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ON THE CONVERGENCE OF A TRIGONOMETRIC INTEGRAL

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In the present paper, we shall first establish a theorem concerning the convergence of a trigonometric integral. Then in the final section, we shall evaluate some known definite integrals with the help of our theorem.

1. DEFINITION. We say that the integral $\int_0^\infty a(u)du$ is summable $(C, 1)$ to sum S, if

$$\lim_{\lambda\to\infty}\int_0^\lambda\left(1-\frac{u}{\lambda}\right)a(u)du = S\,.$$

In [1], a result regarding the $(C, 1)$ summability of a trigonometric integral was proved which is equivalent to

THEOREM A. *Let $f(t)$ be L over $(0, \infty)$. Then, for $0<\alpha<1$, the integral*

$$\int_0^\infty u^\alpha du\int_0^\infty f(t)\sin utdt$$

is summable $(C, 1)$ to

$$\Gamma(\alpha+1)\,\mathrm{Cos}\,\frac{1}{2}\alpha\pi\int_{\to 0}^\infty \frac{f(t)}{t^{1+\alpha}}dt$$

whenever this integral exists and whenever

$$f(t)=0(t^\alpha)\quad as\quad t\to 0\,.$$

In § 2 of the present paper we establish the following theorem.

THEOREM. *Let $t^{-\alpha}f(t)(0<\alpha<1)$ be of bounded variation over $(0, \infty)$ and tend to zero both as $t\to 0$ and $t\to\infty$. If the integral $\int_0^{\to\infty} f(t)\sin utdt$ is uniformly convergent with respect to u over $0<\mu\leqq u\leqq\lambda<\infty$, for every μ and λ, then*

$$(1.1)\qquad \int_{\to 0}^{\to\infty}u^\alpha du\int_0^{\to\infty}f(t)\sin utdt=\Gamma(\alpha+1)\cos\frac{1}{2}\alpha\pi\int_{\to 0}^{\to\infty}\frac{f(t)}{t^{1+\alpha}}dt$$

whenever the last integral exists.

In the present problem $f(t)$ is not necessarily L over $(0, \infty)$. In

§ 3 we shall evaluate some known definite integrals with the help of the above theorem.

2. For the proof of the theorem, we use the following simple lemma.

LEMMA. *If the function $g(t)$ is positive and nonincreasing over the interval (a, ∞), then*

$$\left|\int_a^{\to\infty} g(t)\cos tdt\right| \leqq Ag(a)\,.$$[1]

Proof of the theorem. We write $h(t) = t^{-\alpha}f(t)$. For any $\varepsilon > 0$, there is a δ such that $|h(t)| < \varepsilon$ for all $t < \delta$ and for all $t > 1/\delta$. For the sake of simplicity, we shall drop the sign $\to$ at infinity in the proof. We have

$$\int_\mu^\lambda u^\alpha du \int_0^\infty f(t)\sin utdt = \int_0^\infty f(t)dt\int_\mu^\lambda u^\alpha \sin utdu \tag{2.1}$$

since the inversion of order of integration is justified by the uniform convergence of the inner integral of the left side of (2.1). Using integration by parts, we get

$$\int_\mu^\lambda u^\alpha \sin utdu = \frac{\mu^\alpha}{t}\cos\mu t - \frac{\lambda^\alpha}{t}\cos\lambda t - \frac{\alpha}{t^{\alpha+1}}\int_{\mu t}^{\lambda t}\frac{\cos u}{u^{1-\alpha}}du\,,$$

and then the equation (2.1) becomes

$$\begin{aligned}&\int_\mu^\lambda u^\alpha du\int_0^\infty f(t)\sin utdt\\ &= \mu^\alpha\int_0^\infty \frac{h(t)}{t^{1-\alpha}}\cos\mu tdt - \lambda^\alpha\int_0^\infty\frac{h(t)}{t^{1-\alpha}}\cos\lambda tdt + \alpha\int_0^\infty\frac{h(t)}{t}dt\int_{\mu t}^{\lambda t}\frac{\cos u}{u^{1-\alpha}}du\\ &= I - J + K\,.\end{aligned}$$

