ASYMPTOTIC LOWER BOUNDS FOR THE FUNDAMENTAL FREQUENCY OF CONVEX MEMBRANES

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1. Introduction. Let the bounded, simply connected, open region R of the (x, y)-plane have the boundary curve C. If a uniform ideal elastic membrane of unit density is uniformly stretched upon C with unit tension across each unit length, then λ, the square of the fundamental frequency, satisfies the conditions (subscripts denote differentiation)

$$\text{(1a)} \qquad \begin{cases} \Delta u \equiv u_{xx} + u_{yy} = -\lambda u \quad \text{in} \quad R , \\ \lambda = \text{minimum} , \end{cases}$$

with the boundary condition

$$\text{(1b)} \qquad u(x, y) = 0 \quad \text{on} \quad C .$$

Variational methods of the Rayleigh-Ritz type are frequently used to approximate λ. They always yield upper bounds for λ, and the upper bounds can be made arbitrarily close.

Another common practical method of approximating λ is to calculate the least eigenvalue λ_h of a suitably chosen finite-difference operator Δ_h over a network with small mesh width h. For one choice of Δ_h it was shown by Courant, Friedrichs, and Lewy [3, p. 57] without details that $\lambda_h \to \lambda$ as $h \to 0$. For convex regions R of a special polygonal form the author has shown [4] that a special case of (11) below is valid for a common choice of Δ_h, and hence that λ_h is asymptotically a lower bound for λ as $h \to 0$. For an unusual finite-difference approximation to problem (1) when R is the union of squares of the network, Polya [12] has found that $\lambda_h > \lambda$ for all h, and also for the higher eigenvalues. The author knows of no other study of the sign or order of decrease of $\lambda - \lambda_h$ to 0.

In the present paper the investigation of [4] is extended to a much wider class of regions: those with piecewise analytic boundary curves and convex corners. The new theorems are stated and proved in §§ 3 and 4. Theorem 2 contains the theorem of [4] as a special case. Lemmas used in the proof of Theorem 1 are given in § 5. Identity (31) of Lemma 7 is interesting in itself.

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When C is no longer made up of line segments of the network, it is necessary when using finite-difference methods either to move C or to alter Δ_h near the boundary. The latter procedure is potentially more accurate, and has been adopted in deriving the rather delicate results proved below. The definition of Δ_h given in § 2 is a self-adjoint modification of Mikeladze's approximation [10; 11], and is believed to be new. The cruder approximations to Δ near C proposed by Collatz in 1933 and expounded in [2, p. 357], while easier to compute in practice, appear to introduce an unmanageable term $O(h^2)$ into (19). It is therefore doubted that Theorem 2 would remain valid for these cruder operators.

The technique of the present paper could be applied to study the asymptotic behavior of λ_h also for other difference approximations to Δ in the interior of R—for example, for those associated with a triangular net [2, p. 367].

It is not clear that one could revise the argument of the paper to prove an inequality of the type

$$\frac{\lambda}{\lambda_h} \leq 1 + bh^2 + o(h^2) .$$

2. Definitions. Assume the bounded, simply connected, open region R to have a closed boundary curve C: $x(s)+iy(s)$ $(0 \leq s \leq s_m)$ which is *piecewise analytic*. That is, $x(s)$ and $y(s)$ are real analytic functions of the arc length s of C in each of a finite number m of closed intervals

$$0 = s_0 \leq s \leq s_1, \quad s_1 \leq s \leq s_2, \quad \cdots\cdot, \quad s_{m-1} \leq s \leq s_m .$$

Moreover, we demand that the corners of C be convex; that is, at any point $x(s_j)+iy(s_j)$ $(0 \leq j < m)$ where distinct analytic curves meet, the interior angle of C must be less than π.

For $h > 0$, let a *net* consist of the lines $x = \mu h$, $y = \nu h$ $(\mu, \nu = 0, \pm 1, \pm 2, \cdots)$. The points $(\mu h, \nu h)$ in R are the *interior nodes* R_h of the net. The *boundary nodes* C_h of the net consist of (i) all points $(\mu h, \nu h)$ on C, and (ii) all *isolated* points of intersection of the net with C. Thus each node $(\mu h, \nu h)$ of R_h has two *neighboring nodes* in $R_h \cup C_h$ on the line $x = \mu h$, and two in $R_h \cup C_h$ on the line $y = \nu h$. Moreover, each node in C_h has at least one neighbor in $R_h \cup C_h$.

