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DEGREE OF THE FIRST INTEGRAL OF A PENCIL IN $\mathbb{P}^2$ DEFINED BY LINS NETO

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Abstract: Let $\mathcal{P}_4$ be the linear family of foliations of degree 4 in $\mathbb{P}^2$ introduced by A. Lins Neto, whose set of parameter with first integral $I_p(\mathcal{P}_4)$ is dense and countable. In this work, we will compute explicitly the degree of the rational first integral of the foliations in this linear family, as a function of the parameter.

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1. Introduction

One of the main problems in the theory of planar vector fields is to characterize the ones which admit a first integral. The invariant algebraic curves are a central object in integrability theory since 1878, year when Darboux found connections between algebraic curves and the existence of first integrals of polynomial vector fields. Thus, the first question was to know if a polynomial vector field has or not invariant algebraic curves, which was partially answered by Darboux in [**4**]. The most important improvements of Darboux's results were given by Poincaré in 1891, who tried to answer the following question:

> "Is it possible to decide if a foliation in $\mathbb{P}^2$ has a rational first integral?"

This problem is known as the *Poincaré Problem.* In [**14**], he observed that it is sufficient to bound the degree of a possible algebraic solution. By imposing conditions on the singularities of the foliation he obtains necessary conditions which guarantee the existence of a rational first integral. More recently, this problem has been reformulated as follows: given a foliation on $\mathbb{P}^2$, try to bound the degree of the generic solution using information depending only on the foliation, for example on its degree or on the eigenvalues of its singularities.

Several authors studied this problem, see for instance [**1**], [**2**], [**5**], [**16**].

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In 2002, Lins Neto [**9**] built some notable linear families of foliations (which he later called *pencil* [**10**]) in $\mathbb{P}^2$, where the set of parameters in which the foliation has a first integral is dense and countable. The importance of these families is that there is no bound depending only on the degree and the analytic type of their singularities. One of such families is the pencil $\mathcal{P}_4 = \{\mathcal{F}^4_\alpha\}_{\alpha\in\overline{\mathbb{C}}}$, where $\mathcal{F}^4_\alpha$ is defined by the 1-form $\omega + \alpha\eta$, with

$$\begin{aligned} \omega &= (x^3-1)x\,dy - (y^3-1)y\,dx, \\ \eta &= (x^3-1)y^2\,dy - (y^3-1)x^2\,dx. \end{aligned}$$

It is well known that $I_p(\mathcal{P}_4)$, the set of parameters of foliations in $\mathcal{P}_4$ which have a first integral, is the imaginary quadratic field $\mathbb{Q}(\tau_0)$, where $\tau_0 = e^{2\pi i/3}$.

The purpose of this work is to compute the degree of the foliations $\mathcal{F}_\alpha$ in $\mathcal{P}_4$ with rational first integral as a function of α. For this, we first relate the pencil $\mathcal{P}_4$ with a pencil of linear foliations $\mathcal{P}_4^*$ in a complex torus $E\times E$, where $E = \mathbb{C}/\langle 1,\tau_0\rangle$. Then we derive the formula of the degree using the ideal norm of the ring $\mathbb{Z}[\tau_0]$ as sketched below. Consequently, we are able to address the Poincaré Problem for the foliations in $\mathcal{P}_4$.

Given a foliation $\mathcal{F}_t \in \mathcal{P}_4$, with $t \in I_p(\mathcal{P}_4)$ there exists a unique corresponding foliation $\mathcal{G}_{\alpha(t)} \in \mathcal{P}_4^*$ where $\alpha(t) = \frac{t-1}{-2-\tau_0}$. Then writing $\alpha(t) = \frac{\beta_1}{\alpha_1}$, with $\alpha_1, \beta_1 \in \mathbb{Z}[\tau_0]$ and $(\alpha_1,\beta_1) = 1$, we have proved the following result:

Theorem. *If d_t is the degree of the first integral of $\mathcal{F}^4_t$ then*

$$d_t = N(\beta_1) + N(\alpha_1) + N(\beta_1-\alpha_1) + N(\beta_1+\tau_0\alpha_1),$$

where $N(\beta) = a^2+b^2-ab$, for $\beta = a+\tau_0 b \in \mathbb{Z}[\tau_0]$.

Furthermore, we show that the growth of the counting function $\pi_{\mathcal{P}_4}$ (see Section 4.2), which associates to every $n\in\mathbb{N}$, the number of elements in $I_p(\mathcal{P}_4)$ for which the corresponding foliation has a first integral of degree at most n, has quadratic order, that is

$$\pi_{\mathcal{P}_4}(n) = O(n^2).$$

This is an improvement over a previous result due to Pereira [**13**, Proposition 4].

2. Preliminaries

From now on, τ_0 will denote the complex number $e^{2\pi i/3}$. Let $\mathcal{F}$ be a foliation associated with the 1-form ω. Given a singularity p of $\mathcal{F}$, let λ_1 and λ_2 be the eigenvalues of the linear part of the vector field associated to ω. Recall that p is of *type* $(a:b)$ if $[\lambda_1:\lambda_2] = [a:b] \in \mathbb{P}^1$. Moreover, if p is of type $(1:1)$ then p is called a *radial singularity*.

