|$. Then for any stopping interval $K \in \mathcal{ST}_L^m$, $e^{-1}m_L u \leq m_K u \leq e m_L u$, also $e^{-1}m_L v \leq m_K v \leq e m_L v$.*

Note that the roles of m and n can be interchanged and we get the family \mathcal{ST}_L^n using the same stopping condition (i) as above, but with (ii) replaced by $|K| = 2^{-n}|L|$. Notice that \mathcal{ST}_L^m is a partition of L in dyadic subintervals of length at least $2^{-m}|L|$. Any collection of subintervals of L with this property will be an m -stopping time for L .

Proof: Let K be a maximal stopping time interval; thus no dyadic interval strictly bigger than K can satisfy either stopping criteria. If F is a dyadic interval strictly bigger than K and contained in L , then necessarily $|\Delta_F u|/m_F u \leq (m+n+2)^{-1}$ and $|\Delta_F v|/m_F v \leq (m+n+2)^{-1}$. This is particularly true for the parent of K . Let us denote by \widehat{K} the parent of K , then $|m_K u - m_{\widehat{K}} u|2 \leq m_{\widehat{K}} u/2(m+n+2)$. So, $m_{\widehat{K}} u(1 - 1/2(m+n+2)) \leq m_K u \leq m_{\widehat{K}} u(1 + 1/2(m+n+2))$. Iterating this process until we reach L , we will get that

$$m_L u \left(1 - \frac{1}{2(m+n+2)}\right)^m \leq m_K u \leq m_L u \left(1 + \frac{1}{2(m+n+2)}\right)^m.$$

Remember that $|K| = 2^{-j}|L|$ where $0 \leq j \leq m$ so we will iterate at most m times. We can obtain the same bounds for v . These clearly imply the estimates in the lemma, since $\lim_{k \rightarrow \infty} (1 + 1/k)^k = e$. \square

The following lemma lifts a w -Carleson sequence to m -stopping time intervals with comparable intensity. The lemma appeared in [NV1]

for the particular stopping time \mathcal{ST}_L^m given by the stopping criteria (i) and (ii) in Lemma 3.7, and $w = 1$. This is a property of any stopping time that stops once the m^{th} -generation is reached.

Lemma 3.8. *For each $L \in \mathcal{D}$, let \mathcal{ST}_L^m be a partition of L in dyadic subintervals of length at least $2^{-m}|L|$ (in particular it could be the stopping time intervals defined in Lemma 3.7). Assume $\{\nu_I\}_{I \in \mathcal{D}}$ is a w -Carleson sequence with intensity at most A , let $\nu_L^m := \sum_{K \in \mathcal{ST}_L^m} \nu_K$. Then $\{\nu_L^m\}_{L \in \mathcal{D}}$ is a w -Carleson sequence with intensity at most $(m + 1)A$.*

Proof: In order to show that $\{\nu_L^m\}_{L \in \mathcal{D}}$ is a w -Carleson sequence with intensity at most $(m + 1)A$, it is enough to show that for any $J \in \mathcal{D}$

$$\sum_{L \in \mathcal{D}(J)} \nu_L^m < (m + 1)Aw(J).$$

Observe that for each dyadic interval K inside a fixed dyadic interval J there exist at most $m + 1$ dyadic intervals L such that $K \in \mathcal{ST}_L^m$. Let us denote by K^i the dyadic interval that contains K and such that $|K^i| = 2^i|K|$. If $K \in \mathcal{D}(J)$ then L must be K^0, K^1, \dots or K^m . We just have to notice that if $L = K^i$, for $i > m$ then K cannot be in \mathcal{ST}_L^m because $|K| < 2^{-m}|L|$. Therefore,

$$\begin{aligned} \sum_{L \in \mathcal{D}(J)} \nu_L^m &= \sum_{L \in \mathcal{D}(J)} \sum_{K \in \mathcal{ST}_L^m} \nu_K = \sum_{K \in \mathcal{D}(J)} \sum_{L \in \mathcal{D}(J) \text{ s.t. } K \in \mathcal{ST}_L^m} \nu_K \\ &\leq \sum_{K \in \mathcal{D}(J)} (m + 1)\nu_k \leq (m + 1)Aw(J). \end{aligned}$$

The last inequality follows by the definition of w -Carleson sequence with intensity A . The lemma is proved. □

4. Paraproduct

For $b \in BMO^d$, and $m, n \in \mathbb{N}$, a *dyadic paraproduct of complexity (m, n)* is the operator defined by

$$(4.1) \quad (\pi_b^{m,n} f)(x) := \sum_{L \in \mathcal{D}} \sum_{(I, J) \in \mathcal{D}_m^n(L)} c_{I, J}^L m_I f \langle b, h_I \rangle h_J(x),$$

where $|c_{I, J}^L| \leq \sqrt{|I||J|}/|L|$ for all dyadic intervals L and $(I, J) \in \mathcal{D}_m^n(L)$, where $\mathcal{D}_m^n(L) = \mathcal{D}_n(L) \times \mathcal{D}_m(L)$.

A dyadic paraproduct of complexity $(0, 0)$ is the usual dyadic paraproduct π_b known to be bounded in $L^p(\mathbb{R})$ if and only if $b \in BMO^d$.

A Haar shift operator of complexity (m, n) , $m, n \in \mathbb{N}$, is defined by

$$(S^{m,n}f)(x) := \sum_{L \in \mathcal{D}} \sum_{(I,J) \in \mathcal{D}_m^n(L)} c_{I,J}^L \langle f, h_I \rangle h_J(x),$$

where $|c_{I,J}^L| \leq \sqrt{|I||J|}/|L|$. Notice that the Haar shift operators are automatically uniformly bounded on $L^2(\mathbb{R})$, with operator norm less than or equal to one [LPR], [CMP1].

The dyadic paraproduct of complexity (m, n) is the composition of $S^{m,n}$ and π_b . Therefore, if $b \in BMO^d$ then $\pi_b^{m,n}$ is bounded in $L^2(\mathbb{R})$, since $\pi_b^{m,n} = S^{m,n}\pi_b$, and both π_b (the dyadic paraproduct) and $S^{m,n}$ (the Haar shift operators) are bounded in $L^2(\mathbb{R})$.

Furthermore, π_b and $S^{m,n}$ are bounded in $L^2(w)$ whenever $w \in A_2^d$. Both of them obey bounds on $L^2(w)$ that are linear in the A_2 -characteristic of the weight, immediately providing a quadratic bound in the A_2 -characteristic of the weight for $\pi_b^{m,n}$. We will show that in fact, the dyadic paraproduct of complexity (m, n) obeys the same linear bound in $L^2(w)$ with respect to $[w]_{A_2^d}$ obtained by Beznosova [Be2] for the dyadic paraproduct of complexity $(0, 0)$, multiplied by a polynomial factor that depends on the complexity.