We now move toward a definition of the difference operator Δ_h. Let us denote the neighboring nodes of the node

(2) $\quad (x, y)$ of R_h by $(x-h_1, y)$, $(x+h_2, y)$, $(x, y-h_3)$, and $(x, y+h_4)$,

where $0 < h_i \leq h$ for $i = 1, 2, 3, 4$. For nodes remote from C_h, all $h_i = h$. Let v be any net function defined on the nodes of $R_h \cup C_h$, vanishing on C_h. Define $D_x^{(h)}v$ as the (constant) second derivative of the quadratic polynomial function of x assuming the three values $v(x-h_1, y)$, $v(x, y)$, and $v(x+h_3, y)$. That is,

$$(3)\qquad D_x^{(h)}v(x, y)=\frac{2}{h_1+h_2}\left[\frac{v(x+h_2, y)-v(x, y)}{h_2}-\frac{v(x, y)-v(x-h_1, y)}{h_1}\right].$$

Also, $D_y^{(h)}v(x, y)$ is defined analogously. We next define

$$(4)\qquad \begin{aligned} \Delta^{(h)}v(x, y) &= D_x^{(h)}v(x, y)+D_y^{(h)}v(x, y) \\ &= -\left(\frac{2}{h_1h_2}+\frac{2}{h_3h_4}\right)v(x, y) \\ &\quad +\frac{2}{h_1(h_1+h_2)}v(x-h_1, y)+\frac{2}{h_2(h_1+h_2)}v(x+h_2, y) \\ &\quad +\frac{2}{h_3(h_3+h_4)}v(x, y-h_3)+\frac{2}{h_4(h_3+h_4)}v(x, y+h_4)\ . \end{aligned}$$

The operator $\Delta^{(h)}$ is the approximation to Δ recommended in [10]. It linearly transforms the net function v defined over R_h into the net function $\Delta^{(h)}v$, also defined over R_h. But $\Delta^{(h)}$ is not a self-adjoint linear operator; that is, the matrix $A^{(h)}$ of the linear transformation of v into $\Delta^{(h)}v$ is not symmetric.

We define the matrix A_h as the symmetric part of the matrix $A^{(h)}$:

$$(5)\qquad A_h=\tfrac{1}{2}[A^{(h)}+A^{(h)T}]\ ,$$

where T means transpose. Finally, *we define Δ_h to be the self-adjoint linear operator corresponding to A_h.*

The explicit expressions for Δ_h assume 16 different forms, depending on the location of (x, y) with respect to C_h. Although we shall not need these expressions for the present paper, we describe them briefly. If, in any of the four directions from (x, y), the neighboring node—say $(x-h_1, y)$, for definiteness—is in R_h, then $h_1=h$, and there is another node $(x-h-h_1', y)$ in $R_h\cup C_h$. Then the term $2v(x-h_1, y)/h_1(h_1+h_2)$ of (4) is to be replaced by

$$(6)\qquad \frac{h_1'+2h+h_2}{(h_1'+h)h(h+h_2)}v(x-h, y)\ .$$

For any (x, y), the expression for Δ_h is obtained from (4) by making replacements like (6) corresponding to all neighbors of (x, y) in R_h.

When (x, y) is more than two nodes away from C_h, so that all $h_i=h_i'=h$, the values of both $\Delta^{(h)}$ and Δ_h reduce to the familiar form used in [4]:

$$(7)\qquad \Delta_h v(x, y)=\Delta^{(h)} v(x, y) =\frac{1}{h^2}[v(x-h, y)+v(x+h, y)+v(x, y-h)+v(x, y+h)-4v(x, y)] .$$

Let λ_h satisfy the following difference equation for a net function v defined in $R_h \cup C_h$:

$$(8a)\qquad \begin{cases} \Delta_h v=-\lambda_h v & \text{in } R_h , \\ \lambda_h=\text{minimum} , \end{cases}$$

where v is extended to satisfy the boundary condition

$$(8b)\qquad v=0 \quad \text{on} \quad C_h .$$

It is readily shown that λ_h is the minimum over all net functions v satisfying (8b) of the quotient

$$\rho_h(v)=\frac{-h^2 \sum_{R_h} v\Delta_h v}{h^2 \sum_{R_h} v^2} .$$

(This is simply the minimum principle for a definite quadratic form.) By (5), we can write $\rho_h(v)$ in the following equivalent form, simpler to use:

$$(9)\qquad \rho_h(v)=\frac{-h^2 \sum_{R_h} v\Delta^{(h)} v}{h^2 \sum_{R_h} v^2} .$$

