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ON HOPFIAN GROUPS

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A group G is said to be hopfian if every surjective endomorphism of G is an automorphism of G. Some authors have investigated the problem of forming new hopfian groups in some familiar way from given hopfian groups. This investigation is continued in this paper. Section 1 contains a statement of our main results and a discussion of these results. Sections 2 and 3 contain the proof of these results. We list our main results below.

Let the group G be a semi-direct product of its subgroups H and F; that is, $H \triangle G, G = HF$ and $H \cap F = 1$. If H is a hopfian abelian group, we show that G is hopfian if either of the following holds:

(a) F obeys the maximal condition for subgroups and H does not have an infinite cyclic direct factor.

(b) F is a free abelian group of finite rank.

Let H be a hopfian abelian normal subgroup of G (which is not necessarily a split extension of H). Suppose G/H satisfies the maximal condition for subgroups. Then G is hopfian if any of the following holds:

(c) H is a torsion group.

(d) H is of finite rank and has a hopfian torsion group.

(e) $G = H \cdot F$ where F is a finite group.

Let A be a hopfian group and let $A \times B$ be the direct product of A and B. We will show $A \times B$ is hopfian if either of the following holds:

(f) B is a finite solvable group with exactly n proper normal subgroups which form a chain.

(g) B is a finite group of cube free order.

(h) B is a finite group of order p^3, p a prime.

Finally we give some conditions on $Z(A)$, the center of A, which will guarantee the hopficity of $A \times B$ if B is an infinite cyclic group.

In some respects, the property of hopficity is strange. For example, Gilbert Baumslag and Donald Solitar have constructed a nonhopfian group defined by two generators and a single defining relation and a two generator hopfian group with a normal subgroup of finite index which is not hopfian [2]. A.L.S. Corner has shown that if A is an abelian hopfian group, $A \times A$ need not be hopfian [3].

In seeking to construct new hopfian groups from given hopfian groups one naturally looks at familiar group constructions. For example, I. Dey has shown that under certain conditions the free product of hopfian groups is hopfian [4]. In considering the direct product of the hopfian groups A and B, we see that some further conditions have to be imposed on either A or B if their direct product is to be hopfian. In [8] it is shown that $A \times B$ is hopfian if either of the following holds:

(i) A is abelian and B satisfies the maximal condition for normal subgroups.

(j) B is a finite abelian group.

(k) $Z(A) = 1$ and B has a principal series.

Other types of situations are considered in [8]. For example, it is shown that if H is a super hopfian group (all homomorphic images of H are hopfian) and if $H \triangle L$, $L = HB$ and B has finitely many normal subgroups then L is super hopfian.

In [1], Gilbert Baumslag poses the following problem: Let G be an abelian group with G/H finitely generated, H hopfian. Must G be hopfian?

Our results (c), (d) and (e) and Corollaries (3) and (4) of Theorem 1 answer this question affirmatively for certain restrictions on H, and give information even if G is nonabelian.

A convenient restriction on H is that H have a maximal free abelian direct factor. This is equivalent to expressing H in the form $H = M \times K$ where M is free abelian of finite rank (M may be 1) and where K does not have an infinite cyclic direct factor. In this situation, if G/H obeys the maximal condition for subgroups (G need not be abelian) it turns out as we will show that one need only consider the case when G is abelian and contains a hopfian subgroup of prime index such that G is generated by H and an element $x \in G$ where x is of infinite order. While this special case might appear as an easy one to dispose of, the author has had no success as of this writing in doing so.