2.1. The pencil $\mathcal{P}_4$ in $\mathbb{P}^2$ and the configuration $\mathcal{C}$. In [**9**, §2.2], Lins Neto defines the pencil $\mathcal{P}_4 = \{\mathcal{F}^4_\alpha\}_{\alpha\in\overline{\mathbb{C}}}$ of degree 4 in $\mathbb{P}^2$, where $\mathcal{F}^4_\alpha$ is defined by the 1-form $\omega + \alpha\eta$, with

$$\omega = (x^3-1)x\,dy - (y^3-1)y\,dx,$$
$$\eta = (x^3-1)y^2\,dy - (y^3-1)x^2\,dx.$$

Let us state some properties of the pencil $\mathcal{P}_4$:

(1) The *tangency set* of $\mathcal{P}_4$, given by $\omega \wedge \eta = 0$, is the algebraic curve

$$\Delta(\mathcal{P}_4) = \big\{[x:y:z] \in \mathbb{P}^2 : (x^3-z^3)(y^3-z^3)(x^3-y^3) = 0\big\}.$$

This curve is composed by nine invariant lines such that the set of intersections of these lines is formed by twelve points. We will denote such lines and points by $\mathcal{L} = \{L_1, \ldots, L_9\}$ and $P = \{e_1, \ldots, e_{12}\}$, respectively.

(2) If $\alpha \notin \{1, \tau_0, \tau_0^2, \infty\}$ then $\mathcal{F}^4_\alpha$ has 21 non-degenerated singularities, nine of them are simple singularities of type $(-3:1)$, and the remaining twelve are radial singularities contained in P. Recall that in general, if a foliation has degree k, then it has k^2+k+1 singularities [**11**]. In view of this, $\mathcal{F}^4_\alpha$ has degree 4.

(3) If $\alpha \in \{1, \tau_0, \tau_0^2, \infty\}$ then $\mathrm{Sing}(\mathcal{F}^4_\alpha) = P$.

Consider $\mathcal{C} = \{\mathcal{L}, P\}$ to be the configuration of the above stated twelve points and nine lines in $\mathbb{P}^2$. These are shown graphically in Figure 1.

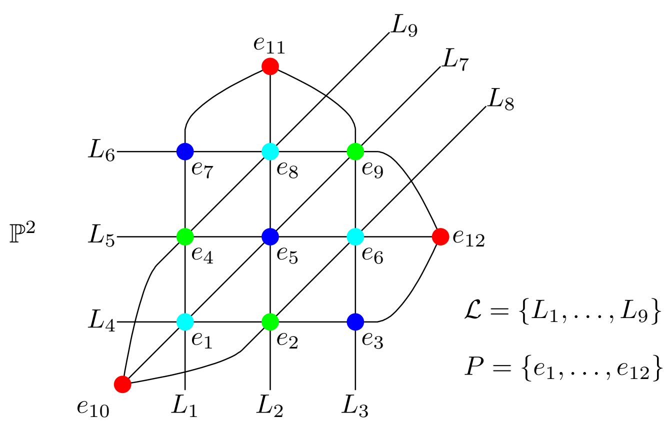

FIGURE 1.

2.2. The pencil $\mathcal{P}_4^*$ and the configuration $\mathcal{C}^*$. Let $E_0 = \mathbb{C}/\Gamma_0$, with $\Gamma_0 = \langle 1, \tau_0 \rangle$, and $X_0 = E_0 \times E_0$. Take a system of coordinates (x, y) in $\mathbb{C}^2$ and let $\phi\colon \mathbb{C}^2 \to X_0$ be the natural projection.

Let $\mathcal{P}_4^*$, using ϕ, the pencil X_0 induced by the linear pencil in $\mathbb{C}^2$

$$\omega_\alpha = dy - \alpha\, dx, \quad \alpha \in \mathbb{C}. \tag{1}$$

Let $\varphi\colon X_0 \to X_0$ be the holomorphic map defined by $\varphi(x, y) = (\tau_0 x, \tau_0 y)$. Denote as $\mathrm{Fix}(\varphi)$, the set fixed points of φ. Consider $p_1 = 0$, $p_2 = \frac{2}{3} + \frac{1}{3}\tau_0$ and $p_3 = \frac{1}{3} + \frac{2}{3}\tau_0$. Then it holds,

(1) $\varphi^3 = \mathrm{id}_X$.

(2) $\mathrm{Fix}(\varphi)$ has nine elements, namely, $\big\{(p_l, p_k)\big\}_{l,k=1}^{3}$.

Now consider the four elliptic curves in X_0,

$$E_{0,1} = \{0\} \times E_0, \quad E_{1,1} = \big\{(x, x) : x \in E_0\big\},$$
$$E_{1,0} = E_0 \times \{0\}, \quad E_{1,-\tau_0} = \big\{(x, -\tau_0 x) : x \in E_0\big\},$$

and let $\mathscr{C}$ be the set of these curves. For $F \in \mathscr{C}$ and $p \in \mathrm{Fix}(\varphi)$, denote $F_p = F + p$. Hence, the set $\mathcal{E} := \{F_p : p \in \mathrm{Fix}(\varphi),\, F \in \mathscr{C}\}$ consists of twelve elliptic curves. Since $\varphi(F_p) = F_p$ and $\mathrm{Fix}(\varphi) \cap F_p = (\mathrm{Fix}(\varphi) \cap F) + p$, we conclude that two different elliptic curves intersect only in three fixed points of φ.