Now

$$\begin{aligned}|I| &= \mu^\alpha\left|\int_0^{\varepsilon/\mu} + \int_{\varepsilon/\mu}^\infty\right|\\ &\leqq A\mu^\alpha\int_0^{\varepsilon/\mu}\frac{dt}{t^{1-\alpha}} + \left|\int_\varepsilon^\infty \frac{h(v/\mu)}{v^{1-\alpha}}\cos vdv\right|\\ &\leqq A\varepsilon^\alpha + o(1) \quad \text{as} \quad \mu\to 0\,,\end{aligned}$$

by applying the lemma for the last integral, after writing $h(t)$ as the difference of two functions which tend to zero monotonically. Similarly

[1] Throughout the present paper we write A for an arbitrary constant which is not necessarily the same at each occurrence.

$$\begin{aligned} |J| &= \lambda^{\alpha}\left|\int_0^{1/\lambda} + \int_{1/\lambda}^{1/\varepsilon\lambda} + \int_{1/\varepsilon\lambda}^{\infty}\right| \\ &\leqq o(1)\lambda^{\alpha}\int_0^{1/\lambda}\frac{dt}{t^{1-\alpha}} + A\int_1^{1/\varepsilon}|h(v/\lambda)|\,dv + \left|\int_{1/\varepsilon}^{\infty}\frac{h(v/\lambda)}{v^{1-\alpha}}\cos v dv\right| \\ &\leqq A\varepsilon^{1-\alpha} + o(1) \quad \text{as} \quad \lambda \to \infty . \end{aligned}$$

Thus it is sufficient to prove that

$$\limsup_{\lambda\to\infty,\,\mu\to 0}\left|k - \alpha\int_{1/\lambda}^{\infty}\frac{h(t)}{t}dt\int_0^{\infty}\frac{\cos u}{u^{1-\alpha}}du\right| \leqq A\varepsilon^{\alpha} , \tag{2.2}$$

since

$$\int_0^{\infty}u^{\alpha-1}\cos u du = \Gamma(\alpha)\cos\frac{1}{2}\alpha\pi .$$

The term inside the absolute value sign of (2.2) is

$$\begin{aligned} k - \alpha\int_{1/\lambda}^{\infty}\frac{h(t)}{t}dt\int_0^{\infty}\frac{\cos u}{u^{1-\alpha}}du &= \alpha\int_0^{1/\lambda}\frac{h(t)}{t}dt\int_{\mu t}^{\lambda t}\frac{\cos u}{u^{1-\alpha}}du \\ &\quad - \alpha\int_{1/\lambda}^{\infty}\frac{h(t)}{t}dt\int_0^{\mu t}\frac{\cos u}{u^{1-\alpha}}du \\ &\quad - \alpha\int_{1/\lambda}^{\infty}\frac{h(t)}{t}dt\int_{\lambda t}^{\infty}\frac{\cos u}{u^{1-\alpha}}du \\ &= \alpha(L - M - N) . \end{aligned}$$

Now,

$$\begin{aligned} |L| &\leqq \int_0^{1/\lambda}\frac{|h(t)|}{t^1}dt\int_0^{\lambda t}\frac{du}{u^{1-\alpha}} \\ &\leqq A\lambda^{\alpha}\int_0^{1/\lambda}\frac{|h(t)|}{t^{1-\alpha}}dt = o(1) \quad \text{as} \quad \lambda \to \infty . \end{aligned}$$

By the formula

$$\int_0^t\frac{\cos\mu v}{v^{1-\alpha}}dv = \frac{\sin\mu t}{\mu t^{1-\alpha}} + \frac{1-\alpha}{\mu}\int_0^t\frac{\sin\mu v}{v^{2-\alpha}}dv ,$$

we get

$$\begin{aligned} M &= \mu^{\alpha}\int_{1/\lambda}^{\infty}\frac{h(t)}{t}dt\int_0^t\frac{\cos\mu v}{v^{1-\alpha}}dv \\ &= \frac{1}{\mu^{1-\alpha}}\int_{1/\lambda}^{\infty}\frac{h(t)\sin\mu t}{t^{2-\alpha}}dt + \frac{1-\alpha}{\mu^{1-\alpha}}\int_{1/\lambda}^{\infty}\frac{h(t)}{t}dt\int_0^t\frac{\sin\mu v}{v^{2-\alpha}}dv \\ &= M_1 + M_2 \end{aligned}$$