Another unresolved "simple" problem is the question whether or not the direct product of a hopfian group (necessarily nonabelian) with an infinite cyclic group is hopfian. We conjecture that for A hopfian, $A \times B$ is not always hopfian for B an infinite cyclic group but is always hopfian if B has a principal series. To explain our bases for this conjecture, let us point out that a group B with a principal series may be cancelled in direct products. That is, if $L \times B = L_1 \times B_1$ and $B \approx B_1$ then $L \approx L_1$ [9]. Hence if α is a surjective endomorphism of $A \times B$, a reasonable plan to show α is an automorphism exists. Namely, first show that α is an isomorphism on B. Next show $A\alpha \cap B\alpha = 1$ so that $A \times B = A\alpha \times B\alpha$. From the cancellation result, deduce that $A \approx A\alpha$ so that α is an isomorphism on A and hence an automorphism. This procedure has worked to get some of the results in [8]. However there is no hope for this idea if B is an infinite cyclic group because as is shown in [9], an infinite cyclic group may not be cancelled in direct products! If our conjecture regarding the infinite cyclic case is false a method of avoiding this anomalous property will have to devised. The best we have been able to do is place some constraints on A. We show that $A \times B$ is hopfian for A hopfian and B infinite cyclic if either of the following holds:

(j) $Z(A)$ is periodic.

(k) $Z(A)$ is divisible.

(l) A has an infinite cyclic direct factor.

In seeking to find some suitable constraints to place upon the factor B and to keep A arbitrary, we note that the cyclic group of order p^{n+1}, p a prime, has an extremely simple normal subgroup structure. Namely there are exactly n normal subgroups other than the whole group and the identity group and these normal subgroups form a chain. We call a group with this normal subgroup structure an n-normal group. For example, the alternating group on 4 elements is 1-normal. We show in addition to the result on n-normality that we have already mentioned that $A \times B$ is hopfian if B is n-normal, $0 \leqq n \leqq 3$.

Finally, it is a pleasure to take this opportunity to express our appreciation to professor Gilbert Baumslag for a generous sharing of his insight which enabled the author to replace a weaker version of Theorem 1 by Theorem 1.

2. Extension of a hopfian abelian group.

LEMMA 1. *Let E be a group, L a normal subgroup of E and α an endmorphism of E onto E. Let $J = gp(L, L\alpha, L\alpha^2, \cdots)$ and suppose that the center Z of J is of finite index in J. Suppose further that E/L satisfies the maximal condition for normal subgroups. Then there exists for every subgroup R of J containing Z a positive integer q* such that $R\alpha^{-q} = R$.

Proof. One uses the fact that the inverse image of J under α is J and that α induces an automorphism of finite order on J/Z.

THEOREM 1. *Let H be a hopfian abelian normal subgroup of the group G and let G/H obey the maximal condition for subgroups. Then G is hopfian if any of the following holds*:

(1) *H is a torsion subgroup*

(2) *H is of finite rank and the torsion subgroup of H is hopfian*

(3) *H has a maximal free abelian direct factor (perhaps the identity subgroup) which is normal in G and G is a semi-direct product of H and a group F.*

(4) $G = H \cdot F$, *where* F *is a periodic group and* $Z(G) \cap F$ *is of finite index in* F.

Proof. We first consider the situation when G is an abelian group generated by a hopfian abelian group H and an element x of finite order such that $x^p \in H$ for some prime p and G/H is cyclic of order p. Suppose x is of order q. Then p/q so we may write $q = p^a r$, $(p, r) = 1$. Hence, without loss of generality we may assume x is of order a power of p (for G is generated by H and $y = x^r$ and y is of order p^a). Now if x is of order p, G is a direct product of H and a cyclic group generated by x so that from the results in [8], G is hopfian.

Assume inductively that G is hopfian if x is of order p^s where $s < k$. Suppose x is of order p^k. Let $x^p = h \in H$. Let $\langle h \rangle$ be the cyclic group generated by h. Now if each element of $\langle h \rangle$ has finite height in H with respect to the prime p, we can find a finite abelian direct factor H_1 of H containing $\langle h \rangle$ (see [5].) Let $H = H_1 \times H_2$. Then $G = \langle x \rangle H_1 \times H_2$ so that G is hopfian. Hence we may assume that some element in $\langle h \rangle$ different from 1, say h^{p^i}, $i < k - 1$, has infinite height in H with respect to p. Hence we may find $h_1 \in H$ such that

$$h^{p^i} = h_1^{p^{i+1}} .$$

Then if we set $y = x h_1^{-1}$ we see $G = \langle y \rangle H$ and y has order which divides p^{i+1} so that G is hopfian by inductive assumption and our assertion is proven.