The reason for not using the least eigenvalue μ_h of $\Delta^{(h)}$ in this investigation is that μ_h does not have the foregoing minimum property and, in fact, might turn out to be complex. On the other hand, it is known [9, p. 27] that $\lambda_h \leq \mathscr{R}(\mu_h)$, so that when μ_h is real it could conceivably be a better approximation to λ than λ_h is. The relative magnitude of $|\lambda_h-\lambda|$ to $|\mu_h-\lambda|$ is not known.

3. The results.

The following new result will be proved in § 4:

THEOREM 1. *Let R be a bounded, open, simply connected region bounded by a piecewise analytic curve C whose corners are convex in the sense of § 2. Let τ be the angle between the tangent to C and the x axis. Let u solve problem* (1) *for R, and let u_n be the normal derivative of u on C. Define λ_h as in § 2. Let*

$$(10)\qquad a=a(R)=\frac{\iint_R (u_{xx}^2+u_{yy}^2)dxdy+\int_C u_n^2 \sin^2 2\tau\, d\tau}{12\iint_R (u_x^2+u_y^2)dxdy} .$$

Then $-\infty < a < \infty$ *and, as* $h \to 0$, *one has*

$$\frac{\lambda_h}{\lambda} \leq 1 - ah^2 + o(h^2) \qquad (h \to 0) . \tag{11}$$

In Theorem 1 the quantity a can probably be negative for certain nonconvex R, because $d\tau$ in (10) will be negative at some points of C. But if R is convex we get a stronger result, as an immediate consequence of Theorem 1.

THEOREM 2. *Under the hypotheses of Theorem* 1, *if* R *is also convex, then* $0 < a < \infty$, *and there exists* $h_0 > 0$ *such that* $\lambda_h < \lambda$ *for all* $h < h_0$.

For the operator Δ_h of § 2 the methods of [3] can undoubtedly be followed to show that $\lambda_h \to \lambda$ as $h \to 0$; the author has not attempted to carry through the details. When $\lambda_h \to \lambda$ as $h \to 0$, the lower bounds λ_{h_0} can be made arbitrarily close by choice of h_0 sufficiently small. Thus for these R the Rayleigh-Ritz methods and the finite-difference methods (8) are theoretically complementary, and together could confine λ to an arbitrarily short interval if one knew an upper bound for h_0.

The author has not developed an upper bound for h_0 in Theorem 2, although it would be desirable to do so by estimating the term $o(h^2)$. One could always make an intelligent guess based on the behavior of λ_h for certain h.

The constant a of (10) is the best possible for certain rectangular regions; see [4]. That the corners of C be convex seems essential to the validity of Theorem 1. Indeed, for one nonconvex polygon some heuristics and an experiment mentioned in [4] make it appear that $\lambda_h = \lambda + Ah^{4/3} + o(h^{4/3})$, where $A > 0$. It would be interesting to know the sign of a for the general case of Theorem 1, or in particular when C is a nonconvex analytic curve.

Corners of angle π are frequent in engineering practice, and it would be desirable to know how λ_h behaves when R has such corners. For such corners Lemma 2 is no longer valid. Lewy [7] provides new tools for an attack on corners of angle π.

4. Proof of Theorem 1.

Let u henceforth be the solution of problem (1) for the fundamental eigenvalue λ. It is known that

$$\lambda \iint_R u^2 dxdy = \iint_R (u_x^2 + u_y^2) dxdy . \tag{12}$$

The proof of Theorem 1, following [4], consists in setting the values of the function u at the nodes of $R_h \cup C_h$ into the Rayleigh quotient (9) of problem (8). It will be shown that

$$(13)\qquad \frac{\rho_h(u)}{\lambda}=1-ah^2+o(h^2) \qquad (h\to 0)\ .$$

Since $\lambda_h \leq \rho_h(u)$, the theorem follows from (13).