Denote $\mathcal{E} = \{E_1, \ldots, E_{12}\}$ and $\mathrm{Fix}(\varphi) = \{l_1, \ldots, l_9\}$. Consider $\mathcal{C}^* = \big(\mathrm{Fix}(\varphi), \mathcal{E}\big)$ to be the configuration of points and elliptic curves in X_0. These are shown graphically in Figure 2.

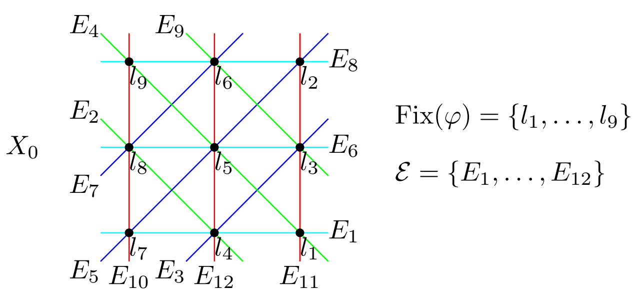

Figure 2.

3. Relation between the pencils $\mathcal{P}_4^*$ and $\mathcal{P}_4$

The relation between the pencils $\mathcal{P}_4^*$ and $\mathcal{P}_4$ was given by McQuillan in [**12**], where he proved the following result.

Proposition 1 ([**12**, Example IV.3.0]). *There exist a rational map* $g\colon X_0 \dashrightarrow \mathbb{P}^2$ *such that* $g^*(\mathcal{P}_4) = \mathcal{P}_4^*$.

We now give an idea of how the function g is constructed. We refer the reader to [**15**] for the details. Let $\pi\colon \widetilde{X} \to X_0$ be obtained from X_0 by blowing-up the nine fixed points of φ, and denote $D_k = \pi^{-1}(l_k)$, for $k = 1, \dots, 9$. So, there is an automorphism $\widetilde{\varphi}\colon \widetilde{X} \to \widetilde{X}$ such that $\pi \circ \widetilde{\varphi} = \varphi \circ \pi$. If we define $\widetilde{Y} = \widetilde{X}/\langle\widetilde{\varphi}\rangle$ then $\widetilde{Y}$ is a smooth rational surface. Moreover, the quotient map $\widetilde{h}\colon \widetilde{X} \to \widetilde{Y}$ is a finite morphism with degree 3 and its ramification divisor is $R = \sum_{i=1}^{9} 3D_k$.

Since $\widetilde{h}|_{D_i}\colon D_i \to \widetilde{D}_i := \widetilde{h}(D_i)$ is a biholomorphism, the rational map $\widetilde{h}$ maps D_i onto a rational curve $\widetilde{D}_i$, with self-intersection -3, for $i = 1, \dots, 9$. Furthermore, the map $\widetilde{h}$ sends π^*E_i, the strict transformation of E_i, into a rational curve denoted by $\widetilde{E}_i$, with self-intersection -1, as shown in Figure 3.

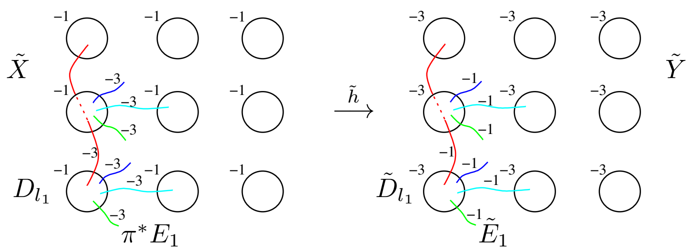

FIGURE 3.

Let $\pi_1\colon \widetilde{Y} \to Y_0$ be the blowing-down of the curves $\widetilde{E}_1, \dots, \widetilde{E}_{12}$. The following lemma holds.

Lemma 2. *With the previous notations we have that* $Y_0 = \mathbb{P}^2$.

Proof: By the Riemann-Hurwitz formula for surfaces we have

$$c_2(\widetilde{X}) = 3c_2(\widetilde{Y}) - \sum_{i=1}^{9} 2\chi(D_k),$$

where $c_2(\widetilde{X}) = 9$ and $\chi(D_k) = 2$, for $k = 1, \ldots, 9$. This implies that $c_2(\widetilde{Y}) = 15$ and $c_2(Y_0) = 3$. Therefore, by the Noether formula (cf. [**6**]), Y_0 is a minimal surface with $c_2(y_0) = 3$. Recall that the only minimal rational surfaces are $\mathbb{P}^2$ and the Hirzebruch surfaces S_n, for $n \neq 1$. We conclude that $Y_0 = \mathbb{P}^2$, since $c_2(\mathbb{P}^2) = 3$ and $c_2(S_n) \geq 4$, for $n \neq 1$. □

Let g the rational map defined by (see Figure 4)

$$g = \pi_1^{-1} \circ \widetilde{h} \circ \pi \colon X_0 \dashrightarrow Y_0 = \mathbb{P}^2.$$

Then g contracts each elliptic curve of $\mathcal{C}^*$ into a point in $\mathbb{P}^2$ and sends each point of $\mathcal{C}^*$ into an algebraic curve L in $\mathbb{P}^2$ with self-intersection one. In particular, L is a line in $\mathbb{P}^2$. Thus, the configuration of lines and points $g(C^*)$ consists in twelve points and nine lines of $\mathbb{P}^2$, denoted by $\mathcal{E}_*$ and $\mathrm{Fix}(\varphi)_*$, respectively, satisfying the following properties:

(1) each line in $\mathrm{Fix}(\varphi)_*$ contains four points of $\mathcal{E}_*$;

(2) each point of $\mathcal{E}_*$ belongs to two lines of $\mathrm{Fix}(\varphi)_*$;

(3) if three points of $\mathcal{E}_*$ are not in a line in $\mathrm{Fix}(\varphi)_*$ then the points are not aligned.

