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ON A CLASS OF NODAL ALGEBRAS

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In this paper it is shown that there do not exist nodal algebras A satisfying the conditions:

(I) $x(xy) + (yx)x = 2(xy)x$

(II) $(xy)x - x(yx)$ is in N, the set of nilpotent elements of A, over any field F of characteristic zero. Also several results regarding algebras satisfying (I) alone are established.

A finite dimensional power-associative algebra A with identity 1 over a field F is called a nodal algebra [7] if every element x of A can be represented in the form $x = \alpha 1 + n$ where α is in F and n is nilpotent and if the set N of nilpotent elements of A is not a subalgebra of A. It is known [5] that there are no nodal flexible algebras over any field F of characteristic zero. (An algebra is said to be flexible if the identity $(xy)x = x(yx)$ is satisfied). There do exist, however, nodal algebras over fields F of characteristic zero in which $(xy)x - x(yx)$ is in N for all elements x, y of the algebra [3]. Algebras satisfying (I) were first studied by Kosier [6]. The concern, however, was for algebras of degree >1.

Throughout, we shall be using the result of Albert [2, p. 526] who proved that there are no commutative nodal algebras over any field F of characteristic zero by showing that N forms a subalgebra. In the noncommutative case we let A^+ be the same vector space as A with multiplication in A^+ given by $x \cdot y = 1/2(xy + yx)$, xy the multiplication in A. Then N is a subalgebra of A^+. In particular, N is a vector space. We use the standard notation, $[x, y]$, for the commutator $xy - yx$ and (x, y, z) for the associator $(xy)z - x(yz)$.

2. It is a well known fact that if an algebra A is power-associative then A^+ is power-associative. For algebras satisfying (I) the converse is also true.

THEOREM 1. *If A is an algebra satisfying* (I) *over a field F of characteristic $\neq 2$ and if A^+ is power-associative then A is power-associative.*

Proof. The following lemma is due to Witthoft [8].

LEMMA 1.1. *$xx^n = x^n x$ for all x in A and for all n.*

The proof is by induction on n. Trivially the lemma holds if

$n = 1$. Assume it holds for $n = k - 1$. Then $xx^{k-1} = x^{k-1}x = x^k$. By (I), however, $x(xx^{k-1}) + (x^{k-1}x)x = 2(xx^{k-1})x$ which reduces to $xx^k = x^k x$ and the lemma holds by mathematical induction.

Now linearize (I) to get:

$$(1) \qquad x(zy) + z(xy) + (yx)z + (yz)x = 2(xy)z + 2(zy)x .$$

Assume inductively that $x^a x^b = x^{a+b}$ for all positive integers a, b such that $a + b < n$. This is certainly true if $n = 3$. The induction hypothesis leads to the following.

LEMMA 1.2. $x^{n-k}x^k = x^k x^{n-k}$ *for all* $k < n$.

Proof of Lemma 1.2. In (1) let $x = x^{n-k}$, $y = x^{k-1}$, and $z = x$. We get:

$$x^{n-k}(xx^{k-1}) + x(x^{n-k}x^{k-1}) + (x^{k-1}x^{n-k})x + (x^{k-1}x)x^{n-k} = 2(x^{n-k}x^{k-1})x + 2(xx^{k-1})x^{n-k} .$$

However, by hypothesis $x^{n-k}x^{k-1} = x^{k-1}x^{n-k} = x^{n-1}$ since the degree of each of these terms is $n - 1 < n$. Also, by Lemma 1.1 $xx^{k-1} = x^{k-1}x = x^k$ and $xx^{n-1} = x^{n-1}x = x^n$. Therefore, the identity is reduced to $x^{n-k}x^k + x^n + x^n + x^k x^{n-k} = 2x^n + 2x^k x^{n-k}$ or $x^{n-k}x^k = x^k x^{n-k}$ as desired.