The denominator $h^2\sum u^2$ of $\rho_h(u)$ differs from a Riemann sum for $\iint_R u^2 dxdy$ at most by the terms corresponding to squares or part-squares at the boundary C. The total contribution of these terms does not exceed the order of magnitude $Lh \max_R u^2$, where L is the length of C. Hence a fortiori

$$(14)\qquad h^2\sum_{R_h} u^2=\iint_R u^2 dxdy+o(1) \qquad (h\to 0)\ .$$

Let the nodes of R_h be divided into three classes:

$$(15)\qquad \begin{cases} R_h^1: & \text{those within a distance } h \text{ of some corner of } C\,; \\ R_h^2: & \text{those not in } R_h^1 \text{ but within a distance } h \text{ of } C\,; \\ R_h^3: & \text{the other nodes of } R_h\,. \end{cases}$$

Split the numerator of $\rho_h(u)$ accordingly:

$$-h^2\sum_{R_h} u\Delta^{(h)}u=\sum_{i=1}^{3}\Big(-h^2\sum_{R_h^i} u\,\Delta^{(h)}u\Big)\equiv\sum_{i=1}^{3} S_h^i(u)\ .$$

There are a fixed number of corners, not exceeding m, and at most two nodes of R_h^1 per corner. Moreover $|\nabla u(x,y)|^2\to 0$ as $(x,y)\to$a corner of C, by Lemma 1 in § 5. At any node (x,y) of R_h^1 with neighbors denoted as in (2), we find from (3) that

$$h^2|u\Delta^{(h)}u|\leq\frac{h^2(u-0)}{\min h_i}\sum_{i=1}^{4}\left|\frac{u-u_i}{h_i}\right|\leq 4h^2\max|\nabla u|^2\ ,$$

where the u_i are the values of u at the four neighbors of (x,y), and where the maximum of $|\nabla u|^2$ is taken over all points within a distance $2h$ of some vertex. Hence

$$(16)\qquad |S_h^1(u)|\leq 8mh^2\max|\nabla u|^2=o(h^2) \qquad (h\to 0)\ .$$

Using the notation and assertion of Lemma 3, we have

$$(17)\qquad S_h^2(u)=-h^2\sum_{R_h^2} u\Delta u-\frac{2h^3}{3}\sum_{R_h^2} u(\theta_x u'_{xxx}+\theta_y u''_{yyy})\ .$$

Since u satisfies (1a),

$$(18)\qquad -h^2\sum_{R_h^2} u\Delta u=\lambda h^2\sum_{R_h^2} u^2\ .$$

By (17), (18), and Lemma 4,

$$|S_h^2(u)-\lambda h^2 \sum_{R_h^2} u^2| \leq \tfrac{2}{3}h^3 \sum_{R_h^2} u(|u'_{xxx}|+|u''_{yyy}|)=o(h^2) \qquad (h\to 0) .$$

Thus

$$S_h^2(u)=\lambda h^2 \sum_{R_h^2} u^2+o(h^2) \qquad (h\to 0) . \tag{19}$$

Similarly, using the notation and assertion of Lemma 5, and by (1a), we have

$$S_h^3(u)=\lambda h^2 \sum_{R_h^3} u^2-\frac{h^4}{12}\sum_{R_h^3} u(u'_{xxxx}+u''_{yyyy}) . \tag{20}$$

Now

$$h^2 \sum_{R_h^2 \cup R_h^3} u^2=h^2 \sum_{R_h} u^2-h^2 \sum_{R_h^1} u^2=h^2 \sum_{R_h} u^2+o(h^2) , \tag{21}$$

since $u(x, y)\to 0$ as $(x, y)\to C$, and since there are at most $2m$ vertices in R_h^1. Adding (19) and (20), and using (21), we find that

$$\begin{aligned} S_h^2(u)+S_h^3(u)&=\lambda h^2 \sum_{R_h} u^2-\frac{h^4}{12}\sum_{R_h^3} u(u'_{xxxx}+u''_{yyyy})+o(h^2) \\ &=\lambda h^2 \sum_{R_h} u^2-\frac{h^4}{12}\iint_R u(u_{xxxx}+u_{yyyy})dxdy+o(h^2) , \end{aligned}$$

by Lemma 6. Adding $S_h^1(u)$ to the above, and dividing by (14), we find that

$$\begin{aligned} \rho_h(u)&=\frac{\sum_{i=1}^{3} S_h^i(u)}{h^2 \sum_{R_h} u^2} \\ &=\lambda-\frac{h^2}{12}\,\frac{\iint_R u(u_{xxxx}+u_{yyyy})dxdy}{\iint_R u^2dxdy}+o(h^2) . \end{aligned} \tag{22}$$

Finally, dividing (22) by λ, and applying Lemma 7 and (12), one proves (13) and hence Theorem 1.