We now consider the situations in (1), (2) and (3). Let $H = M \times K$ where M is a maximal free abelian factor of H. Note that this forces K to be normal in G. For if M_1 and K_1 are the conjugates of M and K respectively under conjugation by an element g of G, then $H = M_1 \times K_1$ and one sees that KK_1/K_1 and KK_1/K are free abelian. Since neither K nor K_1 has an infinite cyclic direct factor we deduce KK_1/K and KK_1/K_1 are the identity subgroups. Now from,

$$\frac{G}{K} \Big| \frac{H}{K} \approx \frac{G}{H}$$

we see that G/K obeys the maximal condition for subgroups. Let α be an endomorphism of G onto G. Since $G \mid K\alpha^i$ obeys the maximal conditions for subgroups, so does $K^i = K\alpha^i / K\alpha^i \cap K\alpha^{i+1}$. Let $K_i = \bigcap K\alpha^i$, $i \geqq 0$, where the intersection is taken over all j, $0 \leqq j \leqq i$. Note that K_i/K_{i+1} is isomorphic to a subgroup of K^i so that K_i/K_{i+1} is finitely generated. This implies that K/K_i is finitely generated for all i. But since K cannot have an infinite cyclic direct factor, K/K_i must be finite.

Choose a positive integer r such that $J = gp(K, K\alpha, K\alpha^2 \cdots) =$

$gp(K, K\alpha, \cdots, K\alpha^r)$. Then K_r is a central subgroup of finite index in J. Hence if Z is the center of J and $R = KZ$ we may apply Lemma 1 and choose $q > 0$ such that $R\alpha^{-q} = R$. Hence α is an automorphism if R is hopfian. Now R is abelian and R/K is a finitely generated abelian group. Hence we may write R in the form

$$R = R_1 \times R_2$$

where $K \subset R_1$ and R_1/K is finite and where R_2 is a free abelian group of finite rank. Hence R is hopfian if and only if R_1 hopfian. However we may find a chain of subgroups $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_s$ with $A_0 = K$ and $A_s = R_1$ and such that $A_{i+1}A_i$ is cyclic of prime order for $0 \leqq i < s$. Now if H is a torsion group we may apply our result on cyclic extension to each A_i successively so that the assertion (1) is proven.

For (2), assume inductively that A_i is a hopfian abelian group of finite rank with a hopfian torsion subgroup. Choose z so that $A_{i+1} = \langle z\rangle A_i$, $A_{i+1}/A_i \approx C_q$ for some prime q. If z is of finite order the torsion subgroup of A_{i+1} is exactly the group generated by z and the torsion subgroup of A_i and so A_{i+1} and its torsion subgroup are hopfian by our result on cyclic extensions. Hence we may assume z is of infinite order. Now suppose there exists $w = z^s a$, $a \in A_i$, $(s, q) = 1$ with w of finite order. Then $A_{i+1} = \langle w\rangle A_i$ and as before we see A_{i+1} is hopfian with a hopfian torsion subgroup. Hence we may assume that the torsion subgroup, A_*, of H_{i+1} is equal to the torsion subgroup of A_i. Now one may easily verify that a torsion free abelian group of finite rank is hopfian. Hence A_{i+1}/A_* is hopfian and since A_* is hopfian we deduce that A_{i+1} is hopfian.

For the proof of (3) it suffices to show that we can write R in the form

$$R = K \times \bar{R}$$

for some subgroup $\bar{R}$ of R. But recall that $G = FH = (FM)K$ and since $M \triangle G$, FM is a subgroup of G and we can take $\bar{R} = R \cap (FM)$.

Finally, suppose that (4) holds. Let α be an endomorphism of G onto G and let $F_1 = Z(G) \cap F$. Apply Lemma 1 by taking $E = G$ and $L = HF_1$. Since L is of finite index in G so is $L\alpha^i$. Choose r so that $J = LL\alpha \cdots L\alpha^r$. Hence,

$$L \cap L\alpha \cap \cdots \cap L\alpha^r$$

is a central subgroup of finite index in J. Let $Z = Z(J)$ and let $R = HZ$. Choose q so that $q \geqq 1$ and $R\alpha^{-q} = R$. Now $R = HF_2$ where $F_1 \subset F_2 \subset F$.

