

ON THE TOWER THEOREM FOR FINITE GROUPS

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Wielandt [2] has given a very ingenious proof of the fact that the tower of automorphisms of a finite group without center ends after a finite number of steps. Using his work as a model a proof of a similar tower theorem for Lie algebras was given in [1]. This depends on the following three facts:

(a) If A (with no center) is a member of the tower of derivation algebras of a Lie algebra L then the centralizer of L in A is (0) .

(b) If L is a subinvariant Lie algebra of A and if the centralizer of L in A is (0) then the centralizer of L^ω in A is contained in A .

(c) If L is subinvariant in A then L^ω is normal in A .

In view of the much sharper estimate obtained in the theorem on Lie algebras it seemed to be of interest to attempt to improve on the results of Wielandt using the method of [1]. The group theory analogue of (a) is to be found in Wielandt's work. I shall prove here the analogue to (b) and then show by a counter-example that the method is not applicable to get the tower theorem even for solvable groups since the analogue to (c) does not hold for groups even under the additional hypothesis of (b).

THEOREM. *If G is a subinvariant subgroup of the finite group A and if the centralizer of G in A is the identity, then the centralizer of G^ω in A is contained in G^ω . It follows that if N is normal in G such that G/N is nilpotent then $N \supset G^\omega$ and the centralizer of N is contained in N .*

Here

$$G^\omega = \bigcap_{k=1}^{\infty} G^k$$

where $G^k = [G^{k-1}, G]$ is the subgroup generated by commutators of the form $[h, g] = hgh^{-1}g^{-1}$, $h \in G^{k-1}$, $g \in G$.

The proof of the Theorem depends on two lemmas.

LEMMA 1. *If G is a finite group then $G = G^\omega H$ where H is a nilpotent subgroup of G .*

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LEMMA 2. *Let G be a group with center E ; then the centralizer W of G^ω in G is contained in G^ω . It follows that if N is normal in G such that G/N is nilpotent then $N \supseteq G^\omega$ and the centralizer of N is contained in N .*

I shall also give an example to show that under the hypotheses of the Theorem G^ω need not be normal in A , even with the added restriction that G be solvable.

Proof of Lemma 1. The proof is based on the fact that a group is nilpotent if and only if its Φ -subgroup contains the commutator subgroup G^2 [3, p. 114]. If G is nilpotent the theorem is trivially true since $G^\omega = E$ and $G = GE$. If G is not nilpotent then the Φ -subgroup does not contain G^2 . Accordingly we can pick a minimal set of generators g_1, \dots, g_k of G where at least one of the generators, g_k for definiteness, is in G^2 . Then g_1, \dots, g_{k-1} generate a proper subgroup K of G ; and $G = KG^2$. On the other hand G/G^ω is nilpotent and hence the Φ -subgroup of G/G^ω contains the commutator subgroup of G/G^ω . Accordingly $g_k G^\omega$ is not essential as a generator of G/G^ω and therefore $g_1 G^\omega, \dots, g_{k-1} G^\omega$ generate G/G^ω . It follows that $G = G^\omega K$.

Now we proceed by induction on the order of the group. Since K is a proper subgroup of G , its order is less than that of G and we can assume that $K = K^\omega H$ where H is a nilpotent group. Then

$$G = G^\omega K = G^\omega K^\omega H = G^\omega H$$

since K^ω is contained in G^ω and the lemma is proved.

Proof of Lemma 2. G^ω is normal in G and hence so also is W . By Lemma 1, $G = G^\omega H$ where H is a nilpotent group. Let $G_1 = WH$. Then G_1 is a group since W is normal and H is a group. Also $G_1 = G_1^\omega H_1$ where H_1 is nilpotent. But $G_1^\omega = (WH)^\omega$ is contained in W since H is nilpotent and W is normal. This can be seen by showing inductively that $(WH)^k \subseteq WH^k$. For let x and y be elements of W , h be in H , and k be in H^k . Then if xh is in WH and if yk is in WH^k

$$[xh, yk] = \underline{x} \underline{h} \underline{y} \underline{h}^{-1} \underline{h} \underline{k} \underline{h}^{-1} \underline{x}^{-1} \underline{h} \underline{k}^{-1} \underline{h}^{-1} \underline{h} \underline{k} \underline{h}^{-1} \underline{k}^{-1} \underline{y}^{-1} [h, k]^{-1} [h, k]$$

which is an element of WH^{k+1} , since the four underlined expressions are in W and $[h, k] \in H^{k+1}$.