5. Some lemmas. The following lemmas are basic to the proof of Theorem 1. In all of them R satisfies the conditions stated at the start of § 2, while $u=u(x, y)$ solves problem (1).

LEMMA 1. *The function u is an analytic function of x and y in $R\cup C$, except possibly at the corners of C. Let r be the distance of (x, y) from a corner P with interior angle π/α, $1<\alpha<\infty$. Then for $m=0, 1, 2, \cdots$, any partial derivative of u of order m has the local representation*

(23) $$\frac{\partial^m u}{\partial x^\mu \partial y^\gamma} = r^{\alpha-m} f_m(x, y) \qquad (\mu+\gamma=m) ,$$

where f_m *is continuous at* P.

Proof. By [**1**, p. 179], u is analytic in R. The representation (27′) below shows that the interior normal derivative u_n is integrable on C. Then the analyticity of u on C (corners excluded) was shown by Hadamard [**5**, p. 25].[1]

Let $t=\xi+i\eta$ and $z=x+iy$. For each $t \in R$ let $w=\Phi(z, t)$ map R conformally onto the circle $|w|<1$, with $\Phi(t, t)=0$. We may assume without loss of generality that P is at $z=0$, and that $\Phi(0, t)=1$. Lichtenstein [**8**, pp. 255–256 and footnote 273] showed[2] that for $m=0$, $1, 2, \cdots$, and $z \in R$,

(24) $$\frac{\partial^m \Phi(z, t)}{\partial z^m} = z^{\alpha-m} \varphi_m(z, t) ,$$

where φ_m is continuous at $z=0$. It follows from (24) that

(25) $$\frac{\partial^m \log \Phi(z, t)}{\partial z^m} = z^{\alpha-m} \psi_m(z, t) ,$$

where ψ_m is continuous at $z=0$. Let $G(z, t)=G(\xi, \eta; x, y)$ be Green's function for Δu in R. Since

$$G(z, t) = -(2\pi)^{-1} \log |f(z, t)| ,$$

it follows from (25) that for $m=0, 1, 2, \cdots$ and $z \in R$,

(26) $$\frac{\partial^m G(z, t)}{\partial x^\mu \partial y^\nu} = r^{\alpha-m} \Psi_m(z, t) \qquad (\mu+\nu=m) ,$$

where Ψ_m is continuous at $z=0$.

Now the function u has the integral representation [**1**, pp. 182–183]

$$u(x, y) = \lambda \iint_R G(x, y; \xi, \eta) u(\xi, \eta) d\xi d\eta .$$

Hence

(27) $$\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x}$$

[1] The author wishes to thank Professor Lewy for this reference.

[2] Lichtenstein actually asserts that (24) is without question true for all α, but that his proof is valid only for irrational α. Warschawski [**13**] has found a simple proof of (24), valid for all α in the range $\frac{1}{2} \leq \alpha < \infty$.

Added in April 1954: For asymptotic expansions of Φ at a corner, see R. Sherman Lehmann, "Development of the mapping function at an analytic corner," Technical Report No. 21, Applied Mathematics and Statistics Laboratory, Stanford University, California, March 31, 1954, 17 pp.