Now since A^+ is power-associative we have $x^n = x^{n-k} \cdot x^k$ for any $k < n$. Since $x^{n-k}x^k = x^k x^{n-k}$ we get $x^n = 2/2x^{n-k}x^k = x^{n-k}x^k$. Suppose now that $a + b = n$. Then $a = n - k, b = k$ for some $k \leqq n$. Then $x^{a+b} = x^n = x^{n-k}x^k = x^a x^b$ and the result holds for $a + b = n$. It follows by mathematical induction that $x^a x^b = x^{a+b}$ for all positive integers a, b and A is power-associative.

Clearly, Theorem 1 would also hold for a ring A in which the equation $2x = a$ is solvable for all a in A. It should be noted that (I) alone is not sufficient to guarantee power-associativity of A since Albert [1, p. 25] has shown that commutativity does not guarantee power-associativity.

3. In this section we shall be considering finite dimensional, power-associative algebras with 1 every element of which is of the form $\alpha 1 + n$ with n nilpotent. We call a nilpotent element w of such an algebra a commutator nilpotent if there are elements u, v in the algebra such that $[u, v] = \alpha 1 + w$ for some α in the base field. We write tr. (T) for the trace of an operator T.

THEOREM 2. *Let A be a finite dimensional algebra satisfying* (I) *over a field F of characteristic zero in which every element z is* *of the form* $z = \alpha 1 + n$ *where* α *is in* F *and* n *is nilpotent. Then a necessary and sufficient condition for the set* N *of nilpotent elements to form an ideal of* A *is that* tr. $(R(w)) = 0$ *for every commutator nilpotent* w. $(R(w))$ *is the operator which takes any* x *into* xw.)

Proof. Gerstenhaber [4, p. 29] has shown that in a commutative power-associative algebra over a field of characteristic zero, the assumption that an element n is nilpotent implies that $R(n)$ is nilpotent. We apply this result to the algebra A^+ so that if a is a nilpotent element of A then $R(a)^+ = 1/2(R(a) + L(a))$ is nilpotent and thus tr. $[R(a)]$ + tr. $[L(a)] = 0$. Writing (1) in terms of operators we get:

$$(2) \qquad R(y)L(x) + R(xy) + L(yx) + L(y)R(x) = 2L(xy) + 2R(y)R(x) \,.$$

If we interchange x and y in (2) and subtract the result from (2) we get $[L(y), R(x)] + [R(y), L(x)] + R([x, y]) + L([y, x]) = 2L([x, y]) + 2[R(y), R(x)]$ which gives rise to:

$$(3) \qquad \text{tr.}\, R([x, y]) + \text{tr.}\, L([y, x]) = 2\text{tr.}\, L([x, y]) \,.$$

Assume that tr. $R(w) = 0$ for all commutator nilpotents w of A. Then tr. $L(w)$ = tr. $R(w) = 0$ also. Let x and y be arbitrary elements of N. Then $[x, y] = \alpha 1 + n$ for some α in F and n in N and n is a commutator nilpotent. Therefore (3) reduces to tr. $[R(\alpha 1)]$ − tr. $[L(\alpha 1)]$ = 2tr. $[L(\alpha 1)]$ or tr. $[R(\alpha 1)]$ = 3tr. $[L(\alpha 1)]$ a contradiction unless $\alpha = 0$. Therefore, $[x, y]$ is in N and by [2], xy and yx are in N. Thus N is an ideal of A.

Conversely, let N be an ideal of A. Therefore $[x, y]$ is in N for all x, y in N and consequently for all x, y in A. Thus if w is a commutator nilpotent of A there is an x, y such that $w = [x, y]$. From (3) we have that tr. $R(w)$ − tr. $L(w)$ = 2tr. $L(w)$. But tr. $R(w)$ + tr. $L(w) = 0$. Therefore tr. $R(w) = 0$ and the result holds.

THEOREM 3. *There are no nodal Lie-admissible algebras satisfying* (I) *over any field* F *of characteristic zero.*

Proof. For if A is such a Lie-admissible algebra then for all u, v in N and w in A we have $[[u, v], w] + [[v, w], u] + [[w, u], v] = 0$. In operator form this becomes:

$$L([u, v]) - R([u, v]) + [L(v), R(u)] + [R(u), R(v)] \\ + [L(u), L(v)] + [R(v), L(u)] = 0 \,.$$

Therefore, tr. $L([u, v])$ = tr. $R([u, v])$.