The proof given by Nazarov and Volberg, in [NV1], of the fact that Haar shift operators with complexity (m, n) are bounded in $L^2(w)$ with a bound that depends linearly on the A_2^d -characteristic of w , and polynomially on the complexity, works, with appropriate modifications, for the dyadic paraproducts of complexity (m, n) . Below we describe those modifications. Beforehand, however, we will present this new and conceptually simpler (in our opinion) proof for the linear bound on the A_2^d -characteristic for the dyadic paraproduct, which will allow us to highlight certain elements of the general proof without dealing with the complexity.

4.1. Complexity $(0, 0)$. The dyadic paraproduct of complexity $(0, 0)$ is defined by $(\pi_b f)(x) := \sum_{I \in \mathcal{D}} c_I m_I f \langle b, h_I \rangle h_I(x)$, where $|c_I| \leq 1$.

It is known that π_b obeys a linear bound in $L^2(w)$ both in terms of the A_2^d -characteristic of the weight w and the BMO -norm of b .

Theorem 4.1 ([Be2]). *There exists $C > 0$, such that for all $b \in BMO^d$ and for all $w \in A_2^d$,*

$$\|\pi_b f\|_{L^2(w)} \leq C[w]_{A_2^d} \|b\|_{BMO^d} \|f\|_{L^2(w)}.$$

Beznosova’s proof is based on the α -Lemma, the Little Lemma (these were the new Bellman function ingredients that she introduced), and

Nazarov-Treil-Volberg’s two-weight Carleson embedding theorem, which can be found in [NTV1]. Below, we give another proof of this result; this proof is still based on the α -Lemma 3.4 (via Lemma 3.5) however it does not make use of the two-weight Carleson embedding theorem. Instead we will use properties of Carleson sequences such as the Little Lemma 3.2, and the Weighted Carleson Lemma 3.1, following the argument in [NV1] for Haar shift operators of complexity (m, n) . The extension of Theorem 4.1 to \mathbb{R}^N can be found in [Ch], and the methods used there can be adapted to extend our proof to \mathbb{R}^N even in the complexity (m, n) case, see [Mo2].

Remark 4.2. Throughout the proofs a constant C will be a numerical constant that may change from line to line.

Proof of Theorem 4.1: Fix $f \in L^2(w)$ and $g \in L^2(w^{-1})$. Define $b_I = \langle b, h_I \rangle$, then $\{b_I^2\}_{I \in \mathcal{D}}$ is a Carleson sequence with intensity $\|b\|_{BMO^d}^2$.

By duality, it suffices to prove:

$$(4.2) \quad |\langle \pi_b(fw), gw^{-1} \rangle| \leq C \|b\|_{BMO^d} [w]_{A_2} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}.$$

Note that $|\langle \pi_b(fw), gw^{-1} \rangle| = |\langle \sum_{I \in \mathcal{D}} c_I b_I m_I(fw) h_I, gw^{-1} \rangle|$. Write $h_I = \alpha_I h_I^{w^{-1}} + \beta_I \chi_I / \sqrt{|I|}$ where $\alpha_I = \alpha_I^{w^{-1}}$ and $\beta_I = \beta_I^{w^{-1}}$ as described in Proposition 2.1. Then

$$(4.3) \quad |\langle \pi_b(fw), gw^{-1} \rangle| \leq \sum_{I \in \mathcal{D}} |b_I| m_I(|f|w) \left| \left\langle gw^{-1}, \alpha_I h_I^{w^{-1}} + \beta_I \frac{\chi_I}{\sqrt{|I|}} \right\rangle \right|.$$

Use the triangle inequality to break the sum in (4.3) into two sums to be estimated separately, $|\langle \pi_b(fw), gw^{-1} \rangle| \leq \Sigma_1 + \Sigma_2$. Where, using the estimates $|\alpha_I| \leq \sqrt{m_I w^{-1}}$, and $|\beta_I| \leq |\Delta_I w^{-1}| / m_I w^{-1}$,

$$\begin{aligned} \Sigma_1 &:= \sum_{I \in \mathcal{D}} |b_I| m_I(|f|w) |\langle gw^{-1}, h_I^{w^{-1}} \rangle| \sqrt{m_I w^{-1}} \\ \Sigma_2 &:= \sum_{I \in \mathcal{D}} |b_I| m_I(|f|w) |\langle gw^{-1}, \chi_I \rangle| \frac{|\Delta_I w^{-1}|}{m_I w^{-1}} \frac{1}{\sqrt{|I|}}. \end{aligned}$$

Estimating Σ_1 : First using that $m_I(|f|w) / m_I w \leq \inf_{x \in I} M_w f(x)$, and that $\langle gv, f \rangle = \langle g, f \rangle_v$; second using the Cauchy-Schwarz inequality and

$m_I w m_I w^{-1} \leq [w]_{A_2^d}$, we get

$$\begin{aligned} \Sigma_1 &\leq \sum_{I \in \mathcal{D}} |b_I| \frac{\inf_{x \in I} M_w f(x)}{\sqrt{m_I w^{-1}}} |\langle g, h_I^{w^{-1}} \rangle_{w^{-1}}| m_I w^{-1} m_I w \\ &\leq [w]_{A_2^d} \left(\sum_{I \in \mathcal{D}} |b_I|^2 \frac{\inf_{x \in I} M_w^2 f(x)}{m_I w^{-1}} \right)^{\frac{1}{2}} \left(\sum_{I \in \mathcal{D}} |\langle g, h_I^{w^{-1}} \rangle_{w^{-1}}|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Using Weighted Carleson Lemma 3.1, with $F(x) = M_w^2 f(x)$, $v = w$, and $\alpha_I = |b_I|^2 / m_I w^{-1}$ (which is a w -Carleson sequence with intensity $4 \|b\|_{BMO^d}^2$, according to Lemma 3.2), together with the fact that $\{h_I^{w^{-1}}\}_{I \in \mathcal{D}}$ is an orthonormal system in $L^2(w^{-1})$, we get

$$\begin{aligned} \Sigma_1 &\leq 4[w]_{A_2^d} \|b\|_{BMO^d} \left(\int_{\mathbb{R}} M_w^2 f(x) w(x) dx \right)^{\frac{1}{2}} \|g\|_{L^2(w^{-1})} \\ &\leq C[w]_{A_2^d} \|b\|_{BMO^d} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}. \end{aligned}$$

In the last inequality we used the fact that M_w is bounded in $L^2(w)$ with operator norm independent of w .