Now R is an abelian group and so R/H is a periodic abelian subgroup of G/H. Hence R/H is finite. Hence we can find a finite group $F_* \subset F_2$ such that $R = HF_*$. The hopficity of R now follows by considering successive cyclic extensions of H terminating in R. Hence α is an automorphism on R and on G so that G is hopfian.

COROLLARY 1. *If H is a hopfian abelian group with a maximal free abelian direct factor and if $H \triangle G$ and G/H satisfies the maximal condition for subgroups then in seeking to prove or disprove that G must be hopfian, we may assume that G is abelian and that G/H is of prime order and that H does not have an infinite cyclic direct factor.*

Proof. Write $H = M \times K$ as in the proof of Theorem 1 and let α be an endomorphism of G onto G. Construct the subgroups R and A_i as in Theorem 1. Without difficulty one may show that none of the subgroups A_i has an infinite cyclic direct factor. Now if G is nonhopfian, then R_1 is nonhopfian and if we choose j as small as possible so that A_j is nonhopfian, then $j \geqq 1$ and A_{j-1} is hopfian, A_{j-1} does not have an infinite cyclic direct factor and A_j/A_{j-1} is of prime order. On the other hand, if for arbitrary abelian groups A, the existence of a hopfian group $\bar{A}$ of prime index in A, $\bar{A}$ without an infinite cyclic direct factor implies that A is hopfian, we see that each of the A_i above are hopfian so that so is G.

COROLLARY 2. *Let F be a group which satisfies the maximal condition for subgroups. Let G be a semi-direct product of a hopfian abelian group H and F. Then if $Z(G) \cap F$ is of finite index in F then G is hopfian.*

Proof. The proof is almost exactly the same as the proof of the fourth assertion of the theorem.

COROLLARY 3. *If H is an abelian super hopfian group with a maximal free abelian direct factor and if G/H satisfies the maximal condition for subgroups then G is super hopfian.*

Proof. We note that an arbitrary homomorphic image H/H_* of H has a maximal free abelian direct factor. For if

$$H/H_* = M_n/H_* \times L_n/H_*$$

with

$$L_i \subset L_{i+1}, \qquad L_n/H_* \text{ free abelian}$$

then $H = M_n \times F_n$ for some $F_n \approx L_n/H_*$. If M is a free abelian group of finite rank and $H = M \times K$ does not have an infinite cyclic direct factor, we see $K \subset M_n$ so that ultimately the groups L_n are identical. Now if B is an arbitrary normal subgroup of G and if $G_1 = G/B$ and $H_1 = HB/B$ then G_1/H_1 satisfies the maximal condition for subgroups and H_1 is a superhopfian abelian group with a maximal free abelian direct factor. Let α be an endomorphism of G onto G. Construct the groups R and A_i relative to G_1 as in the proof of Theorem 1. As in the proof of Theorem 1, we need only show that R_1 is hopfian. But R_1 contains a super hopfian group of finite index so by Theorem 19 of [8], R_1 is super-hopfian.

COROLLARY 4. *Let H be a hopfian abelian group with a maximal free abelian direct factor M. Let $H = K \times M$. Suppose that K has only finitely many subgroups of prime index p for any prime p. If $H \triangle G$ and G/H obeys the maximal condition for subgroups, then G is hopfian.*

Proof. We note that if K is embedded as a subgroup of prime index in an abelian group K_*, then K_* has only finitely many subgroups of prime index p for any prime p. For any subgroup of prime index p in K_* is either K or contains a subgroup $\bar{K}$ of K such that $[K: \bar{K}] = p$. But there are only finitely many possibilities for $\bar{K}$ and for each $\bar{K}$ there are only finitely many subgroups in K_* containing $\bar{K}$. In particular, if $[K_*: K] = q$ then K_* has only finitely many subgroups of index q. By Theorem 1 of [8], K_* is hopfian. Applying this result consecutively to the subgroups A_i constructed in the proof of Theorem 1, we see that each A_i is hopfian. In particular, $A_s = R_1$ is hopfian so that by our remarks in the proof of Theorem 1, G is hopfian.