Then, according to [**9**, Proposition 1], unless an automorphism of $\mathbb{P}^2$, we can suppose that this configuration is $\mathcal{C} = \big(\mathrm{Fix}(\varphi)_*, \mathcal{E}_*\big)$, that has been described in Section 2.2.

Since $g \colon X_0 \to \mathbb{P}^2$ is a rational map, such that $g^*(\mathcal{P}_4) = \mathcal{P}_4^*$, we obtain

$$I_p(\mathcal{P}_4) = I_p(\mathcal{P}_4^*). \tag{2}$$

We will see that is easier to obtain properties of the leaves of $\mathcal{P}_4^*$ (see Section 3.1), because they are elliptic curves.

Recall that, if $\alpha \in \overline{\mathbb{C}}$ is fixed, the foliation $\mathcal{G}_\alpha \in \mathcal{P}_4^*$ in X_0 is induced by $\omega_\alpha = dy - \alpha\, dx$. Since the 1-form ω_α is φ-invariant, $\mathcal{G}_\alpha$ induces a foliation $g_*(\mathcal{G}_\alpha)$ in $\mathbb{P}^2$ as shown in Figure 4. Moreover, all the lines of $\mathrm{Fix}(\varphi)_*$ are invariant by $g_*(\mathcal{G}_\alpha)$. Then, by [**9**, §2.2], there exists $\Lambda(\alpha) \in \overline{\mathbb{C}}$ such that

$$g_*(\mathcal{G}_\alpha) = \mathcal{F}^4_{\Lambda(\alpha)}, \tag{3}$$

where $\mathcal{F}^4_{\Lambda(\alpha)} \in \mathcal{P}_4$. In particular, by (3), Λ is a rational map.

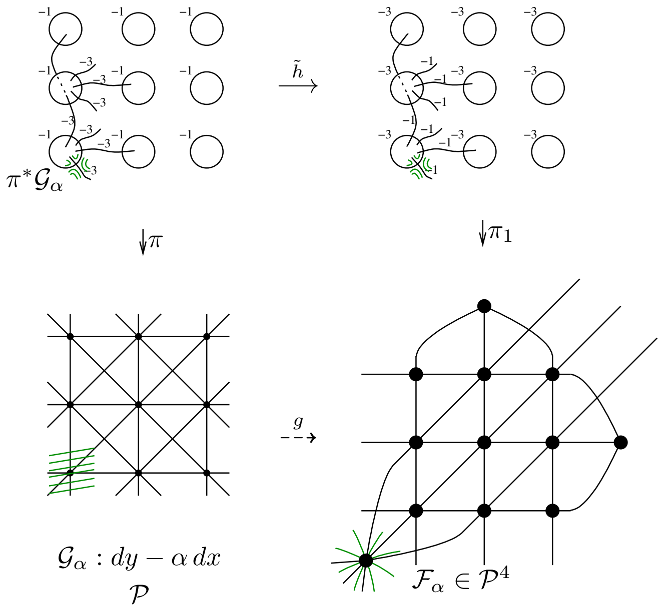

FIGURE 4.

Proposition 3. *The rational function* $\Lambda\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *is a Möbius map defined by* $\Lambda(\alpha) = (\tau_0^2 - 1)\alpha + 1$.

Proof: Since $\mathcal{F}_{\Lambda(0)}$, $\mathcal{F}_{\Lambda(1)}$, $\mathcal{F}_{\Lambda(-\tau_0)}$ and $\mathcal{F}_{\Lambda(\infty)}$ have twelve singularities, we have

$$\big\{\Lambda(0), \Lambda(1), \Lambda(-\tau_0), \Lambda(\infty)\big\} = \{1, \tau_0, \tau_0^2, \infty\}.$$

Moreover, the configurations $\mathcal{C}^*$ in X and $\mathcal{C}$ in $\mathbb{P}^2$ (see Figures 1 and 2), imply

$$g^*(\mathcal{F}^4_\infty) = \mathcal{G}_\infty, \quad g^*(\mathcal{F}^4_1) = \mathcal{G}_0,$$
$$g^*(\mathcal{F}^4_{\tau_0^2}) = \mathcal{G}_1, \quad g^*(\mathcal{F}^4_{\tau_0}) = \mathcal{G}_{-\tau_0}.$$

Let L_α be a generic leaf of $\mathcal{G}_\alpha$. Then $g(L_\alpha)$ is a leaf of $\mathcal{F}^4_\alpha$, that intersects $g(C_*)$ in singularities of $\mathcal{F}^4_\alpha$. Thus, $\Lambda\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is an injective function such that $\Lambda(\infty) = \infty$, $\Lambda(0) = 1$, $\Lambda(1) = \tau_0^2$ and $\Lambda(-\tau_0) = \tau_0$. Therefore $\Lambda(\alpha) = (\tau_0^2 - 1)\alpha + 1 = (-2 - \tau_0)\alpha + 1$. $\square$