Suppose that $[u, v] = \alpha 1 + z$ with α in F and z in N. Then tr. $L(\alpha 1)$ + tr. $L(z)$ = tr. $R(\alpha 1)$ + tr. $R(z)$. Therefore, tr. $R(z)$ = tr. $L(z)$

for all commutator nilpotents z. From [4] we conclude that tr. $R(z) = 0$ and by Theorem 2, N is an ideal of A. Therefore A is not a nodal algebra.

We say that N has nilindex p if p is the smallest positive integer such that $n^p = 0$ for all n in N.

LEMMA 1. *There are no nodal algebras satisfying* (I) *over a field F of characteristic zero for which the nilindex of N is two.*

Proof. For if N has nilindex two, then $xy + yx = 0$ for all x, y in N. Applying (I) to x and y in N we have $x(xy) - (xy)x = 2(xy)x$ or $x(xy) = 3(xy)x$. If $xy = \alpha 1 + z$ with α in F and z in N the preceeding identity becomes $\alpha x + xz = 3\alpha x + 3zx$. But $xz = -zx$. Therefore it reduces to $2\alpha x = 4xz$ and since characteristic $F \neq 2$ to $\alpha x = 2xz$. Multiplying on the left by x we have $0 = \alpha x^2 = 2x(xz)$ or $x(xz) = 0$. But $x[x(xy)] = x[x(\alpha 1 + z)] = x[\alpha x + xz] = \alpha x^2 + x(xz) = 0$. Therefore we have $yL(x)^3 = 0$ for all x, y in N.

Let $\alpha 1 + n$ be a typical element of the algebra A. Then $(\alpha 1 + n)L(x)^3 = \alpha x^3 + nL(x)^3$ and $nL(x)^3 = 0$ as above. Therefore $L(x)^3 = 0$, $L(x)$ is a nilpotent operator of A and tr. $L(x) = 0$. As before, this implies that tr. $R(x) = 0$. By Theorem 2, N is an ideal of A and A is not a nodal algebra.

Anderson [3] has shown the existence of simple nodal algebras over a field of characteristic zero for which the associators (x, y, z) are nilpotent for all x, y, and z. The following theorem shows that no such algebras exist which satisfy (I).

THEOREM 4. *There are no simple nodal algebras satisfying* (I) *and* (II) *over any field F of characteristic zero.*

Proof. We first prove the following lemmas.

LEMMA 4.1. *If x and y are in N then xy^2 and y^2x are also in N.*

For if we let $xy = \alpha 1 + n$ with α in F and n in N, then $yx = 2x \cdot y - \alpha 1 - n$ and $(x, y, x) = 2\alpha x + nx + xn - 2x(x \cdot y)$. But $xn + nx = 2x \cdot n$ is in N, $2\alpha x$ is in N, and by hypothesis (x, y, x) is in N. Therefore, $2x(x \cdot y)$ and consequently $x(x \cdot y)$ is in N. Linearizing this we have:

$$(4) \qquad x(z \cdot y) + z(x \cdot y) \text{ is in } N \text{ if } x, y, z \text{ in } N .$$

Let $z = y$ in (4). Then $xy^2 + y(x \cdot y)$ is in N. But $y(y \cdot x)$ is in N from the previous remark and we conclude that xy^2 is in N. Since $x \cdot y^2$ is in N y^2x is also in N.

It can be further shown by mathematical induction that $x^j y^k$ is in N if $j > 1$ or $k > 1$.

LEMMA 4.2. *For any x, y in N the following elements are in N:* $(xy)x$, $x(xy)$, $(yx)x$, and $x(yx)$.