Estimating Σ_2 : Using arguments similar to the ones used for Σ_1 , we conclude that,

$$\Sigma_2 = \sum_{I \in \mathcal{D}} |b_I| m_I^w |f| m_I^{w^{-1}} |g| \sqrt{\nu_I} \leq \sum_{I \in \mathcal{D}} |b_I| \sqrt{\nu_I} \inf_{x \in I} M_w f(x) M_{w^{-1}} g(x),$$

where $\nu_I = |I| (m_I w)^2 (\Delta_I w^{-1})^2$ as defined in Lemma 3.5, and in the last inequality we used that for any $I \in \mathcal{D}$ and all $x \in I$,

$$m_I^w |f| m_I^{w^{-1}} |g| \leq M_w f(x) M_{w^{-1}} g(x).$$

Since $\{|b_I|^2\}_{I \in \mathcal{D}}$ and $\{\nu_I\}_{I \in \mathcal{D}}$ are Carleson sequences with intensities $\|b\|_{BMO^d}^2$ and $C[w]_{A_2^d}^2$, respectively, by Proposition 2.2, the sequence $\{|b_I| \sqrt{\nu_I}\}_{I \in \mathcal{D}}$ is a Carleson sequence with intensity $C \|b\|_{BMO^d} [w]_{A_2^d}$. Thus, by Lemma 3.1 with $F(x) = M_w f(x) M_{w^{-1}} g(x)$, $\alpha_I = |b_I| \sqrt{\nu_I}$, and $v = 1$,

$$\Sigma_2 \leq C \|b\|_{BMO^d} [w]_{A_2^d} \int_{\mathbb{R}} M_w f(x) M_{w^{-1}} g(x) dx.$$

Using the Cauchy-Schwarz inequality and $w^{\frac{1}{2}}(x)w^{-\frac{1}{2}}(x) = 1$ we get

$$\begin{aligned}\Sigma_2 &\leq C[w]_{A_2^d} \|b\|_{BMO^d} \left(\int_{\mathbb{R}} M_w^2 f(x) w(x) dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} M_{w^{-1}}^2 g(x) w^{-1}(x) dx \right)^{\frac{1}{2}} \\ &= C[w]_{A_2^d} \|b\|_{BMO^d} \|M_w f\|_{L^2(w)} \|M_{w^{-1}} g\|_{L^2(w^{-1})} \\ &\leq C[w]_{A_2^d} \|b\|_{BMO^d} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}.\end{aligned}$$

These estimates together give (4.2), and the theorem is proved. \square

4.2. Complexity (m, n) . In this section, we prove an estimate for the dyadic paraproduct of complexity (m, n) that is linear in the A_2 -characteristic and polynomial in the complexity. The proof will follow the general lines of the argument presented in Subsection 4.1 for the complexity $(0, 0)$ case, with the added refinements devised by Nazarov and Volberg [NV1], adapted to our setting, to handle the general complexity.

Theorem 4.3. *Let $b \in BMO^d$ and $w \in A_2^d$. Then there is $C > 0$ such that*

$$\|\pi_b^{m,n} f\|_{L^2(w)} \leq C(n+m+2)^5 [w]_{A_2^d} \|b\|_{BMO^d} \|f\|_{L^2(w)}.$$

Proof: Fix $f \in L^2(w)$ and $g \in L^2(w^{-1})$, define $b_I = \langle b, h_I \rangle$ and let $C_m^n := (m+n+2)$. By duality, it is enough to show that

$$|\langle \pi_b^{m,n}(fw), gw^{-1} \rangle| \leq C(C_m^n)^5 [w]_{A_2^d} \|b\|_{BMO^d} \|g\|_{L^2(w^{-1})} \|f\|_{L^2(w)}.$$

We write the left-hand side as a double sum, that we will estimate as

$$|\langle \pi_b^{m,n}(fw), gw^{-1} \rangle| \leq \sum_{L \in \mathcal{D}} \sum_{(I,J) \in \mathcal{D}_m^n(L)} |b_I| \frac{\sqrt{|I||J|}}{|L|} m_I(|f|w) |\langle gw^{-1}, h_J \rangle|.$$

As before, we write $h_J = \alpha_J h_J^{w^{-1}} + \beta_J \chi_J / \sqrt{|J|}$, with $\alpha_J = \alpha_J^{w^{-1}}$, $\beta_J = \beta_J^{w^{-1}}$, and break the double sum into two terms to be estimated separately. Then $|\langle \pi_b^{m,n}(fw), gw^{-1} \rangle| \leq \Sigma_1^{m,n} + \Sigma_2^{m,n}$, where

$$\Sigma_1^{m,n} := \sum_{L \in \mathcal{D}} \sum_{(I,J) \in \mathcal{D}_m^n(L)} |b_I| \frac{\sqrt{|I||J|}}{|L|} m_I(|f|w) |\langle g, h_J^{w^{-1}} \rangle_{w^{-1}}| \sqrt{m_J w^{-1}},$$

$$\Sigma_2^{m,n} := \sum_{L \in \mathcal{D}} \sum_{(I,J) \in \mathcal{D}_m^n(L)} |b_I| \frac{\sqrt{|I|}}{|L|} m_I(|f|w) |\langle gw^{-1}, \chi_J \rangle| \frac{|\Delta_J w^{-1}|}{m_J w^{-1}}.$$

For a weight v , and a locally integrable function ϕ we define the following quantities,

$$(4.4) \quad S_L^{v,m} \phi := \sum_{J \in \mathcal{D}_m(L)} |\langle \phi, h_J^v \rangle_v| \sqrt{m_J v} \sqrt{|J|/|L|},$$

$$(4.5) \quad R_L^{v,m} \phi := \sum_{J \in \mathcal{D}_m(L)} \frac{|\Delta_J v|}{m_J v} m_J (|\phi|_v) |J|/\sqrt{|L|},$$

$$(4.6) \quad Pb_L^{v,n} \phi := \sum_{I \in \mathcal{D}_n(L)} |b_I| m_I (|\phi|_v) \sqrt{|I|/|L|}.$$

For $s = 1, 2$ and $w \in A_2^d$, we also define the following Carleson sequences (see Lemma 3.8 and Lemma 3.6):

$$\mu_K^s := (m_K w)^s (m_K w^{-1})^s \left(\frac{|\Delta_K w^{-1}|^2}{(m_K w^{-1})^2} + \frac{|\Delta_K w|^2}{(m_K w)^2} \right) |K|,$$

with intensity $C[w]_{A_2^d}^s$,

$$\mu_L^{m,s} := \sum_{K \in ST_L^m} \mu_K, \text{ with intensity } C(m+1)[w]_{A_2^d}^s,$$

$$\mu_L^{n,s} := \sum_{K \in ST_L^n} \mu_K, \text{ with intensity } C(n+1)[w]_{A_2^d}^s,$$

$$\mu_K^{b,s} := |b_K|^2 (m_K w m_K w^{-1})^s, \text{ with intensity } \|b\|_{BMO^d}^2 [w]_{A_2^d}^s,$$

and

$$\mu_L^{b,n,s} := \sum_{K \in ST_L^n} \mu_K^{b,s}, \text{ with intensity } (n+1) \|b\|_{BMO^d}^2 [w]_{A_2^d}^s.$$