COROLLARY 5. *The statement* (3) *may be made stronger by replacing the condition that the maximal free abelian direct factor M of H be normal in G by the condition that the set $M \cdot F$ be a subgroup of G.*

Proof. Examine the proof of (3)

In considering the semi-direct product of a hopfian abelian group H and a free abelian group, we do not need any restriction on H. We mention first the following lemma which is proved in [8]:

LEMMA 2, *Let G be a nonhopfian group containing the normal hopfian subgroup A. If G/A satisfies the maximal condition for normal subgroups, then there exists an endomorphism of G onto G which is not an isomorphism on A.*

THEOREM 2. *Let H be hopfian and abelian. Let F be a free* *abelian group of finite rank, and let G be a semi-direct product of H and F. Then G is hopfian.*

Proof. Suppose the assertion is false. By Lemma 2 we may find an endomorphism α of G onto G which is not an isomorphism on H. Let

$$R = H\cdot H\alpha^{-1} = H\cdot F_1,\ H\cdot H\alpha = H\cdot F_2 = H\alpha\cdot F_1\alpha = S$$

where $F_i \subset F$, $i = 1, 2$. Now $S\alpha^{-1} = R$ so that

$$F/F_1 \approx G/R \approx G/S \approx F/F_2\ .$$

Hence $F_1 \approx F_2$. Now by considering the mapping α of R onto S, we see

$$H/(H \cap H\alpha^{-1}) \approx F_2\ .$$

Hence,

$$H = H \cap H\alpha^{-1} \times H_2,\ H_2 \approx F_2\ .$$

Since $H/H \cap H\alpha$, is a homomorphic image of F_1, we have

$$H = (H \cap H\alpha)\cdot H_1$$

where H_1 is a homomorphic image of F_1, that is, H_1 is a homomorphic image of H_2. Hence we may define an endomorphism α^* of H onto H by letting α_1 be a homomorphism of H_2 onto H_1 and letting α^* agree with α on $H \cap H\alpha^{-1}$ and with α_1 on H_2. Hence α^* is an automorphism so that α is an isomorphism on $H \cap H\alpha^{-1}$, a contradiction of our assumption.

3. Direct products.

LEMMA 3. *If $G = \langle u\rangle \times H = \langle v\rangle \times K$ and if H and K are isomorphic hopfian groups and $\langle u\rangle$ and $\langle v\rangle$ are infinite cyclic groups and if α is an endomorphism of G onto G such that $u\alpha \in K$ then α is an automorphism.*

Proof. Say $u\alpha = k$ and $(u^r h_1)\alpha = v$, $h_1 \in H$, so that if $w = uh_1$, and $z = vk^{1-r}$ then $w\alpha = z$ and

$$G = \langle w\rangle H = \langle z\rangle \times K$$

so that $\langle z\rangle\alpha^{-1} = \langle w\rangle\cdot H_1$, $H_1 \subset H$. We deduce $H/H_1 \approx K$ and it easily follows that α is an automorphism.

THEOREM 3. *Let F be a finitely generated abelian group. Let A be a hopfian group. Suppose that one of the following conditions holds:*

(a) *A has an infinite cyclic direct factor*
(b) *$Z(A)$ is divisible*
(c) *$Z(A)$ is periodic*
(d) *$Z(A) = A$*
(e) *F is finite.*
Then $A \times F$ is hopfian.

Proof. (d) and (e) are proved in [8] and are stated here for the sake of completeness. Let α be an endomorphism of $A \times F$ onto itself and suppose first that (a) holds. We may assume $F = \langle y \rangle$ is an infinite cyclic group.

