# A THEOREM ON ALTERNATIVES FOR PAIRS OF MATRICES

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The theory of linear inequalities has come into prominence anew in recent years because of its importance in the solution of linear programming problems. In this note we present a simple algebraic proof of an interesting theorem on alternatives for pairs of matrices. This problem was suggested by A. W. Tucker.

Let $A$ and $B$ be matrices, $n$ by $m$ and $n$ by $p$, respectively, and let $x$, $y$, $u$ be column vectors of dimensions $m$, $p$, $n$, respectively.

STATEMENT I. *Either* $A'u>0$, $B'u\geqq 0$ *for some* $u$ *or* $Ax+By=0$ *for some* $x\geq 0$, $y\geqq 0$.[1]

STATEMENT II. *Either* $A'u\geq 0$, $B'u\geqq 0$ *for some* $u$ *or* $Ax+By=0$ *for some* $x>0$, $y\geqq 0$. [7].

We shall prove the following theorem.

THEOREM. *Statement* I *implies, and is implied by, Statement* II.

Note that for the special case when $A=-a$ (column vector) Statement I (or II) reduces to a result of Farkas [2]. If $B=0$, then Statements I and II are two theorems of Stiemke [6]. More importantly, if the matrix $[B, C, -C]$ is substituted for $B$, where $C$ is a $n$ by $q$ matrix, and $y$ is replaced by the vector $\begin{Bmatrix} y \\ y_1 \\ y_2 \end{Bmatrix}$, then Statement I gives the well-known transposition theorem of Motzkin [4, 5]. We refer to [4] for several proofs and further references.

Before proving our theorem, let us make the following preliminary observations. Define the matrix $M=[A, B]$ and the column vector $z=\begin{bmatrix} x \\ y \end{bmatrix}$, and consider the system of equations $Mz=0$. Assume that the vectors $s_1, s_2, \cdots, s_k$ span the linear manifold $\mathscr{S}$ of solutions of this system. Then every solution $z$ can be written in the form $z=S'c$ where $S'=[s_1, s_2, \cdots, s_k]$ and $c$ is a $k$-dimensional (column) vector. Observe that the rows of the matrix $M$ span the orthogonal complement $\mathscr{S}^*$ of $\mathscr{S}$, that is, every solution of the system $Sz^*=0$ can be represented as $z^*=M'd$ where $d$ is a $n$-dimensional (column) vector.

It will be convenient to write $S=[S_1, S_2]$ where $S_1$ and $S_2$ are the $k$ by $m$ and $k$ by $p$ matrices, respectively, into which $S$ can be parti-

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[1] Throughout, transposition is indicated by a dash; also, $x\geq 0$ means $x\geqq 0$ with $x=0$ excluded.

tioned; accordingly, we introduce two column vectors $v$, $w$ with $m$ and $p$ components, respectively, and write $z^*=\begin{bmatrix} v \\ w \end{bmatrix}$.

Clearly, the alternatives in each Statement are mutually exclusive as can be seen by multiplying $Ax+By=0$ on the left by $u'$. To prove the theorem suppose, at first, that $A'u\geq 0$, $B'u\geqq 0$ for no $u$ and $Ax+By=0$ has *no* solution $x>0$, $y\geqq 0$. Then there exists no $c$ such that

$$S_1'c>0\ , \qquad S_2'c\geqq 0\ .$$

Hence, by Statement I, the system $S_1v+S_2w=0$ must be satisfied for some $v\geq 0$, $w\geqq 0$. Since every solution of

$$Sz^*\equiv S_1v+S_2w=0$$

is of the form $z^*=M'd$, there must exist a vector $d$ such that $A'd>0$, $B'd\geqq 0$, which is a contradiction. Thus Statement I implies Statement II. Conversely, if $A'u>0$, $B'u\geqq 0$ for no $u$ and $Ax+By=0$ has *no* solution $x\geq 0$, $y\geqq 0$, then there exists no $c$ such that $S_1'c\geq 0$, $S_2'c\geqq 0$. Hence, by Statement II, the system $S_1v+S_2w=0$ must be satisfied for some $v>0$, $w\geqq 0$, that is, there must exist a vector $d$ such that $A'd>0$, $B'd\geqq 0$; but this is a contradiction. Thus Statement II implies Statement I.

For applications to linear programming Statements I and II are modified by adjoining in them the inequality $u\geqq 0$ to $B'u\geqq 0$, that is, by replacing the matrix $B$ by $[B, I]$; in this form they can be used to prove the duality theorem, [1, 3].

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