$$=\lambda\iint_R \frac{G(x+\Delta x,\,y;\,\xi,\,\eta)-G(x,\,y;\,\xi,\,\eta)}{\Delta x}\,u(\xi,\,\eta)d\xi d\eta$$

$$=\lambda\iint_R \frac{\partial G}{\partial x}(x+\theta\Delta x,\,y;\,\xi,\,\eta)u(\xi,\,\eta)d\xi d\eta\ ,$$

where $0<\theta=\theta(x,\,y,\,\Delta x)<1$. Since $G(z,\,t)=G(t,\,z)$, it is clear that $\partial G/\partial x=\partial G/\partial\xi$ and, as a function of t, $\partial G/\partial x$ behaves like $|t-t_0|^{\alpha-1}$ at any corner t_0 of R, uniformly in z for z bounded away from C. Hence $(\partial G/\partial x)u(\xi,\,\eta)$ in (27) is dominated by an integrable function of ξ, η, uniformly with respect to Δx. By Lebesgue's convergence theorem, letting $\Delta x\to 0$ in (27) proves that

$$\frac{\partial u}{\partial x}=\lambda\iint_R \frac{\partial G}{\partial x}(x,\,y;\,\xi,\,\eta)u(\xi,\,\eta)d\xi d\eta\,. \tag{27$'$}$$

Setting the expression (26) for $m=\mu=1$ into the last equation proves the case $m=\mu=1$ of (23).

In a similar way one can prove all the cases $m=0,\,1,\,2,\,3,\,4$ of (23), and the lemma is established.

LEMMA 2. *The functions* u_{xx}^2, u_xu_{xxx}, uu_{xxxx}, u_{yy}^2, u_yu_{yyy}, *and* uu_{yyyy} *are Lebesgue integrable in* R. *The Lebesgue integrals* $\int_C u_xu_{xx}dy$ *and* $\int_C u_yu_{yy}dx$ *exist.*

Proof. By Lemma 1 the functions $u_{xx}^2,\,\cdots,\,uu_{yyyy}$ are continuous in $R\cup C$ except possibly at the corners, where they are $O(r^{2\alpha-4})$. Since $0<\alpha$, the first sentence follows. The second sentence is proved analogously.

REMARK. The proof of Lemma 2 breaks down for corners of angle $\pi\,(\alpha-1)$, as r^{-2} is not integrable.

LEMMA 3. *At any node* $(x,\,y)$ *of* R_h *whose neighbors are denoted as in* (2), *one has*

$$\Delta^{(h)}u=\Delta u+\tfrac{2}{3}h[\theta_xu'_{xxx}+\theta_yu''_{yyy}]\ ,$$

where $-1<\theta_x<1$, $-1<\theta_y<1$, *and where*

$$\begin{cases} u'_{xxx}=u_{xxx}(x',\,y)\ , & x-h_1<x'<x+h_2\ ,\\ u''_{yyy}=u_{yyy}(x,\,y')\ , & y-h_3<y'<y+h_4\ . \end{cases} \tag{28}$$

Proof. By Lemma 1, u_{xxx} is continuous in the open line segment from $(x-h_1,\,y)$ to $(x+h_2,\,y)$, but may become infinite if the endpoint is a corner of C. Since u is continuous in $R\cup C$, it nevertheless follows from Taylor's formula as stated in [6, p. 357] that, if we fix y and set $\phi(x)=u(x, y)$,

$$\frac{\phi(x+h_2)-\phi(x)}{h_2}=\phi'(x)+\frac{h_2}{2}\phi''(x)+\frac{h_2^2}{6}\phi'''(x+\theta_2 h_2) ,$$

where $0<\theta_2<1$.

Writing a similar formula for h_1 and subtracting, we find in the notation of (3) that

$$D_x^{(h)}\phi(x)=\phi''(x)+\left[\frac{h_2^2}{3}\phi'''(x+\theta_2 h_2)-\frac{h_1^2}{3}\phi'''(x-\theta_1 h_1)\right](h_1+h_2)^{-1}.$$

If one writes $k=\max(h_1, h_2)\leq h$, the last term can be bounded in absolute value by

$$\frac{2k^2}{3k}\max\left[|\phi'''(x+\theta_2 h_2)|, |\phi'''(x-\theta_1 h_1)|\right] ,$$

and hence can be written in the form $(2h/3)\theta_x u'_{xxx}$. Addition of a similar expression for $D_y^{(h)}u(x, y)$ proves the lemma.