Let $A = H \times F_*$, $F_* = \langle z \rangle$, F_* an infinite cyclic group. By Lemma (3) we may assume

$$y\alpha = y^i z^j h,\ h \in H,\ i \neq 0,\ j \neq 0 .$$

Let $d = (i, j)$, $i = rd$, $j = sd$. Let $w = y^r z^s$. Choose $v \in F \times F_*$ so that

$$\langle w \rangle \times \langle v \rangle = F \times F_* .$$

Then

$$F \times F_* \times H = \langle w \rangle \times \langle v \rangle \times H .$$

Hence

$$\langle w \rangle \alpha^{-1} = F \times L,\ L \subset F_* \times H$$

so that

$$(F_* \times H)/L \approx \langle v \rangle \times H \approx F_* \times H \approx A .$$

We deduce $L = 1$ and α is an automorphism.

Now suppose (b) holds. Again we may assume that $F = \langle y \rangle$, an infinite cyclic group.

By Lemma 3 we may assume

$$y\alpha = y^r a,\ r \neq 0,\ a \in Z(A) .$$

Write $a = a_1^r$, $a_1 \in Z(A)$. Hence

$$A \times F = A \times F_1$$

where $F_1 = \langle y a_1 \rangle$, and $F_1 \alpha^{-1} = A_1 \times F$, $A_1 \subset A$. Hence $A/A_1 \approx A$ so that $A_1 = 1$. Hence $|r| = 1$ and we easily see that α is an automorphism.

Finally for (c) we deny the assertion. By Lemma 2 we may assume α is not an isomorphism on A. Let $x_1, x_2, \cdots, x_r$ be a set of generators of F. Since $Z(A)$ is periodic we can find an integer $s \geqq 1$ such that if F_1 is the subgroup of F generated by $x_1^s, x_2^s, \cdots, x_r^s$

$$x_i^s \alpha \subset F_1 .$$

Hence if $M = F_1\alpha^{-1}$, we have, F_1 a proper subgroup of M and

$$(F \times A) \mid F_1 \approx (F \times A) \mid M$$

so that $(F \times A) \mid F_1 \approx (F \mid F_1) \times A$ is not hopfian, contrary to part (e) of this theorem.

DEFINITION. We say that a group B is n-normal, n a nonnegative integer, if B has exactly n-proper normal subgroups such that the normal subgroups of B form a chain. The normal subgroups of B will be designated by

$$B_0, B_1, B_2 \cdots B_n, B_{n+1} ,$$

where $B_0 = 1$, $B_{n+1} = B$ and $B_i \subset B_{i+1}$.

We proceed to show that the direct product of a hopfian group and a finite solvable n-normal group is hopfian.

LEMMA 4. *If $G = A \times L$, A hopfian, and if α is an endomorphism of G onto G and $L \subset A\alpha$, then α is an isomorphism on A.*

Proof. We note,
(1) $A\alpha = L \times A \cap A\alpha$
and since $G = A\alpha \cdot L\alpha = L(A \cap A\alpha)L\alpha$, we see
(2) $A \approx G/L \approx [(A \cap A\alpha)L\alpha]/K$
where $K = [(A \cap A\alpha)L\alpha] \cap L$. Here using (1) and (2) we can define a homomorphism γ of $A\alpha$ onto A so that $\alpha\gamma$ induces an automorphism of A and, a fortiori, α is an isomorphism on A.

The following lemma is easy to verify:

LEMMA 5. *If B is n-normal, then*

(a) *If B is a finite group of prime power order then B is cyclic.*

(b) *All central subgroups of B are cyclic p groups for some fixed prime p.*

(c) *If $B \mid B_e \approx B_f$ then $f = n + 1 - e$.*

(d) *If B is not cyclic B/B_e and B_e cannot both be finite p-groups for any prime p.*

(e) *If B is finite and B_e is nilpotent then B_e is a p group, for some prime p.*

Suppose that there exists a hopfian group A and a k-normal group B with k minimal, such that $G = A \times B$ is not hopfian. Let α be an endomorphism of G onto G.

(3) α not an isomorphism on A.
Then we have

LEMMA 6. *If A, B, α and k are as above then there exists integer i and j with $1 \leqq i < j \leqq k$ such that for $q = k + i - j + 1$*

(a) $A\alpha \cap B\alpha = B_i\alpha$, $B \cap A\alpha = B_j = (B_q\alpha)A\alpha$.