Remark 4. If there exists an automorphism of $\mathbb{P}^2$ preserving the configuration $\mathcal{C} = \big(\mathcal{E}_*, \mathrm{Fix}(\varphi)_*\big)$ of points and lines, then Λ is a Möbius map such that

$$\big\{\Lambda(0), \Lambda(1), \Lambda(-\tau_0), \Lambda(\infty)\big\} = \{1, \tau_0, \tau_0^2, \infty\}.$$

3.1. The set $I_p(\mathcal{P}_4^*)$ and some notions on elliptic curves. Fix an elliptic curve $E = \mathbb{C}/\Gamma$, where $\Gamma = \langle 1, \tau\rangle$ and let $X = E \times E$. Let $\mathcal{P} = \{\mathcal{H}_\alpha\}_{\alpha\in\overline{\mathbb{C}}}$ be the pencil of linear foliations in X induced by the pencil of linear foliations in $\mathbb{C}^2$, given by the 1-forms

$$\omega_\alpha = dy - \alpha\, dx, \quad \alpha \in \mathbb{C}. \tag{4}$$

Let us define the set $I_p(\mathcal{P})$ as

$$I_p(\mathcal{P}) := \{\alpha \in \overline{\mathbb{C}} : \mathcal{H}_\alpha \text{ has an meromorphic first integral}\}.$$

For $\alpha \in \mathbb{C}\setminus\{0\}$, let $L_\alpha = \big\{\big(\pi(x), \pi(\alpha x)\big) : x \in \mathbb{C}\big\}$ be the leaf of $\mathcal{H}_\alpha$ passing through $(0,0)$. Thus,

$$\begin{aligned}\#\big(L_\alpha \cap (\{0\}\times E)\big) < \infty &\iff \exists\, k \in \mathbb{N} : k\alpha(m+\tau n) \in \Gamma,\ \forall\, m,n \in \mathbb{Z},\\ &\iff \exists\, k \in \mathbb{N} : k\Gamma(\alpha) \subset \Gamma,\ \text{where } \Gamma(\alpha) = \alpha\Gamma.\end{aligned}$$

By Jouanolou's Theorem [**8**], the latter implies that, for $\alpha \in \mathbb{C}\setminus\{0\}$, $\mathcal{H}_\alpha$ has an meromorphic first integral if, and only if, there exists $k \in \mathbb{N}$ such that $k\Gamma(\alpha) \subset \Gamma$. So we have proved the following proposition.

Proposition 5. *Let $\mathcal{P} = \{\mathcal{H}_\alpha\}_{\alpha\in\overline{\mathbb{C}}}$ be a pencil of linear foliations in X, as above. Then*

$$I_p(\mathcal{P}) = (\mathbb{Q} + \tau\mathbb{Q}) \cup \{\infty\}.$$

In particular,

$$I_p(\mathcal{P}_4^*) = (\mathbb{Q} + \tau_0\mathbb{Q}) \cup \{\infty\} = \mathbb{Q}(\tau_0) \cup \{\infty\} = I_p(\mathcal{P}_4).$$

To calculate the degree of the first integral of a foliation in $\mathcal{F}_\alpha^4$ it is sufficient to calculate the autointersection of two leaves of $\mathcal{G}_\alpha$. Thus, we need to recall some properties of elliptic curves.

Let $K \subset \mathbb{C}$ be an *algebraic number field* and let $\mathcal{O}_K$ be the ring of algebraic integers contained in K. Given an ideal I of $\mathcal{O}_K$, we consider the quotient ring $\mathcal{O}_K/I$ which is finite (cf. [**17**, p. 106]). The *ideal norm* of I, denoted by $N_{\mathcal{O}_K}(I)$, is the cardinality of $\mathcal{O}_K/I$.

The *Dedekind Zeta function* of K is defined, for a complex number s with $\mathrm{Re}(s) > 1$, by the Dirichlet series

$$\zeta_K(s) = \sum_{I\subset\mathcal{O}_K} \frac{1}{N_{\mathcal{O}_K}(I)^s},$$

where I ranges through the non-zero ideals of the ring of integers $\mathcal{O}_K$ of K. This sum converges absolutely for all complex numbers s with $\operatorname{Re}(s) > 1$. Note that $\zeta_{\mathbb{Q}}$ coincides with the Riemann zeta function.

Let $E = \mathbb{C}/\Gamma$ be an elliptic curve, where $\Gamma = \langle 1, \tau\rangle$ and define $\operatorname{End}(E) := \operatorname{Hom}(E, E)$. It is well known that the field $\operatorname{End}(E) \otimes \mathbb{Q}$ is isomorphic to a number field K such that $\mathcal{O}_K \simeq \operatorname{End}(E)$. For $\alpha, \beta \in \operatorname{End}(E)$, let us define the morphism $\varphi_{\alpha,\beta}\colon E \to E \times E$ as

$$\varphi_{\alpha,\beta}(x) = (\alpha x, \beta x),$$

where $\alpha x := \alpha(x)$. Note that the image $E_{\alpha,\beta}$ of $\varphi_{\alpha,\beta}$ is an elliptic curve. Given $\alpha, \beta, \gamma, \delta \in \operatorname{End}(E)$, then the *intersection number* of the elliptic curves $E_{\alpha,\beta}$ and $E_{\gamma,\delta}$ is given by

$$E_{\alpha,\beta} \cdot E_{\gamma,\delta} = \frac{N_{\mathfrak{O}_K}\left(\det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)}{N_{\mathfrak{O}_K}(\alpha,\beta) N_{\mathfrak{O}_K}(\gamma,\delta)}, \tag{5}$$

where $N_{\mathcal{O}_K}(a_1, \ldots, a_r)$ is the norm of the ideal generated by $a_1, \ldots, a_r \in \operatorname{End}(E)$ (cf. [**7**, Lemma 3]).