Note that

$$\Sigma_1^{m,n} \leq \sum_{L \in \mathcal{D}} Pb_L^{w,n} f S_L^{w^{-1},m} g \quad \text{and} \quad \Sigma_2^{m,n} \leq \sum_{L \in \mathcal{D}} Pb_L^{w,n} f R_L^{w^{-1},m} g.$$

In order to estimate $\Sigma_1^{m,n}$ and $\Sigma_2^{m,n}$ we will use the following estimates for $S_L^{w^{-1},m}g$, $R_L^{w^{-1},m}g$, and $Pb_L^{w,n}f$,

$$(4.7) \quad S_L^{w^{-1},m}g \leq \left(\sum_{J \in \mathcal{D}_m(L)} |\langle g, h_J^{w^{-1}} \rangle_{w^{-1}}|^2 \right)^{\frac{1}{2}} (m_L w^{-1})^{\frac{1}{2}},$$

$$(4.8) \quad R_L^{w^{-1},m}g \leq C C_m^n (m_L w)^{-\frac{s}{2}} (m_L w^{-1})^{1-\frac{s}{2}} \inf_{x \in L} (M_{w^{-1}}(|g|^p)(x))^{\frac{1}{p}} \sqrt{\mu_L^{m,s}},$$

$$(4.9) \quad Pb_L^{w,n}f \leq C C_m^n (m_L w)^{1-\frac{s}{2}} (m_L w^{-1})^{-\frac{s}{2}} \inf_{x \in L} (M_w(|f|^p)(x))^{\frac{1}{p}} \nu_L^{n,s},$$

where $\nu_L^{n,s} = \|b\|_{BMO^d} \sqrt{\mu_L^{n,s}} + \sqrt{\mu_L^{b,n,s}}$, and $p = 2 - (C_m^n)^{-1}$ (note that $1 < p < 2$). In the proof it will become clear why this is a good choice; the reader is invited to assume first that $p = 2$ and reach a point of no return in the argument.

Estimate (4.7) is easy to show. We just use the Cauchy-Schwarz inequality and the fact that $\mathcal{D}_m(L)$ is a partition of L .

$$S_L^{w^{-1},m}g \leq \left(\sum_{J \in \mathcal{D}_m(L)} |\langle g, h_J^{w^{-1}} \rangle_{w^{-1}}|^2 \right)^{\frac{1}{2}} (m_L w^{-1})^{\frac{1}{2}}.$$

Estimate (4.8) was obtained in [NV1]. With a variation on their argument we prove estimate (4.9) in Lemma 4.4. Let us first use estimates (4.7), (4.8) and (4.9) to estimate $\Sigma_1^{m,n}$ and $\Sigma_2^{m,n}$.

Estimate for $\Sigma_1^{m,n}$: Use estimates (4.7) and (4.9) with $s = 2$, the Cauchy-Schwarz inequality and the fact that $\{h_J^{w^{-1}}\}_{J \in \mathcal{D}}$ is an orthonormal system in $L^2(w^{-1})$ and $\mathcal{D} = \cup_{L \in \mathcal{D}} \mathcal{D}_m(L)$. Then

$$\Sigma_1^{m,n} \leq C C_m^n \left(\sum_{L \in \mathcal{D}} \frac{(\nu_L^{n,2})^2}{m_L w^{-1}} \inf_{x \in L} (M_w(|f|^p)(x))^{\frac{2}{p}} \right)^{\frac{1}{2}} \|g\|_{L^2(w^{-1})}.$$

We will now use the Weighted Carleson Lemma 3.1 with

$F(x) = (M_w(|f|^p)(x))^{2/p}$, $v = w$, and $\alpha_L = (\nu_L^{n,2})^2/m_L w^{-1}$. Recall that $\nu_L^{n,2} = \|b\|_{BMO^d} \sqrt{\mu_L^{n,2}} + \sqrt{\mu_L^{b,n,2}}$. By Proposition 2.2, $\{(\nu_L^{n,2})^2\}_{L \in \mathcal{D}}$ is a Carleson sequence with intensity at most $C C_m^n \|b\|_{BMO^d}^2 [w]_{A_2^d}^2$. By

Lemma 3.2, $\{(\nu_L^{n,2})^2/m_L w^{-1}\}_{L \in \mathcal{D}}$ is a w -Carleson sequence with comparable intensity. Thus we will have that

$$\begin{aligned} \Sigma_1^{m,n} &\leq C (C_m^n)^{\frac{3}{2}} [w]_{A_2^d} \|b\|_{BMO^d} \|g\|_{L^2(w^{-1})} \left\| M_w(|f|^p) \right\|_{L^{\frac{2}{p}}(w)}^{\frac{1}{p}} \\ &\leq C [(2/p)']^{\frac{1}{p}} (C_m^n)^{\frac{3}{2}} [w]_{A_2^d} \|b\|_{BMO^d} \|g\|_{L^2(w^{-1})} \left\| |f|^p \right\|_{L^{\frac{2}{p}}(w)}^{\frac{1}{p}} \\ &= C (C_m^n)^{\frac{5}{2}} [w]_{A_2^d} \|b\|_{BMO^d} \|g\|_{L^2(w^{-1})} \|f\|_{L^2(w)}. \end{aligned}$$

We used in the first inequality that M_w is bounded in $L^q(w)$ for all $q > 1$, more specifically we used that $\|M_w f\|_{L^q(w)} \leq C q' \|f\|_{L^q(w)}$. In our case $q = 2/p$ and $q' = 2/(2-p) = 2C_m^n$.