(b) $B/B_j \approx B_q \mid B_i$ *and* $B \cap B\alpha^s = 1$ *for all* $|s| \geqq 1$.

(c) B_q *is a central p group for some prime p.*

Proof. The fact that $j \leqq k$ follows from (3) and from Lemma 4. The rest of the assertion follows are in Lemma 11 and Theorem 17 of [8].

THEOREM 4. *The direct product of a hopfian group and a finite solvable n-normal group is hopfian.*

Proof. Suppose the assertion is false. Let B be k normal, k minimal with respect to $A \times B$ being nonhopfian. It suffices to show that if r is any prime divisor of $|B|$, we can find an integer f such that $K = B_{f+1}/B_f$ is an r group and a subgroup L such that

$$B \mid B_f = K \cdot L,\ K \cap L = 1,\ L \neq 1\ .$$

For in particular, if we choose r to be the smallest prime divisor of $|B|$, and if we choose $J \subset K$, $[K: J] = r$, then $J \cdot L$ is of index r in $B \mid B_f$ so that $J \cdot L \triangle B \mid B_f$ but neither of $J \cdot L$ nor K is a subgroup of the other.

Suppose then first that p is a divisor of $|Z(B)|$. Recall that any of the factors in a principal series of a solvable finite group are abelian of prime power order ([6], p. 139). Hence, the groups B_{t+1}/B_t are abelian. Let $B_e = \mathrm{Fr}(B)$ designate the Frattini subgroup of B. Since B is not nilpotent and the Frattini subgroup of a finite group is nilpotent, $B_e \neq B$. Since $\mathrm{Fr}(B/B_e) = 1$ we may conclude from our above remarks and Theorem 7.4.14 of [10] that there exists a subgroup M of B/B_e such that for $K = B_{e+1}/B_e$

$$B/B_e = K \cdot M,\ K \cap M = 1\ .$$

Moreover by Theorem 7.4.8 of [10] B_{e+1} is nilpotent. With the aid of Lemma 5 we may conclude that B_{e+1} is a p group. Hence $M \neq 1$ or else B is a p group and hence a cyclic group which would contradict Theorem 3.

Now let r be a prime divisor of B, $r \neq p$. Choose f maximal so that B_{f+1}/B_f is an r group. By Lemma 6, $f < k$ and p is a divisor of $|B: B_f|$. Also recall that if a prime divides the order of a group it divides the order of the group modulo its Frattini group. (Exercise

9.3.19, [10]). Hence, $\mathrm{Fr}(B/B_f) = 1$ so that as in the preceding paragraph if $L = B_{f+1}/B_f$ we may find a subgroup N of B/B_f such that

$$B/B_f = L \cdot N,\ L \cap N = 1 .$$

Since $p/[B\colon B_f]$, we see that $N \neq 1$, which completes the proof. If we now invoke the Thompson-Feit theorem that a group of odd order is solvable, we have the

COROLLARY. *If B is an n-normal group of odd order, A hopfian, then $A \times B$ is hopfian.*

We now obtain a result which gives some information regardless of whether B is finite or infinite.

THEOREM 5. *The direct product of a hopfian group with an n-normal group, where $0 \leqq n \leqq 3$ is hopfian.*

Proof. Let A, B, k, and α be as defined previously. Let M be the subgroup of G generated by the subgroups $B\alpha^f, f \neq 0$.

If $A \cdot B\alpha^s$ is a maximal subgroups of G among the groups $A \cdot B\alpha^f$ $f \neq 0$, we have for some integer e

$$A \cdot M = A \cdot B\alpha^s = A \times B_e . \tag{4}$$

Furthermore, since $B \cdot A\alpha \subset A\alpha \cdot M\alpha = A\alpha \cdot B_e\alpha$, we see $e \geqq q$ and from (4)

$$B_e \approx B/B_{k+1-e} . \tag{5}$$

Since (4) and Lemma 6 guarantee B_e is central, we see B_{k+1-e} is not central. Hence, $k + 1 - e > e$. Hence,

$$k + 1 + i - j = q \leqq e < (k + 1)/2 .$$

Hence,

$$k + 1 > j > (k + 1)/2 + i \geqq [(k + 1)/2] + 1 . \tag{6}$$

But clearly (6) is impossible for $0 \leqq k \leqq 3$. Hence $k \geqq 4$, which completes the proof.