where

$$|M_1| = \left|\int_{\mu/\lambda}^{\infty} \frac{h(v/\mu)\sin v}{v^{2-\alpha}} dv\right|$$
$$\leqq \left|\int_{\mu/\lambda}^{\epsilon}\right| + \left|\int_{\varepsilon}^{\infty}\right| \leqq A\varepsilon^{\alpha} + o(1) \quad \text{as} \quad \mu \to 0$$

and

$$M_2 = \frac{A}{\mu^{1-\alpha}}\int_{1/\lambda}^{\infty}\frac{h(t)}{t}dt\int_0^{1/\lambda}\frac{\sin\mu v}{v^{2-\alpha}}dv$$
$$+ \frac{A}{\mu^{1-\alpha}}\int_{1/\lambda}^{\infty}\frac{\sin\mu v}{v^{2-\alpha}}dv\int_v^{\infty}\frac{h(t)}{t}dt$$
$$= o(1) + A\int_{\mu/\lambda}^{\infty}\frac{\sin\omega}{\omega^{2-\alpha}}d\omega\int_{\omega/\mu}^{\infty}\frac{h(t)}{t}dt$$

where the change of order of integration is easily proved, and then

$$|M_2| \leqq A\varepsilon^{\alpha} + o(1) .$$

Finally

$$|N| \leqq \frac{A}{\lambda^{1-\alpha}}\int_{1/\lambda}^{\delta}\frac{|h(t)|}{t^{2-\alpha}}dt + \int_{\delta}^{1/\delta}\frac{|h(t)|}{t}dt\int_{\lambda t}^{\infty}\frac{\cos v}{v^{1-\alpha}}dv$$
$$+ \frac{A}{\lambda^{1-\alpha}}\int_{1/\delta}^{\infty}\frac{|h(t)|}{t^{2-\alpha}}dt$$
$$\leqq A\varepsilon + o(1) \quad \text{as} \quad \lambda \to \infty .$$

Thus we get the required inequality (2.2) and the theorem is completely proved.

3. Evaluation of integrals. Let us consider the function

$$f(t) = t/(1+t^2) \qquad (0, \infty) .$$

Then the integral of the left side of (1.1) for the present function reduces to

$$\int_0^{\infty} u^{\alpha}du\int_0^{\infty}\frac{t}{1+t^2}\sin utdt$$
$$= \frac{\pi}{2}\int_0^{\infty}u^{\alpha}e^{-u}du = \frac{\pi}{2}\Gamma(\alpha+1) .$$

Obviously, the function satisfies all the conditions of the theorem, so we have

$$\frac{\pi}{2}\Gamma(\alpha+1) = \Gamma(\alpha+1)\cos\frac{1}{2}\alpha\pi\int_0^{\infty}\frac{t^{-\alpha}}{1+t^2}dt$$

i.e.,

(3.1) $$\int_0^\infty \frac{t^{-\alpha}}{1+t^2}dt = \frac{\pi}{2\cos\frac{1}{2}\alpha\pi} \quad \text{for} \quad 0<\alpha<1 .$$

Next we consider the function

$$f(t) = t^3/(1+t^4) \qquad (0, \infty) .$$

Obviously, this function satisfies the hypotheses of the theorem of the present paper. The integral on the left side of (1.1) for the present function reduces to

$$\int_0^\infty u^\alpha du \int_0^\infty \frac{t^3}{1+t^4} \sin utdt$$
$$= \frac{\pi}{2}\int_0^\infty u^\alpha e^{-u/\sqrt{2}} \cos \frac{u}{\sqrt{2}} du = \frac{\pi}{2}\Gamma(\alpha+1)\cos(\alpha+1)\frac{\pi}{4} .$$

Now by the theorem of the present note, we have

$$\frac{\pi}{2}\Gamma(\alpha+1)\cos(\alpha+1)\frac{\pi}{4} = \Gamma(\alpha+1)\cos\frac{1}{2}\alpha\pi\int_0^\infty \frac{t^{2-\alpha}}{1+t^4}dt .$$

Therefore

$$\int_0^\infty \frac{t^{2-\alpha}}{1+t^4}dt = \frac{\pi}{2}\frac{\cos(\alpha+1)\pi/4}{\cos\frac{1}{2}\alpha\pi} \quad \text{for} \quad 0<\alpha<1 .$$

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