Estimate for $\Sigma_2^{m,n}$: Use estimates (4.8) and (4.9) with $s = 1$ in both cases, together with the facts that $(m_I w m_I w^{-1})^{-1} \leq 1$, and that the product of the infimum of positive quantities is smaller than the infimum of the product. Then

$$\Sigma_2^{m,n} \leq C (C_m^n)^2 \sum_{L \in \mathcal{D}} \nu_L^{n,1} \sqrt{\mu_L^{m,1}} \inf_{x \in L} (M_w(|f|^p)(x))^{\frac{1}{p}} (M_{w^{-1}}(|g|^p)(x))^{\frac{1}{p}}.$$

Since $(\nu_L^{n,1})^2$ and $\mu_L^{m,1}$ have intensity at most $C(n+1)[w]_{A_2^d} \|b\|_{BMO}^2$ and $C(m+1)[w]_{A_2^d}$, by Proposition 2.2, we have that $\nu_L^{n,1} \sqrt{\mu_L^{m,1}}$ is a Carleson sequence with intensity at most $C C_m^n \|b\|_{BMO^d} [w]_{A_2^d}$. If we now apply

Lemma 3.1 with $F^p(x) = M_w(|f|^p)(x) M_{w^{-1}}(|g|^p)(x)$, $\alpha_L = \nu_L^{n,1} \sqrt{\mu_L^{m,1}}$, and $v = 1$, we will have, by the Cauchy-Schwarz inequality and the boundedness of M_v in $L^q(v)$ for $q = p/2 > 1$,

$$\begin{aligned} \Sigma_2^{m,n} &\leq C (C_m^n)^3 [w]_{A_2^d} \|b\|_{BMO^d} \int_{\mathbb{R}} (M_w(|f|^p)(x))^{\frac{1}{p}} (M_{w^{-1}}(|g|^p)(x))^{\frac{1}{p}} dx \\ &\leq C (C_m^n)^3 [w]_{A_2^d} \|b\|_{BMO^d} \left\| M_w(|f|^p) \right\|_{L^{\frac{2}{p}}(w)}^{\frac{1}{p}} \left\| M_{w^{-1}}(|g|^p) \right\|_{L^{\frac{2}{p}}(w^{-1})}^{\frac{1}{p}} \\ &\leq C [(2/p)']^{\frac{2}{p}} (C_m^n)^3 [w]_{A_2^d} \|b\|_{BMO^d} \left\| |f|^p \right\|_{L^{\frac{2}{p}}(w)}^{\frac{1}{p}} \left\| |g|^p \right\|_{L^{\frac{2}{p}}(w^{-1})}^{\frac{1}{p}} \\ &= C (C_m^n)^5 [w]_{A_2^d} \|b\|_{BMO^d} \|f\|_{L^2(w)} \|g\|_{L^2(w^{-1})}. \end{aligned}$$

Together these estimates prove the theorem, under the assumption that estimate (4.9) holds. \square

4.3. Key Lemma. The missing step in the previous proof is estimate (4.9), which we now prove. The argument we present is an adaptation of the argument used in [NV1] to obtain estimate (4.8).

Lemma 4.4. *Let $b \in BMO^d$, and let ϕ be a locally integrable function. Then*

$$Pb_L^{w,n} \phi \leq C C_m^n (m_L w)^{1-\frac{s}{2}} (m_L w^{-1})^{\frac{-s}{2}} \inf_{x \in L} (M_w(|\phi|^p)(x))^{\frac{1}{p}} \nu_L^{n,s},$$

where $\nu_L^{n,s} = \|b\|_{BMO^d} \sqrt{\mu_L^{n,s}} + \sqrt{\mu_L^{b,n,s}}$, and $p = 2 - (C_m^n)^{-1}$.

Proof: Let \mathcal{ST}_L^n be the collection of stopping time intervals defined in Lemma 3.7. Noting that $\mathcal{D}_n(L) = \cup_{K \in \mathcal{ST}_L^n} (\mathcal{D}(K) \cap \mathcal{D}_n(L))$, we get

$$Pb_L^{w,n} \phi = \sum_{K \in \mathcal{ST}_L^n} \sum_{I \in \mathcal{D}(K) \cap \mathcal{D}_n(L)} |b_I| m_I(|\phi|w) \sqrt{|I|/|L|}.$$

Note that if K is a stopping time interval by the first criterion then

$$\begin{aligned} Pb_L^{w,n} \phi &\leq \|b\|_{BMO^d} m_K(|\phi|w) |K| / \sqrt{|L|} \\ &\leq C_m^n \|b\|_{BMO^d} m_K(|\phi|w) (\sqrt{|K|/|L|}) \sqrt{2\mu_K^s} (m_K w m_K w^{-1})^{\frac{-s}{2}}. \end{aligned}$$

The first inequality is true because $|b_I|/\sqrt{|I|} \leq \|b\|_{BMO^d}$ and the second one because

$$1 \leq C_m^n \left(\frac{|\Delta_K w|}{m_K w} + \frac{|\Delta_K w^{-1}|}{m_K w^{-1}} \right) \sqrt{|K|} \leq C_m^n \sqrt{2\mu_K^s} (m_K w m_K w^{-1})^{\frac{-s}{2}}.$$

Now we use the fact, proved in Lemma 3.7, that we can compare the averages of the weights on the stopping intervals with their averages in L , paying a price of a constant e , and continue estimating by

$$\sqrt{2} C_m^n e^s \|b\|_{BMO^d} m_K(|\phi|w) \sqrt{|K|/|L|} \sqrt{\mu_K^s} (m_L w m_L w^{-1})^{\frac{-s}{2}}.$$

If K is a stopping time interval by the second criterion, then the sum collapses to just one term

$$\begin{aligned} &\sum_{I \in \mathcal{D}(K) \cap \mathcal{D}_n(L)} |b_I| m_I(|\phi|w) \sqrt{|I|/|L|} \\ &= |b_K| m_K(|\phi|w) \sqrt{|K|/|L|} \\ &= m_K(|\phi|w) \sqrt{|K|/|L|} \sqrt{\mu_K^{b,s}} (m_K w m_K w^{-1})^{\frac{-s}{2}} \\ &\leq C_m^n e^s m_K(|\phi|w) \sqrt{|K|/|L|} \sqrt{\mu_K^{b,s}} (m_L w m_L w^{-1})^{\frac{-s}{2}}. \end{aligned}$$

Let $\Xi_1(L) := \{K \in \mathcal{ST}_L^n : K \text{ is a stopping time interval by criterion 1}\}$, and $\Xi_2(L) := \{K \in \mathcal{ST}_L^n : K \text{ is a stopping time interval by criterion 2}\}$. Note that $\Xi_1(L) \cup \Xi_2(L)$ is a partition of L . We then have

$$(4.10) \quad Pb_L^{w,n} \leq \sqrt{2} C_n^m e^s (m_L w m_L w^{-1})^{-\frac{s}{2}} \left(\|b\|_{BMO^d} \Sigma_{Pb}^1 + \Sigma_{Pb}^2 \right),$$

where the terms Σ_{Pb}^1 and Σ_{Pb}^2 are defined as follows,

$$\Sigma_{Pb}^1 := \sum_{K \in \Xi_1(L)} m_K (|\phi|w) \sqrt{|K|/|L|} \sqrt{\mu_K^s},$$

$$\Sigma_{Pb}^2 := \sum_{K \in \Xi_2(L)} m_K (|\phi|w) \sqrt{|K|/|L|} \sqrt{\mu_K^{b,s}}.$$