As an application let us consider the following example:

Example 6. Let $\tau_0 = e^{2\pi i/3}$. If $E = \mathbb{C}/\langle 1, \tau_0\rangle$ then $\operatorname{End}(E) \simeq \mathbb{Z}[\tau_0]$. For $\alpha = a + \tau_0 b \in \mathbb{Z}[\tau_0]$, the norm of the ideal $\langle\alpha\rangle$ is given by $N_{\mathbb{Z}[\tau_0]}(\alpha) = |\alpha|^2 = a^2 + b^2 - ab$. From equation (5), for $\alpha, \beta, \gamma, \delta \in \mathbb{Z}[\tau_0]$ such that $(\alpha, \beta) = 1$ and $(\gamma, \delta) = 1$, the intersection number of the elliptic curves $E_{\alpha,\beta}$ and $E_{\gamma,\delta}$ is

$$E_{\alpha,\beta} \cdot E_{\gamma,\delta} = N_{\mathbb{Z}_{\tau_0}}(\alpha\delta - \beta\gamma). \tag{6}$$

4. Degree of the first integral of a foliation $\mathcal{F}_t^4$, $t \in \mathbb{Q}(\tau_0)$

4.1. Computing the degree of the first integral of $\mathcal{F}_t^4$, $t \in \mathbb{Q}(\tau_0)$. Let $\mathcal{F}_t^4 \in \mathcal{P}_4$, with $t \in \mathbb{Q}(\tau_0)$. Then the foliation $g^*(\mathcal{F}_t^4) \in \mathcal{P}_4^*$ is equal to $\mathcal{G}_{\Lambda^{-1}(t)}$. Since $\mathbb{Z}[\tau_0]$ is a unique factorization domain, we can choose $\alpha_1, \beta_1 \in \mathbb{Z}[\tau_0]$ such that $(\alpha_1, \beta_1) = 1$ and $\alpha = \frac{\beta_1}{\alpha_1}$. In particular, $\mathcal{G}_\alpha$ is induced by the 1-form $\omega = \alpha_1\, dy - \beta_1\, dx$. Furthermore, $f_{\alpha_1,\beta_1} = \alpha_1 y - \beta_1 x$ is a first integral of $\mathcal{G}_\alpha$ and

$$E_{\alpha_1,\beta_1} = \big\{(\alpha_1 x, \beta_1 x) : x \in E\big\}$$

is the leaf of $\mathcal{G}_\alpha$ passing through $(0,0)$.

Let F_t be the rational first integral of $\mathcal{F}_t^4$ and d_t be the degree of F_t. We want to determine d_t. For this, take a generic irreducible fiber C of F_t, of degree d_t. We can suppose that $C^* := g^*(C) = E_{\alpha_1,\beta_1} + p$, where $p \notin \operatorname{Fix}(\varphi)$. Set $C^*_{1,0} := E_{1,0} + p$ in X_0 and let $C_{1,0} = g(C^*_{1,0})$

be the curve obtained in $\mathbb{P}^2$. The idea for computing d_t is to find the relation between the intersection of C and $C_{1,0}$ in $\mathbb{P}^2$ and the intersection of C^* and $C^*_{1,0}$ in X_0 (see Figure 5).

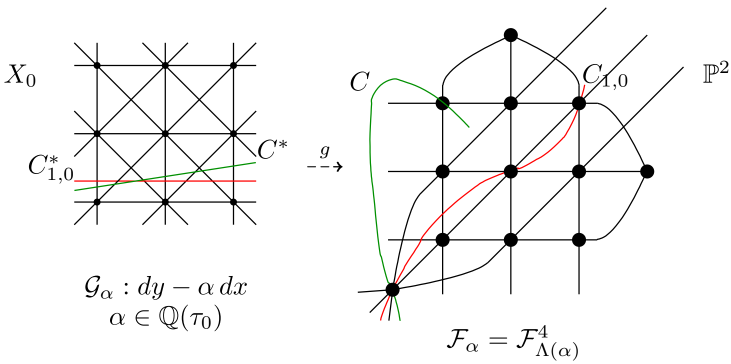

FIGURE 5.

We observe that

$$d_t \deg(C_{1,0}) = C \cdot C_{1,0} = \pi_1^*(C) \cdot \pi_1^*(C_{1,0}). \tag{7}$$

Let $\widetilde{C}$ and $\widetilde{C}_{1,0}$ the strict transforms of C and $C_{1,0}$ by π_1, respectively, then

$$\pi_1^*(C) = \widetilde{C} + \sum_{p \in \mathcal{E}_* \cap C} m_p D_p, \tag{8}$$

where m_p is the multiplicity of C in p and $D_p = \pi_1^{-1}(p)$. Furthermore,

$$\pi_1^*(C_{1,0}) = \widetilde{C}_{1,0} + \sum_{p \in \mathcal{E}_* \cap C_{1,0}} D_p, \tag{9}$$

where $\mathcal{E}_* \cap C_{1,0} = \mathcal{E}_* \setminus \{e_1, e_6, e_8\}$.

