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ON THE BAKER'S MAP AND THE SIMPLICITY OF THE HIGHER DIMENSIONAL THOMPSON GROUPS nV

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Abstract

We show that the baker's map is a finite product of transpositions (particularly pleasant involutions), and conclude from this that an existing very short proof of the simplicity of Thompson's group V applies with equal brevity to the higher dimensional Thompson groups nV.

1. Introduction

Of the original groups $F \subseteq T \subseteq V$ of Richard Thompson (see [**5**]), all are infinite and finitely presented, and the last two are simple. An infinite family of groups nV, $n \in \{1, 2, 3, \ldots, \omega\}$, of which $1V = V$, is introduced in [**2**] where it is shown that $2V$ is infinite and simple and not isomorphic to V. A finite presentation for $2V$ is given in [**3**], it is shown that nV and mV are isomorphic only when $m = n$ in [**1**], and metric properties of $2V$ are studied in [**4**].

A very short argument that $V = 1V$ is generated by transpositions is given in Section 12 of [**2**], followed by an equally short argument based on this fact (due to Rubin) that V is simple. It is also shown in that section that the baker's map (an element of $2V$) prevents the first argument from showing that $2V$ is generated by transpositions. As a result, the proof in [**2**] of the simplicity of $2V$ is rather involved and is based on calculations which show that the abelianization of $2V$ is trivial.

Here we give a short proof that the baker's maps in $2V$ are finite products of transpositions in $2V$. We also give some of the very brief details from [**2**] that show that this implies that $2V$ is simple. The

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extension of these arguments to show that all nV with $n \leq \omega$ are simple is left as an easy exercise for the reader.

Longer arguments for simplicity exist. Presentations (finite when $n < \omega$) for the nV, $n \leq \omega$, are given in [**6**], and one can calculate from these presentations that each of the nV, $n \leq \omega$, has trivial abelianization. From the arguments in Section 3 of [**2**] (which are about $2V$ but, as noted in 4.1 of [**2**], they apply as well to the nV, $n \leq \omega$) it then follows that each of the nV, $n \leq \omega$, is simple.

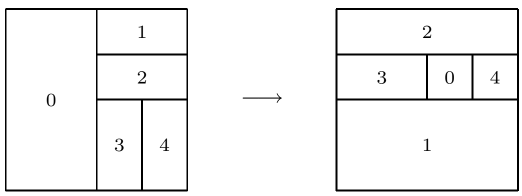
2. The group $2V$

We give a brief description of the group $2V$. See [**2**] for more detail.

The elements of $2V$ are permutations of the unit square $[0,1]^2$. They are almost homeomorphisms in that they are discontinuous only on a finite number of straight line segments. Each element is defined by a pair of numbered patterns in the unit square. We start by defining a single pattern, and then give an example of a pair of patterns that show how the pair is used to define an element of $2V$.

A pattern is obtained from the unit square by adding to the unit square a finite sequence of line segments so that each is either vertical or horizontal, and so that as each line segment in the sequence is added, it divides a rectangle created by previous line segments in the sequence exactly in half. We require that the number of rectangles in the two patterns be the same. Each pattern then has its rectangles numbered consecutively from 0 arbitrarily.

We illustrate with an example. Consider the following pair of patterns.



The pair of patterns above represents the following permutation of $[0,1] \times [0,1]$. For $i = 0, 1, 2, 3, 4$, the rectangle numbered i on the left is taken affinely to the rectangle numbered i on the right so that the right side goes to the right side, left side to left side, top to top and bottom to bottom. The map this defines on the entire unit square is ambiguous on the dividing lines and obviously cannot be continuous there. So we declare that we do not care what the function does on the dividing lines and consider two functions the same if they agree off a set of measure zero. Since the movement is from the left pattern to the right pattern, we call the left pattern the *domain pattern* and the right pattern the *range pattern.*

The group $2V$ is the set of all such (equivalence classes of) permutations given by pairs of diagrams. That this forms a group is an easy exercise. The group nV, $n \leq \omega$, is defined by patterns on $[0,1]^n$ where the divisions of rectangles (always exactly in half) are by codimension-1 rectangles, each parallel to one of the sides.

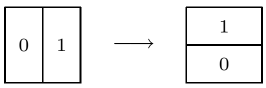
It is clear that a single function can be described by many pairs of patterns. For example both pairs below give the identity element.

(1)

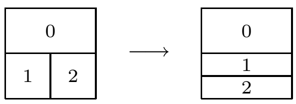
In the group $2V$, a *permutation* is an element that is given by two patterns that are identical except for the numbering. It is clear that such an element will have finite order in the group.

In $2V$, a *transposition* is given by a pair of identical patterns that are numbered identically except for a switch of two of the numbers. The transposition is *proper* if there are more than two rectangles in each pattern.

The following pair is not a permutation. It is called the (primary) *baker's map.* It has infinite order and is interesting for the well known property that for any positive integer n there is an orbit of the baker's map of size n. See [**2**].



A (secondary) baker's map is given by a pair of patterns that are identical and identically numbered with one exception: for one singly divided rectangle in the domain and for the corresponding singly divided rectangle in the range, the division is vertical in the domain and horizontal in the range. An example is below.



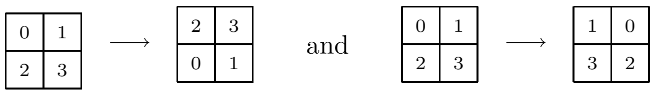
We refer to the smallest rectangle containing the non-identity part of the baker's map as the *support* of the baker's map.