Now estimate Σ_{Pb}^1 using the Cauchy-Schwarz inequality, noting that we can move a power $p/2 < 1$ from outside to inside the sum, and that $\mu_L^{n,s} := \sum_{I \in \mathcal{ST}_L^n} \mu_K^s \geq \sum_{I \in \Xi_1(L)} \mu_K^s$,

$$(4.11) \quad \begin{aligned} \Sigma_{Pb}^1 &\leq \left(\sum_{K \in \Xi_1(L)} (m_K (|\phi|w))^2 |K|/|L| \right)^{\frac{1}{2}} \left(\sum_{K \in \Xi_1(L)} \mu_K^s \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{K \in \Xi_1(L)} (m_K (|\phi|w))^p (|K|/|L|)^{\frac{p}{2}} \right)^{\frac{1}{p}} \sqrt{\mu_L^{n,s}}. \end{aligned}$$

By the second stopping criterion $|K|/|L| = 2^{-j}$ for $0 \leq j \leq m$, then

$$(4.12) \quad (|K|/|L|)^{\frac{p}{2}} = 2^{-j + \frac{j}{2(m+n+2)}} < 2 \cdot 2^{-j} = 2|K|/|L|.$$

Plugging (4.12) into (4.11) gives

$$\Sigma_{Pb}^1 \leq \left(2 \sum_{K \in \Xi_1(L)} (m_K (|\phi|w))^p |K|/|L| \right)^{\frac{1}{p}} \sqrt{\mu_L^{n,s}}.$$

Use Hölder's inequality inside the sum, then Lift Lemma 3.7, to get

$$\begin{aligned} \Sigma_{Pb}^1 &\leq \left(\sum_{K \in \Xi_1(L)} (m_K (|\phi|^p w))(m_K w)^{p-1} |K|/|L| \right)^{\frac{1}{p}} \sqrt{\mu_L^{n,s}} \\ &\leq 2^{\frac{1}{p}} (e m_L w)^{1-\frac{1}{p}} \left(\frac{1}{|L|} \sum_{K \in \Xi_1(L)} \int_K |\phi(x)|^p w(x) dx \right)^{\frac{1}{p}} \sqrt{\mu_L^{n,s}}. \end{aligned}$$

Observe that the intervals $K \in \Xi_1(L)$ are disjoint subintervals of L , therefore, $\sum_{K \in \Xi_1(L)} \int_K |\phi(x)|^p w(x) dx \leq \int_L |\phi(x)|^p w(x) dx$, thus,

$$(4.13) \quad \Sigma_{Pb}^1 \leq 2e m_L w \inf_{x \in L} \left(M_w(|\phi|^p)(x) \right)^{\frac{1}{p}} \sqrt{\mu_L^{n,s}}.$$

Similarly we estimate Σ_{Pb}^2 , to get

$$\begin{aligned} \Sigma_{Pb}^2 &\leq \left(\sum_{K \in \Xi_2} (m_K(|\phi|w))^2 |K|/|L| \right)^{\frac{1}{2}} \left(\sum_{K \in \Xi_2} \mu_K^{b,s} \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{K \in \mathcal{ST}_L^n} (m_K(|\phi|w))^p (|K|/|L|)^{\frac{p}{2}} \right)^{\frac{1}{p}} \sqrt{\mu_L^{b,n,s}}. \end{aligned}$$

Following the same steps as we did in the estimate for Σ_{Pb}^1 , we will have

$$(4.14) \quad \Sigma_{Pb}^2 \leq 2e m_L w \inf_{x \in L} \left(M_w(|\phi|^p)(x) \right)^{\frac{1}{p}} \sqrt{\mu_L^{b,n,s}}.$$

Insert estimates (4.13) and (4.14) into (4.10). Altogether, we can bound $Pb_L^{w,n}$ by

$$C C_n^m e^{s+1} (m_L w)^{1-\frac{s}{2}} (m_L w^{-1})^{\frac{-s}{2}} \inf_{x \in L} \left(M_w(|\phi|^p)(x) \right)^{\frac{1}{p}} \nu_L^{n,s}.$$

The lemma is proved. □

Remark 4.5. In [NV2], Nazarov and Volberg extend the results that they had for Haar shift operators in [NV1] to metric spaces with geometric doubling. One can extend Theorem 4.3 to this setting as well, see [Mo2].

5. Haar Multipliers

For a weight w , $t \in \mathbb{R}$, and $m, n \in \mathbb{N}$, a t -Haar multiplier of complexity (m, n) is the operator defined as

$$(5.1) \quad T_{t,w}^{m,n} f(x) := \sum_{L \in \mathcal{D}} \sum_{(I,J) \in \mathcal{D}_m^n(L)} c_{I,J}^L \left(\frac{w(x)}{m_L w} \right)^t \langle f, h_I \rangle h_J(x),$$

where $|c_{I,J}^L| \leq \sqrt{|I||J|}/|L|$.

Note that these operators have symbols, namely $c_{I,J}^L(w(x)/m_L w)^t$, that depend on: the space variable x , the frequency encoded in the dyadic interval L , and the complexity encoded in the subintervals $I \in$

$\mathcal{D}_n(L)$ and $J \in \mathcal{D}_m(L)$. This makes these operators akin to pseudodifferential operators where the trigonometric functions have been replaced by the Haar functions.

Observe that $T_{t,w}^{m,n}$ is different from both $S^{m,n}T_w^t$ and $T_w^tS^{m,n}$ and, that, unlike $T_{t,w}^{m,n}$, both $S^{m,n}T_w^t$ and $T_w^tS^{m,n}$ obey the same bound that T_w^t obeys in $L^2(\mathbb{R})$, because the Haar shift multipliers have L^2 -norm less than or equal to one.

5.1. Necessary conditions. Let us first show a necessary condition on the weight w so that the Haar multiplier $T_{w,t}^{m,n}$ with $c_{I,J}^I = \sqrt{|I||J|}/|L|$ is bounded on $L^p(\mathbb{R})$. This necessary C_{tp}^d -condition is the same condition found in [KP] for the t -Haar multiplier of complexity $(0, 0)$.