Since $C_{1,0} \cap L_7 = \{e_{10}, e_5, e_9\}$ (see Figure 1), where e_{10}, e_5, e_9 are radial singularities of $\mathcal{F}^4_1$ and $\widetilde{C}_{1,0}$ is a regular curve, we obtain that the intersection multiplicity of the points in $C_{1,0} \cap L_7$ is 1. So, by Bezout's Theorem, $\deg(C_{1,0}) \deg(L_7) = 3 = \deg(C_{1,0})$. Thus, combining (8) and

(9) in (7), we obtain

$$3d_t = \widetilde{C}\cdot\widetilde{C}_{1,0} + \sum_{p\in\mathcal{E}_*\cap C_{1,0}} \widetilde{C}\cdot D_p + \sum_{p\in\mathcal{E}_*\cap C} m_p\widetilde{C}_{1,0}\cdot D_p - \sum_{p\in\mathcal{E}_*\cap C_{1,0}} m_p$$

$$= \widetilde{C}\cdot\widetilde{C}_{1,0} + \sum_{p\in\mathcal{E}_*\cap C_{1,0}} \widetilde{C}\cdot D_p + \sum_{p\in\mathcal{E}_*\cap C_{1,0}} m_p\widetilde{C}_{1,0}\cdot D_p - \sum_{p\in\mathcal{E}_*\cap C_{1,0}} m_p.$$

Now, given $p \in \mathcal{E}_* \cap C_{1,0}$, we have

$$\widetilde{C}_{1,0}\cdot D_p = C^*_{1,0}\cdot E_p = 1,$$
$$\widetilde{C}\cdot D_p = C^*\cdot E_p = m_p,$$

where $E_p \in \mathcal{E}$ is an elliptic curve in X_0 such that $g(E_p) = p$. Hence,

$$3d_t = \widetilde{C}\cdot\widetilde{C}_{1,0} + \sum_{p\in\mathcal{E}_*\cap C_{1,0}} C^*\cdot E_p. \tag{10}$$

Let $\widetilde{C^*}_{1,0}$ and $\widetilde{C^*}$, be the strict transforms of $C^*_{1,0}$ and C^* by π, respectively. Thus, $C^*\cdot C^*_{1,0} = \widetilde{C^*}\cdot\widetilde{C^*}_{1,0}$. Since $\widetilde{h}^*\widetilde{C} = 3\widetilde{C^*}$ and $\widetilde{h}^*\widetilde{C}_{1,0} = 3\widetilde{C^*}_{1,0}$, by the Projection Formula, we have

$$3C^*\cdot 3C^*_{1,0} = 3\widetilde{C^*}\cdot 3\widetilde{C^*}_{1,0} = (\widetilde{h}^*\widetilde{C}\cdot\widetilde{h}^*\widetilde{C}_{1,0}) = 3\widetilde{C}\cdot\widetilde{C}_{1,0},$$

where we conclude that

$$3C^*\cdot C^*_{1,0} = \widetilde{C}\cdot\widetilde{C}_{1,0}.$$

Using this in (10), we obtain

$$3d_t = 3C^*\cdot C^*_{1,0} + \sum_{p\in\mathcal{E}_*\cap C^*_{1,0}} C^*\cdot E_p$$

$$= 3C^*\cdot E_{1,0} + 3C^*\cdot E_{0,1} + 3C^*\cdot E_{1,1} + 3C^*\cdot E_{1,-\tau_0}.$$

Therefore

$$3d_t = 3E_{\alpha_1,\beta_1}\cdot E_{1,0} + 3E_{\alpha_1,\beta_1}\cdot E_{0,1} + 3E_{\alpha_1,\beta_1}\cdot E_{1,1} + 3E_{\alpha_1,\beta_1}\cdot E_{1,-\tau_0}.$$

Now, let us set $N(\alpha) := N_{\mathbb{Z}[\tau_0]}(\alpha)$, for $\alpha\in\mathbb{Z}(\tau_0)$. Using Example 6, we conclude

$$3d_t = 3N(-\beta_1) + 3N(\alpha_1) + 3N(\alpha_1-\beta_1) + 3N(-\alpha_1\tau_0-\beta_1).$$

Hence,

$$d_t = N(\beta_1) + N(\alpha_1) + N(\beta_1-\alpha_1) + N(\beta_1+\tau_0\alpha_1), \tag{11}$$

where $\Lambda(t) = \alpha = \frac{\beta_1}{\alpha_1}$. This proves our main result.

Remark 7. In (11), if we take $\alpha = a + \tau_0 b$ and $\beta = c + \tau_0 d$ then

$$d_t = d_t(a,b,c,d) = 3(a^2 + b^2 + c^2 + d^2 - ab - ac + ad - bd - cd).$$

In particular, d_t is a multiple of 3.