LEMMA 4. *For each node* (x, y) *of* R_h^2 *defined in* (15) *use the notation of* (28). *Then, as* $h\to 0$, *one has*

$$h\sum_{R_h^2} u(|u'_{xxx}|+|u''_{yyy}|)=o(1) \qquad (h\to 0) . \tag{29}$$

Proof. The lemma is proved much like Lemma 6 of [4]. The functions $u|u_{xxx}|$ and $u|u_{yyy}|$ are continuous in $R\cup C$, except at a corner of interior angle $\pi\alpha$, where Lemma 1 states that they behave like $r^{2\alpha-3}$ with $2\alpha-3>-1$. The sum (29) can be majorized by the Lebesgue integral of a step function over a polygonal arc in R which converges in length to C as $h\to 0$. The integrability of $r^{2\alpha-3}$ in (0, 1) permits the application of Lebesgue's convergence theorem as $h\to 0$. Since $u=0$ on C, (29) follows. Details are omitted.

LEMMA 5. *At each node in* R_h^3, *defined in* (15), *one has*

$$\Delta^{(h)}u=\Delta u+\frac{1}{12}h^2(u'_{xxxx}+u''_{yyyy}) ,$$

where

$$\left\{\begin{array}{ll} u'_{xxxx}=u_{xxxx}(x+\theta' h, y) , & -1<\theta'<1 , \\ u''_{yyyy}=u_{yyyy}(x, y+\theta'' h) , & -1<\theta''<1 . \end{array}\right. \tag{30}$$

Proof. In [4]; the points of R_h^3 all have four neighbors in R_h^3, each at a distance h.

LEMMA 6. *At each node of R_h^3, defined in* (15), *use the notation of* (30). *Then, as $h\to 0$, one has*

$$h^2 \sum_{R_h^3} u(u'_{xxxx}+u''_{yyyy}) = \iint_R u(u_{xxxx}+u_{yyyy})dxdy + o(1) \qquad (h\to 0).$$

Proof. In [4].

LEMMA 7. *Define u_n and τ as in Theorem* 1. *One then has*

$$\iint_R u(u_{xxxx}+u_{yyyy})dxdy = \iint_R (u_{xx}^2+u_{yy}^2)dxdy + \int_C u_n^2 \sin^2 2\tau d\tau ,$$

where the latter is a Riemann-Stieltjes integral.

Proof. The proof repeats that of Lemma 7 in [4] down to (29) of that paper. It then remains only to prove for smooth convex curves C that

$$\int_C u_{yy}(u_y dx + u_x dy) = \int_C u_n^2 \sin^2 2\tau d\tau . \tag{31}$$

Let s denote arclength on C, and let primes denote d/ds. Differentiating the relations $u_x = -u_n \sin\tau$, $u_y = u_n \cos\tau$, we find that, on C,

$$\begin{cases} u_x' = -u_n' \sin\tau - u_n\tau' \cos\tau = u_{xy} \sin\tau + u_{xx}\cos\tau , \\ u_y' = \quad u_n' \cos\tau - u_n\tau' \sin\tau = u_{xy}\cos\tau + u_{yy}\sin\tau . \end{cases} \tag{32}$$

Changing u_{xx} to $-u_{yy}$ by (1), we can solve (32) for u_{yy} on C:

$$u_{yy} = u_n' \sin 2\tau + u_n\tau' \cos 2\tau .$$

Since $dx = ds\cos\tau$ and $dy = ds\sin\tau$, we obtain

$$\begin{aligned} \int_C u_{yy}(u_y dx + u_x dy) &= \int_C (u_n' \sin 2\tau + u_n\tau' \cos 2\tau)(u_n \cos 2\tau)ds \\ &= \int_C u_n^2\tau' \cos^2 2\tau\, ds + \int_C u_n u_n' \cos 2\tau \sin 2\tau\, ds . \end{aligned} \tag{33}$$

By partial integration, we have

$$\begin{aligned} \int_C u_n u_n' \cos 2\tau \sin 2\tau\, ds &= \tfrac{1}{4}\int_C (u_n^2)' \sin 4\tau\, ds \\ &= \tfrac{1}{4}[u_n^2 \sin 4\tau]_C - \int_C u_n^2\tau' \cos 4\tau\, ds . \end{aligned} \tag{34}$$

Since $\cos^2 2\tau - \cos 4\tau \equiv \sin^2 2\tau$, substitution of (34) into (33) shows that

$$\int_C u_{yy}(u_y dx + u_x dy) = \int_C u_n^2 \tau' \sin^2 2\tau \, ds .$$

Since $\tau' ds = d\tau$, the identity (31) is proved, and with it, the lemma.

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