Theorem 5.1. *Let w be a weight, m, n positive integers and t a real number. If $T_{t,w}^{m,n}$ is the t -Haar multiplier with $c_{I,J}^I = \sqrt{|I||J|}/|L|$ and is a bounded operator in $L^p(\mathbb{R})$, then w is in C_{tp}^d .*

Proof: Assume that $T_{t,w}^{m,n}$ is bounded in $L^p(\mathbb{R})$ for $1 < p < \infty$. Then there exists $C > 0$ such that for any $f \in L^p(\mathbb{R})$ we have $\|T_{t,w}^{m,n}f\|_p \leq C\|f\|_p$. Thus for any $I_0 \in \mathcal{D}$ we should have

$$(5.2) \quad \|T_{t,w}^{m,n}h_{I_0}\|_p^p \leq C^p\|h_{I_0}\|_p^p.$$

Let us compute the norm on the left-hand side of (5.2). Observe that

$$(5.3) \quad T_{t,w}^{m,n}h_{I_0}(x) = \sum_{L \in \mathcal{D}} \sum_{(I,J) \in \mathcal{D}_m^n(L)} \sqrt{|I||J|}/|L| \left(\frac{w(x)}{m_L w}\right)^t \langle h_{I_0}, h_I \rangle h_J(x).$$

We have $\langle h_{I_0}, h_I \rangle = 1$ if $I_0 = I$ and $\langle h_{I_0}, h_I \rangle = 0$ otherwise. Also, there exists just one dyadic interval L_0 such that $I_0 \subset L_0$ and $|I_0| = 2^{-n}|L_0|$. Therefore we can collapse the sums in (5.3) to just one sum, and calculate the L^p -norm as follows,

$$\|T_{t,w}^{m,n}h_{I_0}\|_p^p = \int_{\mathbb{R}} \left| \sum_{J \in \mathcal{D}_m(L_0)} \sqrt{|I_0||J|}/|L_0| \left(\frac{w(x)}{m_{L_0} w}\right)^t h_J(x) \right|^p dx.$$

Furthermore, since $\mathcal{D}_m(L_0)$ is a partition of L_0 , the power p can be put inside the sum, and we get

$$(5.4) \quad \|T_{t,w}^{m,n}h_{I_0}\|_p^p = (|I_0|^{\frac{n}{2}}/|L_0|^{p-1})(m_{L_0}w^{tp}/(m_{L_0}w)^{pt}).$$

Inserting $\|h_{I_0}\|_p^p = |I_0|^{1-\frac{2}{p}}$ and (5.4) in (5.2), we will have that for any dyadic interval I_0 there exists C such that

$$(|I_0|^{\frac{n}{2}}/|L_0|^{p-1})(m_{L_0}w^{tp}/(m_{L_0}w)^{pt}) \leq C^p|I_0|^{1-\frac{2}{p}}.$$

Thus, $m_{L_0} w^{tp}/(m_{L_0} w)^{pt} \leq C^p |I_0|^{1-p} |L_0|^{p-1} = C^p 2^{n(p-1)} =: C_{n,p}$. Now observe that this inequality should hold for any $L_0 \in \mathcal{D}$, we just have to choose as I_0 any of the descendants of L_0 in the n -th generation, and that n is fixed. Therefore,

$$[w]_{C_{2t}^d} = \sup_{L \in \mathcal{D}} (m_L w^{tp}) (m_L w)^{-pt} \leq C_{n,p}.$$

We conclude that $w \in C_{tp}^d$; moreover $[w]_{C_{tp}^d} \leq 2^{n(p-1)} \|T_{t,w}^{m,n}\|_p^p$. \square

5.2. Sufficient condition. For most $t \in \mathbb{R}$, the C_{2t}^d -condition is not only necessary but also sufficient for a t -Haar multiplier of complexity (m, n) to be bounded on $L^2(\mathbb{R})$; this was proved in [KP] for the case $m = n = 0$. Here we are concerned not only with the boundedness but also with the dependence of the operator norm on the C_{2t}^d -constant. For the case $m = n = 0$ and $t = 1, \pm 1/2$ this was studied in [Pe2]. Beznosova [Be1] was able to obtain estimates, under the additional condition on the weight: $w^{2t} \in A_p^d$ for some $p > 1$, for the case of complexity $(0, 0)$ and for all $t \in \mathbb{R}$. We generalize her results when $w^{2t} \in A_2^d$ for complexity (m, n) . Our proof differs from hers in that we are adapting the methods of Nazarov and Volberg [NV1] to this setting as well. Both proofs rely on the α -Lemma (Lemma 3.4) and on the Little Lemma (Lemma 3.2). See also [BMP].

Theorem 5.2. *Let t be a real number and w a weight in C_{2t}^d , such that $w^{2t} \in A_2^d$. Then $T_{t,w}^{m,n}$, a t -Haar multiplier with depth (m, n) , is bounded in $L^2(\mathbb{R})$. Moreover,*

$$\|T_{t,w}^{m,n} f\|_2 \leq C(m+n+2)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_2^d}^{\frac{1}{2}} \|f\|_2.$$

Proof: Fix $f, g \in L^2(\mathbb{R})$. By duality, it is enough to show that

$$|\langle T_{t,w}^{m,n} f, g \rangle| \leq C(m+n+2)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_2^d}^{\frac{1}{2}} \|f\|_2 \|g\|_2.$$

The inner product on the left-hand-side can be expanded into a double sum that we now estimate,

$$|\langle T_{t,w}^{m,n} f, g \rangle| \leq \sum_{L \in \mathcal{D}} \sum_{(I, J) \in \mathcal{D}_m^n(L)} (\sqrt{|I||J|}/|L|) \frac{|\langle f, h_I \rangle|}{(m_L w)^t} |\langle g w^t, h_J \rangle|.$$

Decompose h_J into a linear combination of a weighted Haar function and a characteristic function, $h_J = \alpha_J h_J^{w^{2t}} + \beta_J \chi_{J/\sqrt{|J|}}$, where $\alpha_J = \alpha_J^{w^{2t}}$, $\beta_J = \beta_J^{w^{2t}}$, $|\alpha_J| \leq \sqrt{m_J w^{2t}}$, and $|\beta_J| \leq |\Delta_J(w^{2t})|/m_J w^{2t}$. Now we break this sum into two terms to be estimated separately so that,

$$|\langle T_{t,w}^{m,n} f, g \rangle| \leq \Sigma_3^{m,n} + \Sigma_4^{m,n},$$

where

$$\Sigma_3^{m,n} := \sum_{L \in \mathcal{D}} \sum_{(I,J) \in \mathcal{D}_m^n(L)} \frac{\sqrt{|I||J|}}{|L|} \frac{\sqrt{m_J(w^{2t})}}{(m_L w)^t} |\langle f, h_I \rangle| |\langle gw^t, h_J^{w^{2t}} \rangle|,$$

$$\Sigma_4^{m,n} := \sum_{L \in \mathcal{D}} \sum_{(I,J) \in \mathcal{D}_m^n(L)} \frac{|J| \sqrt{|I|}}{|L| (m_L w)^t} \frac{|\Delta_J(w^{2t})|}{m_J(w^{2t})} |\langle f, h_I \rangle| m_J(|g|w^t).$$