3. Arguments

Lemmas 1 and 2 below are from [**2**]. Lemma 1 is a very mild extension of Lemmas 12.1 and 12.2 of [**2**].

Lemma 1. *The group* $2V$ *is generated by proper transpositions and baker's maps.*

Proof: The only non-proper transpositions are

(2) 

which are shown as products of proper transpositions. The permutations are generated by the transpositions, so we can use permutations in our argument.

Given an $\alpha \in 2V$, we will show that α can be multiplied by transpositions and baker's maps to become trivial.

(†) In a pair of patterns, the numbers in one of the patterns can be permuted by composing (before or after, as appropriate) by a permutation.

Each pattern in the pair for α is given by a sequence of line segments, and each pattern has a last rectangle that is divided in half by the last line segment. By renumbering using (†), we can assure that the two halves of the last rectangle divided in the domain pattern have the same numbering as the two halves of the last rectangle divided in the range pattern. Further if the last division is vertical in both, or is horizontal in both, then the numbers can agree left-right in first case, or top-bottom in the second. This will allow the last line segment to be eliminated from the domain and range patterns in a manner indicated in (1). If the last division is vertical in one pattern and horizontal in another, then multiplying by a baker's map will reduce to the case where both are vertical or both horizontal. An induction on the number of line segments completes the proof. □

The short proof of the next lemma can be found in the proof of Proposition 12.3 of [**2**], or the reader can show as an exercise that for any non-trivial $\alpha \in 2V$ some element in the form $[\gamma, [\beta, \alpha]]$ is a non-trivial proper permutation.

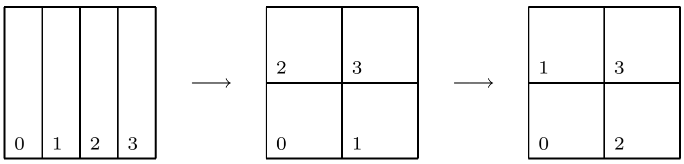
Lemma 2. *The normal closure in* $2V$ *of any non-trivial element contains a non-trivial proper transposition.* □

Since all proper transpositions in $2V$ are conjugate, we have that any normal subgroup of $2V$ contains all the permutations. As indicated in the Introduction, the only new fact needed to prove simplicity is the following.

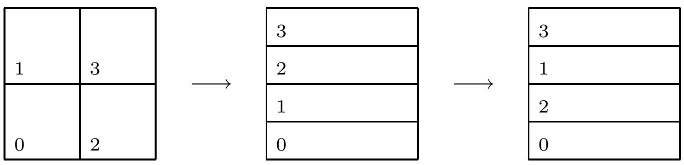
Lemma 3. *Any baker's map in $2V$ is a product of finitely many proper transpositions from $2V$.*

Proof: In the following, we write $\alpha \sim \beta$, or say α equals β modulo transpositions, to mean that elements α and β of $2V$ are equal modulo the normal subgroup generated by the proper transpositions. Since a conjugate of a proper transposition is a permutation and thus a product of proper transpositions, we have $\alpha \sim 1$ if and only if α is a product of proper transpositions.

1. *For any baker's map β whose support is a rectangle R there is a baker's map β_1 whose support is the left half of R and a baker's map β_2 whose support is the right half of R so that $\beta \sim \beta_1\beta_2$.* In the pictures below, we show only the support of β. The first arrow is the baker's map β in a reducible form. The second is a proper transposition. The composition is the promised product of two baker's maps.



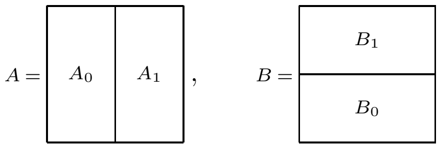
2. *For any baker's map β whose support is a rectangle R there is a baker's map β_1 whose support is the top half of R and a baker's map β_2 whose support is the bottom half of R so that $\beta \sim \beta_1\beta_2$.* The relevant pictures follow and the comments are as in 1.



3. *Any baker's map is, modulo transpositions, a product of arbitrarily small baker's maps.* By "arbitrarily small" we mean having support with

diameter smaller than an arbitrarily chosen positive real. This follows from 1 and 2.

4. *A product of a baker's map and an inverse of a baker's map with disjoint supports is a product of transpositions.* Let A and B be the disjoint supports. We refer to the rectangles in figures below to describe a sequence of transpositions. (a) Switch A_0 with B_0. (b) Switch A_1 with B_1. (c) Switch A with B. The composition of (a) with (b) with (c) in that order is the desired result.



5. *If R is a rectangle in a pattern so that neither side of R has length more than $\frac{1}{2}$, then the baker's map with support R is a product of transpositions.* The assumptions make R one half of a rectangle A that is not all of the unit square. Thus there is a rectangle B that is disjoint from A. Let S be the rectangle that is the "other half" of A. Let α be a product of a baker's map on A with an inverse of a baker's map on B. By 4, $\alpha \sim 1$. If α' is a baker's map on each of R and S and an inverse of a baker's map on B, then by 1 or 2 we have $\alpha' \sim \alpha$ so $\alpha' \sim 1$. Let β be a product of a baker's map on B and an inverse of a baker's map on S. By 4, $\beta \sim 1$. Now $\alpha'\beta$ is a baker's map on R and $\alpha'\beta \sim 1$.

The lemma follows from 3 and 5. □

It follows from Lemmas 1, 2 and 3 that $2V$ is generated by transpositions and is simple. The following generalization to all nV is left to the reader.

Theorem 1. *The nV, $n \leq \omega$, are generated by transpositions and are simple.*

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