For, since A is power-associative we have

$$(x, x, y) + (y, x, x) + (x, y, x) = 0 .$$

But (x, y, x) is in N. So we have that $(x, x, y) + (y, x, x)$ is in N for all x, y in A or: $x^2y - x(xy) + (yx)x - yx^2$ is in N for all x, y in A. If x and y are in N then by Lemma 4.1, $x^2y - yx^2$ is in N. Thus,

$$(5) \qquad (yx)x - x(xy) \text{ is in } N \text{ for all } x, y \text{ in } N .$$

We write $x(xy) - (yx)x = n$ for some n in N. Adding (I) to this we get that $2x(xy) = 2(xy)x + n$. But characteristic $F \neq 2$. Therefore, $x(xy) - (xy)x$ is in N. But $x \cdot (xy)$ is in N. Thus, $x(xy)$ and $(xy)x$ are in N if x and y are in N. Applying (I) again $(yx)x = 2(xy)x - x(xy)$. By the previous remark the right side is in N. We conclude, therefore, that $(yx)x$ and hence $x(yx)$ is in N completing the proof of the lemma.

Since $x(xy)$ is in N, it follows that:

$$(6) \qquad x(zy) + z(xy) \text{ is in } N \text{ if } x, y, z \text{ are in } N .$$

Also $(yx)x$ in N implies that:

$$(7) \qquad (yx)z + (yz)x \text{ is in } N \text{ if } x, y, z \text{ are in } N .$$

Now, let y be an element of N. Then y^2 is in N. We shall analyze the ideal I generated by the element y^2. I is the set of all sums of terms, each term being a product of elements of A at least one element of which is the element y^2. Consider the number of multiplications on y^2 in a typical summand. If we multiply y^2 by a single element in N, say z, we have either y^2z or zy^2 which are in N by Lemma 4.1.

We prove by mathematical induction that any number of multiplications on y^2 by elements of N maintains nilpotency. The result has been shown for one multiplication. Assume that n multiplications on y^2 maintains nilpotency and consider $n+1$ multiplications by elements $q_1, q_2, \cdots, q_n, q_{n+1}$ of N. There are only four cases to consider:

$$(1) \quad \{[(((\cdots (y^2) \cdots)))]q_n\}q_{n+1} \qquad (2) \quad q_{n+1}\{[(((\cdots (y^2) \cdots)))]q_n\}$$

$$(3) \quad q_{+1}\{q_n[(((\cdots (y^2) \cdots)))]\} \qquad (4) \quad \{q_n[(((\cdots (y^2) \cdots)))]\}q_{n+1}$$

for all other arrangements would involve n or less multiplications. Let

$b = (((\cdots(y^2)\cdots)))$. By hypothesis b is in N. We must show then, that

$$(1)\ (bq_n)q_{n+1} \qquad (2)\ q_{n+1}(bq_n) \qquad (3)\ q_{n+1}(q_nb) \qquad (4)\ (q_nb)q_{n+1}$$

are all in N.

In (6) let $x = q_{n+1}$, $z = b$, and $y = q_n$. Then we have that $q_{n+1}(bq_n) + b(q_{n+1}q_n)$ is in N. But $b(q_{n+1}q_n)$ involves only n multiplications on y^2. Therefore, by the induction hypothesis it is in N and we conclude that $q_{n+1}(bq_n)$ and therefore by [2] $(bq_n)q_{n+1}$ are in N. Similarly, in (7) let $x = b$, $y = q_n$, and $z = q_{n+1}$. Then we have $(q_nb)q_{n+1} + (q_nq_{n+1})b$ are in N. As before this implies that $(q_nb)q_{n+1}$ and consequently $q_{n+1}(q_nb)$ are in N. Therefore $n+1$ multiplications on y^2 by elements of N maintains nilpotency and the result holds for any number of multiplications. It follows easily that any number of multiplications on y^2 by elements of A preserve nilpotency.

Now every element of I is a sum of terms of the above type and consequently nilpotent. Thus $I \subsetneqq N$. Hence, I is an ideal of A which does not encompass all of A and by the simplicity of A, $I = 0$. But y^2 is in I. Therefore $y^2 = 0$. This holds for all y in N and so the nilindex of N is two. By Lemma 1, A is not nodal.

THEOREM 5. *There are no nodal algebras satisfying* (I) *and* (II) *over any field F of characteristic zero.*

Proof. For let A be such an algebra. By Theorem 4, A is not simple. Let B be a maximal ideal of A. Then A/B is a simple nodal algebra satisfying (I) and (II) contradicting Theorem 4.

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