Again, let $p = 2 - (C_m^n)^{-1}$, and define as in (4.4) and (4.5), the quantities $S_L^{v,m} \phi$ and $R_L^{v,m} \phi$, with $v = w^{2t}$. Let

$$P_L^n \phi := \sum_{I \in \mathcal{D}_n(L)} |\langle f, h_I \rangle| \sqrt{|I|/|L|},$$

and

$$\eta_I := m_I(w^{2t}) m_I(w^{-2t}) \left(\frac{|\Delta_I(w^{2t})|^2}{|m_I w^{2t}|^2} + \frac{|\Delta_I(w^{-2t})|^2}{|m_I w^{-2t}|^2} \right) |I|.$$

By Lemma 3.6 with $s = 1$, $\{\eta_I\}_{I \in \mathcal{D}}$ is a Carleson sequence with intensity $C[w^{2t}]_{A_2^d}$. Let $\eta_L^m := \sum_{I \in \mathcal{ST}_L^m} \eta_I$, where the stopping time \mathcal{ST}_L^m is defined as in Lemma 3.7 (with respect to the weight w^{2t}). By Lemma 3.8, $\{\eta_L^m\}_{L \in \mathcal{D}}$ is a Carleson sequence with intensity $C(m+1)[w^{2t}]_{A_2^d}$.

Observe that on the one hand $\langle gw^t, h_J^{w^{2t}} \rangle = \langle gw^{-t}, h_J^{w^{2t}} \rangle_{w^{2t}}$, and on the other $m_J(|g|w^t) = m_J(|gw^{-t}|w^{2t})$. Therefore,

$$\Sigma_3^{m,n} = \sum_{L \in \mathcal{D}} (m_L w)^{-t} S_L^{w^{2t},m}(gw^{-t}) P_L^n f,$$

$$\Sigma_4^{m,n} = \sum_{L \in \mathcal{D}} (m_L w)^{-t} R_L^{w^{2t},m}(gw^{-t}) P_L^n f.$$

Estimates (4.7) and (4.8) with $s = 1$ hold for $S_L^{w^{2t},m}(gw^{-t})$ and $R_L^{w^{2t},m}(gw^{-t})$, with w^{-1} and g replaced by w^{2t} and gw^{-t} :

$$S_L^{w^{2t},m}(gw^{-t}) \leq (m_L w^{2t})^{\frac{1}{2}} \left(\sum_{J \in \mathcal{D}_m(L)} |\langle gw^{-t}, h_J^{w^{2t}} \rangle_{w^{2t}}|^2 \right)^{\frac{1}{2}},$$

$$R_L^{w^{2t},m}(gw^{-t}) \leq C C_m^n (m_L w^{2t})^{\frac{1}{2}} (m_L w^{-2t})^{\frac{-1}{2}} F^{\frac{1}{2}}(x) \sqrt{\eta_L^m},$$

where $F(x) = \inf_{x \in L} (M_{w^{2t}}(|gw^{-t}|^p)(x))^{\frac{2}{p}}$. Estimating $P_L^n f$ is simple:

$$P_L^n f \leq \left(\sum_{I \in \mathcal{D}_n(L)} |I|/|L| \right)^{\frac{1}{2}} \left(\sum_{I \in \mathcal{D}_n(L)} |\langle f, h_I \rangle|^2 \right)^{\frac{1}{2}} = \left(\sum_{I \in \mathcal{D}_n(L)} |\langle f, h_I \rangle|^2 \right)^{\frac{1}{2}}.$$

Estimating $\Sigma_3^{m,n}$: Plug in the estimates for $S_L^{w^{2t},m}(gw^{-t})$ and $P_L^n f$, observing that $(m_L w^{2t})^{\frac{1}{2}}/(m_L w)^t \leq [w]_{C_{2t}^d}^{\frac{1}{2}}$. Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \Sigma_3^{m,n} &\leq \sum_{L \in \mathcal{D}} [w]_{C_{2t}^d}^{\frac{1}{2}} \left(\sum_{J \in \mathcal{D}_m(L)} |\langle gw^{-t}, h_J^{w^{2t}} \rangle_{w^{2t}}|^2 \right)^{\frac{1}{2}} \left(\sum_{I \in \mathcal{D}_n(L)} |\langle f, h_I \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_2 \|gw^{-t}\|_{L^2(w^{2t})} = [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_2 \|g\|_2. \end{aligned}$$

Estimating $\Sigma_4^{m,n}$: Plug in the estimates for $R_L^{w^{2t},m}(gw^{-t})$ and $P_L^n f$, where $F(x) = (M_{w^{2t}}(|gw^{-t}|^p)(x))^{2/p}$. Using the Cauchy-Schwarz inequality and considering again that $(m_L w^{2t})^{\frac{1}{2}}/(m_L w)^t \leq [w]_{C_{2t}^d}^{\frac{1}{2}}$, then

$$\Sigma_4^{m,n} \leq C C_m^n [w]_{C_{2t}^d}^{\frac{1}{2}} \|f\|_2 \left(\sum_{L \in \mathcal{D}} \frac{\eta_L^m}{m_L w^{-2t}} \inf_{x \in L} F(x) \right)^{\frac{1}{2}}.$$

Now, use the Weighted Carleson Lemma 3.1 with $\alpha_L = \eta_L^m/m_L(w^{-2t})$ (which by Lemma 3.2 is a w^{2t} -Carleson sequence with intensity at most $C C_m^n [w^{2t}]_{A_2^d}$). Let $F(x) = (M_{w^{2t}}|gw^{-t}|^p(x))^{2/p}$, and $v = w^{2t}$, then

$$\Sigma_4^{m,n} \leq C(C_m^n)^2 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_2^d}^{\frac{1}{2}} \|f\|_2 \|M_{w^{2t}}(|gw^{-t}|^p)\|_{L^{\frac{2}{p}}(w^{2t})}^{\frac{1}{2}}.$$

Using (2.1), that is the boundedness of $M_{w^{2t}}$ in $L^{\frac{2}{p}}(w^{2t})$ for $2/p > 1$, and $(2/p)' = 2C_m^n$, we get

$$\begin{aligned} \Sigma_4^{m,n} &\leq C(C_m^n)^2 (2/p)' [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_2^d}^{\frac{1}{2}} \|f\|_2 \| |gw^{-t}|^p \|_{L^{\frac{2}{p}}(w^{2t})}^{\frac{1}{2}} \\ &\leq C(C_m^n)^3 [w]_{C_{2t}^d}^{\frac{1}{2}} [w^{2t}]_{A_2^d}^{\frac{1}{2}} \|f\|_2 \|g\|_2. \end{aligned}$$

The theorem is proved. \square

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Primera versió rebuda el 16 d'agost de 2011,
darrera versió rebuda el 12 de febrer de 2013.