Now if W is contained in G_1^ω then $W \subseteq G^\omega$ since $G_1^\omega \subseteq G^\omega$. Hence the lemma is false only if $W \not\subseteq G_1^\omega$. We need only consider therefore if there is an element w in W , w not in G_1^ω . We shall write $w = gh$ where g is in G_1^ω and h in H_1 . Of course $h \neq e$ since then w would be in G_1^ω . It follows that $h = g^{-1}w$ is in W , since $W \supseteq G_1^\omega$ and therefore $H_1 \cap W \neq E$. But $H_1 \cap W$ is normal in H_1 since W is normal in G . Thus $H_1 \cap W$

has intersection $P \neq E$ with the center of H_1 . But this will imply that P is in the center of G . For $G = G^\omega H = G^\omega H_1$ since $H \subseteq G_1^\omega H_1$; and P is in the centralizer of G^ω and in the center of H_1 . We have shown that if the centralizer of G^ω is not contained in G^ω then G has center not equal to E contradicting the hypothesis of the lemma.

Proof of the Theorem. By Lemma 2 we know that if Z is the centralizer of G^ω in A then $Z \cap G \subseteq G^\omega$ since otherwise G would have a non-trivial center. Now if Z is not contained in G^ω let K be the group generated by G and Z . G^ω is normal in K since

$$[G^\omega, G] \subseteq G^\omega \quad \text{and} \quad [G^\omega, Z] = E \subseteq G^\omega.$$

It follows that Z is normal in K and hence $K = ZG$. But G is sub-invariant in A , and hence in K . That is, G is a proper normal subgroup of G_1 , G_1 contained in K . Pick g_1 in G_1 but not in G . Since $K = ZG$, $g_1 = gz$ where g is in G , and z in Z . Furthermore $z = g^{-1}g_1$ is in G_1 and not in G . Now G and z generate a group $L = G\langle z \rangle$ since G is normal in G_1 . Also $L^\omega = G^\omega$; for

$$L/G^\omega = G/G^\omega \times \langle z \rangle G^\omega/G^\omega$$

and hence is nilpotent.

Now since $z \notin G^\omega = L^\omega$ it follows by Lemma 2 that L has a non-trivial center; but this is a contradiction of the fact that G has centralizer E in A . This completes the proof of the Theorem.

The counter-example mentioned earlier is as follows. Let H be the non-Abelian group of order 27 all of whose elements are of order 3; and let a and b be generators of H . Let σ and τ be automorphisms of H defined by $a^\sigma = a^2, b^\sigma = b$; and $a^\tau = a, b^\tau = b^2$. Let B be the holomorph of H with σ and τ and let G be the subgroup of B containing σ, a , and $[a, b]$. Then G is invariant in the subgroup containing G and b which subgroup in turn is invariant in B ; and it is easy to check that G^ω is the group generated by a and $[a, b]$.

Now let ρ be the automorphism of order 2 of B defined by $b^\rho = a, a^\rho = b, \sigma^\rho = \tau, \tau^\rho = \sigma$, and let A be the holomorph of B and ρ . Then G is subinvariant in A ; the centralizer of G in A is the identity, but clearly G^ω is not normal in A since $a^\rho = b$.

Omitting the hypothesis of solvability Professor Zassenhaus kindly furnished me with a similar example; in fact, an example of a group $G = G^2$ with trivial center such that the group of automorphisms of G is not complete.

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