4.2. The growth of the pencil $\mathcal{P}_4$. In [**13**], Pereira defines the counting function π_C of an algebraic curve C included in $\mathbb{F}ol(2,d)$, the space of foliations in $\mathbb{P}^2$ of degree d. Take $\mathcal{P} = \{\mathcal{F}_\alpha\}_{\alpha\in\overline{\mathbb{C}}}$ a line in $\mathbb{F}ol(2,d)$, that is a pencil of foliations in $\mathbb{P}^2$. For $n \in \mathbb{N}$, denote

$$E_n = \{\alpha \in \overline{\mathbb{C}} : \mathcal{F}_\alpha \text{ have a first integral of degree at most } n\}.$$

Thus, the counting function of $\mathcal{P}$, $\pi_{\mathcal{P}}\colon \mathbb{N} \to \mathbb{N}\cup\{+\infty\}$ is defined, for $n \in \mathbb{N}$, as

$$\pi_{\mathcal{P}}(n) = \#E_n.$$

Also in [**13**], the author observes the importance of study the function $\pi_{\mathcal{P}}$ and shows the following example (cf. [**13**, Example 3]).

Example 8. Let $\mathcal{P} = \{\mathcal{F}_\alpha\}_{\alpha\in\overline{\mathbb{C}}}$ be a pencil in $\mathbb{P}^2$, where $\mathcal{F}_\alpha$ is given by

$$\alpha x\, dy - y\, dx.$$

In this case,

$$\alpha \in I_p(\mathcal{P}) \setminus \{\infty\} \quad \Longleftrightarrow \quad \alpha \in \mathbb{Q}.$$

Take $\alpha = \frac{p}{q} \in \mathbb{Q}$, with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $(p,q) = 1$. If $f_{p,q}$ is the first integral of $\mathcal{F}_\alpha$ of degree $d_{p,q}$ then

$$d_{p,q} = \begin{cases} \max\{p,q\}, & \text{if } p \geq 0, \\ |p| + q, & \text{if } p < 0. \end{cases}$$

Thus,

$$\pi_{\mathcal{P}}(n) = 2 + 3\sum_{j=1}^{n} \varphi(j),$$

where φ is the Euler totient function. Now, since

$$\sum_{j=1}^{n} \varphi(j) = \frac{3n^2}{\pi^2} + O\Big(n \ln(n)^{2/3} \ln\big(\ln(n)\big)^{4/3}\Big),$$

(cf. [**18**, p. 178]), we have

$$\lim_{n\to\infty} \frac{\pi_{\mathcal{P}}(n)}{n^2} = \frac{3}{\pi^2}.$$

Now, we will estimate $\pi_{\mathcal{P}_4}(n)$, for $n \in \mathbb{N}$, and see that the counting function $\pi_{\mathcal{P}^4}$ has the same behavior as in Example 8.

Corollary 9. *The function $\pi_{\mathcal{P}_4}(n)$ satisfies*

$$\pi_{\mathcal{P}_4}(n) = O(n^2).$$

Proof: Indeed, in this case

$$t \in I_p(\mathcal{P}_4) \quad \Longleftrightarrow \quad \Lambda^{-1}(t) = \alpha \in \mathbb{Q}(\tau_0) \cup \{\infty\},$$

where $\Lambda(\alpha) = (\tau_0^2 - 1)\alpha + 1$. Suppose that $\alpha = \frac{\beta_1}{\alpha_1}$, $\alpha_1, \beta_1 \in \mathbb{Z}[\tau_0]$. Then

$$\pi_{\mathcal{P}_4}(n) = \#\Big\{\big(\alpha_1, \beta_1\big) \in (\mathbb{Z}[\tau_0] \times \mathbb{Z}[\tau_0]) \setminus \{0\} : (\alpha_1, \beta_1) = 1,\, d_t \leq n\Big\},$$

where $d_t = N(\beta_1) + N(\alpha_1) + N(\beta_1 - \alpha_1) + N(\beta_1 + \tau_0\alpha_1)$. Let

$$\mathcal{E}_n = \#\left\{\big(\alpha_1, \beta_1\big) \in (\mathbb{Z}[\tau_0] \times I_p(\mathcal{P})) \setminus \{0\} : t = \frac{\beta_1}{\alpha_1}, \right.$$
$$\left. (\alpha_1, \beta_1) = 1,\, N(\alpha_1) \leq n,\, N(\beta_1) \leq n\right\},$$

then

$$\pi_{\mathcal{P}_4}(n) \leq \mathcal{E}_n, \quad \forall\, n \in \mathbb{N}.$$

Let $H(n) = \#\big\{I \text{ ideal in } \mathbb{Z}[\tau_0] : N_{\mathbb{Z}[\tau_0]}(I) \leq n\big\}$ then, by [**3**], we have

(1) $H(n) = cn + O(n^{1/2})$, where c is a constant;

(2) $\lim\limits_{n\to\infty} \frac{\mathcal{E}(n)}{H(n)^2} \leq \frac{1}{\zeta_{\mathbb{Q}(\tau_0)}(2)}$, where $\zeta_{\mathbb{Q}(\tau_0)}$ is the Dedekind Zeta function of $\mathbb{Q}(\tau_0)$ (see Section 2).

Therefore, by item (2), we conclude

$$\lim_{n\to\infty} \frac{\pi_{\mathcal{P}_4}(n)}{H(n)^2} \leq \frac{1}{\zeta_{\mathbb{Q}(\tau_0)}(2)}.$$

In particular, $\pi_{\mathcal{P}_4}(n) = O(n^2)$. □

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