We now investigate the direct product of a hopfian group A with a finite group E, such that $|E|$ is not divisible by a cube different from 1, or such that $|E| = p^3$. We will show $A \times E$ is hopfian.

LEMMA 7. *Let D be a finite group and suppose that there exists a hopfian group A such that $A \times D$ is not hopfian. Then there exists a homomorphic image C of D such that $|Z(C)|$ is not square free and such that $A \times C$ is not hopfian.*

Proof. Choose a homomorphic image C of D such that $A \times C$ is not hopfian and $|C|$ is minimal. As in Theorem 13 of 8, $Z(C) \neq 1$ and we may choose an endomorphism α of $A \times C$ onto itself such that $C \cap C\alpha^i = 1$ for integers $i \neq 0$, and such that α is an isomorphism on C but not on A. We show that $|Z(C)|$ is divisible by a square. Suppose that this is not the case. Then $Z(C)$ is cyclic. If we write

$$C(A\alpha) = (A\alpha)(C_1\alpha),\ C_1 \subset C\ . \tag{7}$$

We see that C_1 is a central subgroup of C so that we may write (7) as

$$C\cdot A\alpha = A\alpha \times C_2\alpha,\ C_2 \subset Z(C)\ .$$

Write $C_2 = \langle c \rangle$ and say $|C_2| = q$, so that we may find $c_* \in C$ with

$$C = \langle c_* \rangle (C \cap A\alpha)$$

where the order of c_* mod $A\alpha$ is q. Let $c\alpha = a_1c_1$, $a_1 \in A$, $c_1 \in C$. Then either a_1 or c_1 has order q mod $A\alpha$. We claim a_1 has order q mod $A\alpha$. If $q = 1$, this is obvious. If $q > 1$ and c_1 has order q mod $A\alpha$ then,

$$C = \langle c_1 \rangle \times C \cap A\alpha$$

and then by Theorem 3 $A \times C \cap A\alpha$ is not hopfian contradicting the minimality of $|C|$. Hence in any case,

$$C\cdot A\alpha = \langle a_1 \rangle \times A\alpha\ .$$

Hence

$$C_* = a_1^i(a\alpha),\ a \in A,\ (i, q) = 1\ . \tag{8}$$

If

$$L = \langle a\alpha \rangle (C \cap A\alpha)$$

we see, that if γ is the projection of $C\cdot A\alpha$ into itself which maps $\langle a_1 \rangle$ into 1, and leaves $A\alpha$ fixed, then L is just the image of C under γ. It follows that $L \approx C$ and $L \triangle (C\cdot A\alpha)$ and since $L \subset A\alpha$ we have $L \triangle (A \times C)$. Moreover, with the aid of (8), we see $L \cap A = 1$, so that $A \times C = A \times L$, $L \subset A\alpha$, a contradiction of Lemma 4.

THEOREM 6. *The direct product of a hopfian group with a finite group D, $|D| = p^3$, p a prime, is hopfian.*

Proof. Suppose the assertion is false. Choose A and C as in Lemma 12. Hence we may assume p^2 is a divisor of $|Z(C)|$. This implies C is abelian contrary to Theorem 3.

THEOREM 7. *Let D be a finite group whose order is not divisible* *by the cube of an integer different from* 1. *Then the direct product of a hopfian group and D is hopfian.*

Proof. Suppose the assertion is false. Let C be as in Lemma 12 and choose a prime p such that p^2 is a divisor of $|Z(C)|$. Hence if C^* is a central subgroup of C of order p^2, we may write ([11], p. 162, Th. 25)

$$C = C^* \times C_1$$

which with Theorem 3 contradicts the minimality of C.

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